Technische Universität Graz
Institut für Angewandte Mathematik
Assoc. Prof. Dr. Günther Of

Numerical Mathematics 4

Exercise sheet 7, January 30, 2025

Exercise 27: We consider the equilibrium equations

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) = f_i(x) \qquad x \in \Omega, \ i = 1, 2, 3,$$

where $\vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}$ describes the displacement of an elastic body and where $\sigma_{ij}(\vec{u})$ $(i, j = 1, 2, 3)$ are the components of the stress tensor σ which is linked to the linearized strain tensor ε, where $\varepsilon_{ij}(\vec{u}) := \frac{1}{2}\left[\frac{\partial}{\partial x_i} u_j(x) + \frac{\partial}{\partial x_j} u_i(x)\right]$ for $i, j = 1, 2, 3$, by Hooke's law:

$$\sigma_{ij}(\vec{u}) = \lambda \mathrm{tr}(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij}(\vec{u}).$$

Derive Betti's first formula

$$-\int_\Omega \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \sigma_{ij}(\vec{u}) v_i(x) dx = \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(\vec{u})\varepsilon_{ij}(\vec{v}) dx - \int_\Gamma \sum_{i,j=1}^{3} \sigma_{ij}(\vec{u}) n_j(x) v_i(x) ds_x,$$

by multiplying the equilibrium equations with a test function and applying integration by parts.

Insert the strain tensor as well as Hooke's law into the bilinear form to get

$$a(\vec{u}, \vec{v}) = \sum_{i,j=1}^{3} \int_\Omega \sigma_{ij}(\vec{u})\varepsilon_{ij}(\vec{v}) dx = 2\mu \int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v}) dx + \lambda \int_\Omega \mathrm{div}(\vec{u})\mathrm{div}(\vec{v}) dx.$$

Exercise 28: Prove the boundedness of the bilinear form of linear elasticity, see Ex. 27.

Exercise 29: Consider the bilinear form

$$a(\vec{u}, \vec{v}) = 2\mu \int_\Omega \sum_{i,j=1}^{d} \varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v}) dx + \lambda \int_\Omega \mathrm{div}(\vec{u})\mathrm{div}(\vec{v}) dx \qquad \text{and} \qquad \mathrm{div}(\vec{u}) = 0$$

of linear elasticity (exercise 27) with $\mu = 1$ and the bilinear form of the Stokes system (exercise 23)

$$a(\vec{u}, \vec{v}) = \int_\Omega \sum_{i=1}^{d} (\nabla u_i \cdot \nabla v_i) \, dx.$$

Show that the two bilinear forms are equivalent for $\vec{u} \in [H^1(\Omega)]^d$, $\vec{v} \in [H_0^1(\Omega)]^d$, $d \in \{2, 3\}$ and div $\vec{u} = 0$.
Hint: Show that

$$\sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} \, dx = \sum_{i=1}^{d} \langle \mathrm{div} \ \vec{u}, \frac{\partial v_i}{\partial x_i} \rangle_\Omega = 0, \qquad \forall v \in [H_0^1(\Omega)]^d.$$

Consider smooth functions firstly and then apply completion.

Exercise 30: Consider the matrix M and the (expensive) preconditioner P given by

$$M = \begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \quad P^{-1} = \begin{pmatrix} A^{-1} & \\ & S^{-1} \end{pmatrix}$$

where $S = C + BA^{-1}B^\top$. Assume that A and C are symmetric and that A and S are positive definite. For symmetric and positive definite matrices D we can define $D^{-1/2}$ such that $D^{-1} = D^{-1/2}D^{-1/2}$.

Show that $P^{-1}M$ and $P^{-1/2}MP^{-1/2}$ have the same spectrum. Compute all eigenvalues of the matrix $P^{-1/2}MP^{-1/2}$ for $C = 0$. Gain similar insight into the spectrum of $P^{-1/2}MP^{-1/2}$ for $C \neq 0$.