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Numerical Mathematics 4

Exercise sheet 6, January 9, 2025

Exercise 23: Derive the variational problem (5.1.1) of the Dirichlet boundary value problem (5.0.1) of the Stokes equations for sufficiently smooth functions in detail.

Exercise 24: Consider the instable pair $[Q_1]^2 \times P_0$ of finite elements for the quadrilaterals of a Cartesian grid of the square $(0,1) \times (0,1)$. Show that

$$\begin{aligned}\int p_h \operatorname{div} \vec{u}_h dx = &\frac{h}{2} \sum_{i,j=1,\dots,N-1} v_{i,j} \left(-p_{i+1/2,j+1/2} + p_{i-1/2,j-1/2} + p_{i-1/2,j+1/2} - p_{i+1/2,j-1/2}\right) \\ &+ \frac{h}{2} \sum_{i,j=1,\dots,N-1} w_{i,j} \left(-p_{i+1/2,j+1/2} + p_{i-1/2,j-1/2} - p_{i-1/2,j+1/2} + p_{i+1/2,j-1/2}\right)\end{aligned}$$

for $\vec{u}_h = \begin{pmatrix} v_h \\ w_h \end{pmatrix} \in [H_0^1(\Omega)]^2$ using the notation of the lecture.

Exercise 25: Consider the Dirichlet boundary value problem of Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma.$$

Investigate the unique solvability of the mixed problem (1.10).

Exercise 26: Consider the $L_2(T)$ projection $Q_T : L_2(T) \to P_0(T)$ for every element $T \in \mathcal{T}_h$. Prove

$$\begin{aligned}\|Q_T v\|_{L_2(T)} &\le \|v\|_{L_2(T)} \\ \|v - Q_T v\|_{L_2(T)} &\le c h_T |v|_{H^1(T)}\end{aligned}$$

for every $v \in H^1(T)$ by means of scaling arguments and the Bramble Hilbert Lemma.