Technische Universität Graz
Institut für Angewandte Mathematik
Assoc. Prof. Dr. Günther Of

Numerical Mathematics 4

Exercise sheet 3, November 14, 2024

Exercise 9:

a) Consider the quadratic B-splines $B_{i,2}$ on the interval (0,1) using the (slightly modified) definition: Let $x_{-2} = x_{-1} = x_0 = 0 < x_1 < \ldots < 1 = x_N = x_{N+1} = x_{N+2}$ and set

$$B_{i,0}(x) = \begin{cases} 1 & \text{for } x \in [x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$
$$B_{i,p}(x) = \frac{x - x_{i-p}}{x_i - x_{i-p}} B_{i-1,p-1}(x) + \frac{x_{i+1} - x}{x_{i+1} - x_{i+1-p}} B_{i,p-1}(x).$$

Terms related to empty intervals are considered to be zero in the definition. Set up the B-splines $B_{1,2}(x)$ and $B_{2,2}(x)$ for sufficiently many points x_i.

b) Consider the discrete version of Example 2.1.16 on the interval $\Omega = (0,1)$. The considered mesh consist of N segments $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$. Check the discrete inf sup condition (2.2.3) for $Y_h = S_h^{0,-1}(0,1)$ and $X_h = \operatorname{span}\{B_{i,2}\}_{i=1}^N$. Keep in mind the equivalent norm of Exercise 8.

Exercise 11: Let X_1, X_2, Y_1, and Y_2 be Hilbert spaces. Let $f \in X_2'$ and $g \in Y_2'$. Consider the generalized mixed problem to find $(u, \lambda) \in X_1 \times Y_1$:

$$\begin{aligned} a(u,v) + b_1(v,\lambda) &= \langle f, v \rangle \qquad \text{for all } v \in X_2, \\ b_2(u,\mu) &= \langle g, \mu \rangle \qquad \text{for all } \mu \in Y_2 \end{aligned}$$

for continuous bilinear forms $a(\cdot,\cdot) : X_1 \times X_2 \to \mathbb{R}$, $b_1(\cdot,\cdot) : X_2 \times Y_1 \to \mathbb{R}$, $b_2(\cdot,\cdot) : X_1 \times Y_2 \to \mathbb{R}$.
Prove that the conditions

- $A_{K_2 K_1'}$ is an isomorphism from $K_2 = \ker B_2$ to K_1' for $K_1 = \ker B_1$
- $\operatorname{Im}_{X_1} B_2 = Y_2'$
- $\operatorname{Im}_{X_2} B_1 = Y_1'$

imply that the generalized mixed problem has a unique solution.
Find appropriate conditions on the bilinear forms to satisfy above conditions.

Exercise 12: Prove that $H(\operatorname{div}, \Omega) := \{\vec{v} \in [L^2(\Omega)]^d, \operatorname{div} \vec{v} \in L_2(\Omega)\}$ equipped with the inner product

$$\langle \vec{\sigma}, \vec{\tau} \rangle_{\operatorname{div},\Omega} := \int_\Omega (\vec{\sigma} \cdot \vec{\tau} + \operatorname{div} \vec{\sigma} \operatorname{div} \vec{\tau})\, dx$$

is a Hilbert space. Do not forget to show that $\langle \vec{\sigma}, \vec{\tau} \rangle_{\operatorname{div},\Omega}$ is an inner product in $H(\operatorname{div}, \Omega)$.

Exercise 13: Consider the Neumann boundary value problem of the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma$$

for $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ and the solvability condition

$$\int_\Omega f(x)dx + \int_\Gamma g(x)ds_x = 0.$$

Use the constraint

$$\int_\Omega u\,dx = 0$$

to define a unique solution. Formulate this problem as mixed problem by using a Lagrangian multiplier and investigate the unique solvability of the continuous problem and of asuitably chosen discrete problem.

Exercise 14: Let X and Y be Hilbert spaces. Let $A\colon X \to X'$ be a linear, bounded, X elliptic and self-adjoint operator, inducing an equivalent norm in X' by $\|\cdot\|_{A^{-1}} = \langle A^{-1}\cdot, \cdot\rangle^{1/2}$.

a) Derive the variational problem equivalent to the minimization problem

$$J(u) = \min_{v\in Y} \frac{1}{2}\|Bv - f\|^2_{A^{-1}}$$

where $B\colon Y \to X'$, $f \in X'$.

b) Rewrite the variational problem as a mixed problem to avoid the application of A^{-1}.

c) Formulate sufficient conditions on the unique solvability of the mixed problem and a related discrete version.

d) Assume that there exist a unique solution of $Bu = f$. Prove

$$\|p_h\|_X \le c\|u - u_h\|_Y.$$

e) Consider the specific example $X = Y = H^1_0(\Omega)$ and $A = B = -\Delta$ and provide favorable bilinear forms.