Eigenvalues of Schrödinger operators and Dirichlet-to-Neumann maps

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Eigenvalues and eigenspaces of selfadjoint Schrödinger operators on $\mathbb{R}^n$ are expressed in terms of Dirichlet-to-Neumann maps corresponding to Schrödinger operators on the upper and lower half space.

1 Introduction

It is known that the eigenvalues of a Schrödinger operator A_D with Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^n$ with a bounded, real-valued potential V coincide with the poles of the meromorphic operator function $\mu \mapsto M^\Omega(\mu)$, where $M^\Omega(\mu)$ is the Dirichlet-to-Neumann map of $-\Delta + V - \mu$, see, e.g., [1,2]. Moreover, for each eigenvalue λ the map

$$\tau : \ker(A_D - \lambda) \to \operatorname{ran} \operatorname{Res}_\lambda M^\Omega, \qquad u \mapsto \partial_\nu u|_{\partial\Omega}$$

(where $\partial_\nu u|_{\partial\Omega}$ denotes the trace of the normal derivative of u at the boundary $\partial\Omega$) is an isomorphism between the eigenspace and the range of the residue of M^Ω at λ; cf. [2]. Such a result is also desirable for a selfadjoint Schrödinger operator $A = -\Delta + V$ in $L^2(\mathbb{R}^n)$, $n \geq 2$. In order to define an operator function which plays the role of M^Ω we introduce the artificial "boundary" $\Sigma := \mathbb{R}^{n-1} \times \{0\}$, which separates $\mathbb{R}^n$ into $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, \infty)$ and $\mathbb{R}^n_- := \mathbb{R}^{n-1} \times (-\infty, 0)$, and consider the Dirichlet-to-Neumann maps $M^\pm(\mu)$ in $L^2(\Sigma)$ corresponding to the Schrödinger operators $-\Delta + V - \mu$ on $\mathbb{R}^n_\pm$, respectively. A natural candidate for the description of the eigenvalues of A is $M(\mu) := (M^+(\mu) + M^-(\mu))^{-1}$; cf. [3] for a similar function defined in the case that Σ is a sphere. In Theorem 2.1 of this note we show that each pole of M is an eigenvalue of A but in general the analog of the map τ is not bijective. We indicate in Theorem 2.2 that this drawback can be avoided by considering a certain 2×2 block operator matrix function with entries formed by $M^\pm$ and M.

2 Characterization of eigenvalues and eigenspaces with Dirichlet-to-Neumann maps

Let $n \geq 2$ and denote by $H^s(\mathbb{R}^n)$ and $H^s(\Sigma)$ the Sobolev spaces of order $s > 0$ on $\mathbb{R}^n$ and Σ, respectively. Moreover, let $V \in L^\infty(\mathbb{R}^n)$ be a real-valued potential. We consider the selfadjoint Schrödinger operator

$$Au = -\Delta u + Vu, \qquad \operatorname{dom} A = H^2(\mathbb{R}^n),$$

in $L^2(\mathbb{R}^n)$. For μ in the resolvent set $\rho(A)$ of A we define

$$\mathcal{N}^\pm_\mu := \{u^\pm_\mu \in H^2(\mathbb{R}^n_\pm) : (-\Delta + V - \mu)u^\pm_\mu = 0\},$$
$$\mathcal{N}_\mu := \{u^+_\mu \oplus u^-_\mu \in \mathcal{N}^+_\mu \oplus \mathcal{N}^-_\mu : u^+_\mu|_\Sigma = u^-_\mu|_\Sigma\},$$

where $v|_\Sigma$ denotes the trace of a Sobolev function v at Σ. Let $\partial_n v := \frac{\partial v}{\partial x_n}$. One can show, that for every $g \in H^{\frac{1}{2}}(\Sigma)$ there exists a unique element $u_\mu \in \mathcal{N}_\mu$ with $\partial_n u^-_\mu|_\Sigma - \partial_n u^+_\mu|_\Sigma = g$. Hence the operator-valued function M defined via

$$\rho(A) \ni \mu \mapsto M(\mu), \qquad M(\mu)\big(\partial_n u^-_\mu|_\Sigma - \partial_n u^+_\mu|_\Sigma\big) := u_\mu|_\Sigma$$

is well-defined. $M(\mu)$ is a bounded operator in $L^2(\Sigma)$ with domain $H^{\frac{1}{2}}(\Sigma)$ and range in $H^{\frac{3}{2}}(\Sigma)$ for every $\mu \in \rho(A)$. Moreover, for every $g \in H^{\frac{1}{2}}(\Sigma)$ the function $\mu \mapsto M(\mu)g$ is holomorphic and has poles of at most order one; cf. [2]. Note that for $\mu \in \mathbb{C} \setminus \mathbb{R}$ the operator $M(\mu)$ coincides with $(M^+(\mu) + M^-(\mu))^{-1}$, where $M^\pm(\mu)$ denotes the Dirichlet-to-Neumann map with respect to $-\Delta + V - \mu$ on $\mathbb{R}^n_\pm$, i.e. $M^\pm(\mu)u^\pm_\mu|_\Sigma = \mp\partial_n u^\pm_\mu|_\Sigma$ for $u^\pm_\mu \in \mathcal{N}^\pm_\mu$, respectively.

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Theorem 2.1 *If $\lambda \in \mathbb{R}$ is a pole of M then λ is an eigenvalue of A, but in general* $\dim \operatorname{ran} \operatorname{Res}_\lambda M \lneq \dim \ker(A - \lambda)$.

