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Numerical Mathematics 4

Exercise sheet 2, October 31, 2024

Exercise 6: Prove the following statement without using the continuous equivalent:
Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $m, n \in \mathbb{N}$, and $K = \ker B$.
Then

$$M = \begin{pmatrix} A & B^\top \\ B & 0 \end{pmatrix}$$

is non-singular if and only if:

- $A_{KK} : K \to K$ is surjective (or equivalently injective).
- $B : \mathbb{R}^n \to \mathbb{R}^m$ is surjective (or equivalently $B^\top$ is injective).

The restriction A_{KK} of the matrix A to a subset $Z \subset \mathbb{R}^n$ is defined by applying the orthogonal projection $\pi_Z : \mathbb{R}^n \to Z$ where $\pi_Z \underline{v} \in Z$ is the unique solution of $\underline{w}^\top \pi_Z \underline{v} = \underline{w}^\top \underline{v}$ for all $\underline{w} \in Z$.

Exercise 7: Prove the following statement:
Let X, Y be real Hilbert spaces. Let $f \in X', g \in Y'$ and $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$ be a bounded bilinear form, $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a bounded, symmetric and positive semi-definite bilinear form. Then $(u, \lambda) \in X \times Y$ is a solution of the variational problem (3.1.1) iff

$$\mathcal{L}(u,\mu) \le \mathcal{L}(u,\lambda) \le \mathcal{L}(v,\lambda) \qquad \text{for all } v \in X, \text{ for all } \mu \in Y.$$

The Lagrangian functional is defined as

$$\mathcal{L}(v,\mu) = J(v) + b(v,\mu) - \langle g,\mu\rangle \qquad \text{where} \quad J(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle.$$

Exercise 8:

a) Show that $\|\Delta v\|_{L_2(\Omega)}$ is an equivalent norm in $X = \{v \in H_0^1(\Omega) : \Delta v \in L_2(\Omega)\}$. Hints: Integration by parts and Friedrichs/Poincaré inequality.

b) Derive the variational formulation equivalent to the minimization problem

$$\inf_{v\in X} \frac{1}{2}\|\Delta u + f\|_{L_2(\Omega)}$$

and show its wellposedness for $f \in L_2(\Omega)$.

Exercise 9:

a) Consider the quadratic B-splines $B_{i,2}$ on the interval (0,1) using the (slightly modified) definition: Let $x_{-2} = x_{-1} = x_0 = 0 < x_1 < \ldots < 1 = x_N = x_{N+1} = x_{N+2}$ and set

$$B_{i,0}(x) = \begin{cases} 1 & \text{for } x \in [x_i, x_{i+1}) \\ 0 & \text{else} \end{cases}$$

$$B_{i,p}(x) = \frac{x - x_{i-p}}{x_i - x_{i-p}} B_{i-1,p-1} + \frac{x_{i+1} - x}{x_{i+1} - x_{i+1-p}} B_{i,p-1}.$$

Terms related to empty intervals are considered to be zero in the definition. Set up the B-splines $B_{1,2}(x)$ and $B_{2,2}(x)$ for sufficiently many points x_i.

b) Consider the discrete version of Example 2.1.16 on the interval $\Omega = (0,1)$. The considered mesh consist of N segments $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$. Check the discrete inf sup condition (2.2.3) for $Y_h = S_h^{0,-1}(0,1)$ and $X_h = \operatorname{span}\{B_{i,2}\}_{i=1}^N$. Keep in mind the equivalent norm of Exercise 8.

Exercise 10: Consider the initial value problem

$$u'(x) = f(x) \quad \text{for } x \in \Omega = (0,T), \qquad u(0) = 0$$

and the related variational formulation

$$u \in X: \quad \int_0^T u'(x)v(x)dx = \int_0^T f(x)v(x)dx \quad \text{for all } v \in L_2(0,T)$$

where $X = H^1_{0,}(0,T) = \{v \in H^1(0,T) : v(0) = 0\}$.

a) Show that there exists a unique solution of the variational problem. Hint: Consider derivatives and antiderivatives.

b) Provide appropriate finite-dimensional spaces for the discretization. Check the wellposedness of the related discrete problem. Work out the details of the related system of linear equations.