Proof. Let $\lambda \in \mathbb{R}$ be a pole of M. We show $\dim \ker(A - \lambda) \geq \dim \operatorname{ran} \operatorname{Res}_\lambda M$, from which, in particular, the first assertion follows. Let $\mu, \nu, z \in \mathbb{C} \setminus \mathbb{R}$ be distinct and let $g \in H^{\frac{1}{2}}(\Sigma)$. For $j, k \in \{\mu, \nu, z\}$ denote by u_j the unique element in $\mathcal{N}_j$ with $\partial_n u_j^-|_\Sigma - \partial_n u_j^+|_\Sigma = g$ and choose u_k analogously. Due to $u_j - u_k \in \operatorname{dom} A$ and

$$(A - j)(u_j - u_k) = (-\Delta + V - j)(u_j^+ - u_k^+) \oplus (-\Delta + V - j)(u_j^- - u_k^-) = (j - k)u_k$$

we obtain $(A - j)^{-1} u_k = \frac{u_j - u_k}{j - k}$ if $j \neq k$. Hence we get

$$\begin{aligned} \big((A - \mu)^{-1}(A - z)^{-1} u_\nu\big)\big|_\Sigma &= \frac{1}{z - \nu} \big((A - \mu)^{-1}(u_z - u_\nu)\big)\big|_\Sigma = \frac{1}{z - \nu} \left[\frac{u_\mu - u_z}{\mu - z} - \frac{u_\mu - u_\nu}{\mu - \nu}\right]\Bigg|_\Sigma \\ &= \frac{1}{z - \nu} \left[\frac{M(\mu)g - M(z)g}{\mu - z} - \frac{M(\mu)g - M(\nu)g}{\mu - \nu}\right]. \end{aligned}$$

By the spectral theorem one gets $iPu_\nu = \lim_{\eta \searrow 0} \eta(A - (\lambda + i\eta))^{-1} u_\nu$, where P denotes the orthogonal projection in $L^2(\mathbb{R}^n)$ onto $\ker(A - \lambda)$. As the map $v \mapsto [(A - \mu)^{-1} v]|_\Sigma$ is continuous from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma)$ we get for $z = \lambda + i\eta$

$$\begin{aligned} \big(Pu_\nu\big)|_\Sigma &= \big[(A - \mu)^{-1}(\lambda - \mu)Pu_\nu\big]|_\Sigma = (\lambda - \mu) \lim_{\eta \searrow 0} \frac{\eta}{i} \big[(A - \mu)^{-1}(A - (\lambda + i\eta))^{-1} u_\nu\big]\big|_\Sigma \\ &= \lim_{\eta \searrow 0} \frac{(\lambda - \mu)\eta}{(z - \nu)i} \left[\frac{M(\mu)g - M(z)g}{\mu - z} - \frac{M(\mu)g - M(\nu)g}{\mu - \nu}\right] = \lim_{\eta \searrow 0} \frac{i\eta}{\lambda - \nu} M(z)g = \frac{\operatorname{Res}_\lambda Mg}{\lambda - \nu}. \end{aligned}$$

We have shown $\big\{u|_\Sigma : u \in P\mathcal{N}_\nu\big\} = \operatorname{ran} \operatorname{Res}_\lambda M$, hence $\dim \ker(A - \lambda) \geq \dim \operatorname{ran} \operatorname{Res}_\lambda M$. In general equality does not hold. For example for a potential V reflection symmetric with respect to Σ (i.e., $V(x', x_n) = V(x', -x_n)$) eigenfunctions with vanishing traces on Σ may exist. □

In order to characterize all eigenvalues and eigenspaces of A we define the block operator matrix function $\mathcal{M}$ via

$$\mu \mapsto \mathcal{M}(\mu) := \begin{bmatrix} M(\mu) & -M(\mu)M^-(\mu) \\ -M^-(\mu)M(\mu) & -M^-(\mu)M(\mu)M^+(\mu) \end{bmatrix}, \qquad \mu \in \mathbb{C} \setminus \mathbb{R}.$$

$\mathcal{M}(\mu)$ is an operator in $L^2(\Sigma) \times L^2(\Sigma)$ with domain $H^{\frac{1}{2}}(\Sigma) \times H^{\frac{3}{2}}(\Sigma)$ and range in $H^{\frac{3}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$. The function $\mathcal{M}$ is holomorphic in the strong sense and can be extended to a strongly holomorphic function (also denoted by $\mathcal{M}$) defined on $\rho(A)$. Similar functions were already considered in, e.g., [5] for the ODE case and in [6, 7] in an abstract setting.

Theorem 2.2 $\lambda \in \mathbb{R}$ *is a pole of* $\mathcal{M}$ *and* $\operatorname{ran} \operatorname{Res}_\lambda \mathcal{M}$ *is finite-dimensional if and only if λ is an isolated eigenvalue of A with finite multiplicity. In this case the map*

$$\mathcal{T} : \ker(A - \lambda) \to \operatorname{ran} \operatorname{Res}_\lambda \mathcal{M}, \qquad u \mapsto \big[u|_\Sigma, -\partial_n u|_\Sigma\big]^\top.$$

is bijective.

We omit the proof of Theorem 2.2, which uses methods similar to the proof of Theorem 2.1 and a unique continuation argument; cf. [4] for a similar reasoning.

Remark 2.3 With the help of the function $\mathcal{M}$ one can even characterize all (embedded and isolated) eigenvalues and the corresponding eigenspaces of A; cf. [4] for the case of a Schrödinger operator on an exterior domain.

Acknowledgements The authors gratefully acknowledge financial support by the Austrian Science Fund (FWF): P 25162-N26.

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