Optimal Dirichlet boundary control for the Navier–Stokes equations

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Optimal control problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded Lipschitz domain with $\Gamma = \partial\Omega$ satisfying $\overline{\Gamma}_\mathrm{D} \cap \overline{\Gamma}_\mathrm{c} \neq \emptyset$ and $\overline{\Gamma}_\mathrm{N} \cap \overline{\Gamma}_\mathrm{c} = \emptyset$. We consider an optimal Dirichlet boundary control problem for the Navier–Stokes equations, see [2], given by:
Minimize the cost functional

$$\mathcal{J}(\underline{u}, \underline{z}) := \frac{1}{2} \|\underline{u} - \overline{\underline{u}}\|^2_{L_2(\Omega)} + \frac{1}{2} \varrho \, |\underline{z}|^2_{H^{1/2}(\Gamma_\mathrm{c})} \tag{1}$$

under the constraint

$$\begin{aligned} -\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}. \end{aligned} \tag{2}$$

We realize the $H^{1/2}(\Gamma_\mathrm{c})$ semi–norm by the duality product

$$|\underline{z}|^2_{H^{1/2}(\Gamma_\mathrm{c})} = \langle S\underline{z}, \underline{z}\rangle_{\Gamma_\mathrm{c}},$$

for all $\underline{z} \in [\widetilde{H}^{1/2}(\Gamma_\mathrm{c})]^n$, where $S : [\widetilde{H}^{1/2}(\Gamma_\mathrm{c})]^n \to [H^{-1/2}(\Gamma_\mathrm{c})]^n$ is the so called Steklov–Poincaré operator of the mixed boundary value problem of the vector valued Laplace equation, see [2].

From the standard theory, see [4], we obtain for (1)–(2) the first order necessary optimality conditions (optimality system).

Primal problem

$$\begin{aligned} -\nu\Delta\underline{u} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p &= \underline{f} && \text{in } \Omega, \\ \nabla\cdot\underline{u} &= 0 && \text{in } \Omega, \\ \underline{u} &= \underline{g} && \text{on } \Gamma_\mathrm{D}, \\ \nu(\nabla\underline{u})\underline{n} - p\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \\ \underline{u} &= \underline{z} && \text{on } \Gamma_\mathrm{c}, \end{aligned}$$

Adjoint problem

$$\begin{aligned} -\nu\Delta\underline{w} - (\nabla\underline{w})\underline{u} - (\nabla\underline{w})^\top\underline{u} - \nabla r &= \underline{u} - \overline{\underline{u}} && \text{in } \Omega, \\ \nabla\cdot\underline{w} &= 0 && \text{in } \Omega, \\ \underline{w} &= \underline{0} && \text{on } \Gamma_\mathrm{D}\cup\Gamma_\mathrm{c}, \\ \nu(\nabla\underline{w})\underline{n} + (\underline{u}\cdot\underline{w})\underline{n} + (\underline{u}\cdot\underline{n})\underline{w} + r\underline{n} &= \underline{0} && \text{on } \Gamma_\mathrm{N}, \end{aligned}$$

Optimality condition

$$-\nu(\nabla\underline{w})\underline{n} - (\underline{u}\cdot\underline{w})\underline{n} - (\underline{u}\cdot\underline{n})\underline{w} - r\underline{n} + \varrho S\underline{z} = \underline{0} \qquad \text{on } \Gamma_\mathrm{c}.$$

Discretization

For our problem we apply a finite element method using finite dimensional subspaces $V_h \subset [H^1_0(\Omega, \Gamma_\mathrm{D}\cup\Gamma_\mathrm{c})]^n$ and $Q_h \subset L_2(\Omega)$ of lowest order, e.g. $\mathcal{P}_1$–$\mathcal{P}_1$, stabilized by the Dohrmann–Bochev method, see [1]. For that, the stabilization term is given by

$$c(p_h, q_h) = \frac{1}{\nu}\int_\Omega (p_h - \Pi_h p_h)(q_h - \Pi_h q_h)\, dx$$

with $L_2(\Omega)$–projection $\Pi_h : L_2(\Omega) \to Q_h^0$.

The Galerkin matrix of the Steklov–Poincaré operator S is given by the Schur complement of the stiffness matrix, i.e.

$$S_h = A_{CC} - A_{CI}A_{II}^{-1}A_{IC}.$$

This representation makes the semi–norm easy to compute.

Numerical results

Let $\Omega = (0,1)^2$, $\Gamma = \Gamma_\mathrm{c}$ with data $\overline{\underline{u}} = (x_2(x_2-1)+1, x_1(x_1-1)+1)^\top$, $\underline{f} = \underline{1}$, $\nu = 1$, $\varrho = 1$. We compare the errors for the $L_2(\Gamma)$ control and the $H^{1/2}(\Gamma)$ control.

	$L_2(\Gamma)$ control		$H^{1/2}(\Gamma)$ control	
L	$\|\underline{z}_{h_9} - \underline{z}_h\|_{L_2(\Gamma)}$	eoc	$\|\underline{z}_{h_9} - \underline{z}_h\|_{L_2(\Gamma)}$	eoc
0	$3.55102\,e-01$	–	$9.64011\,e-03$	–
1	$2.43439\,e-01$	0.54	$3.47652\,e-03$	1.47
2	$1.70947\,e-01$	0.51	$1.72964\,e-03$	1.01
3	$1.19022\,e-01$	0.52	$5.30675\,e-04$	1.70
4	$8.29968\,e-02$	0.52	$1.77762\,e-04$	1.58
5	$5.75723\,e-02$	0.53	$7.22122\,e-05$	1.30
6	$3.92801\,e-02$	0.55	$3.21299\,e-05$	1.17
7	$2.57017\,e-02$	0.61	$1.36679\,e-05$	1.23
8	$1.48376\,e-02$	0.79	$4.55947\,e-06$	1.58

Table 1: Errors and estimated order of convergence for $L_2(\Gamma)$ and $H^{1/2}(\Gamma)$ control.

For the example above we obtain 0.6 order of convergence for the $L_2(\Gamma)$ control and nearly 1.3 order of convergence for the $H^{1/2}(\Gamma)$ approach.

Application to arterial blood flow

We consider the control of the inflow in a real geometry aneurysm–bypass. The blood flow is described by the Navier–Stokes equations, with $Re \approx 100$ corresponding to the kinematic viscosity $\nu = 0.04$, see [3].

We present the differences in numerical results for the control in $L_2(\Gamma_\mathrm{c})$ and the energy space $\widetilde{H}^{1/2}(\Gamma_\mathrm{c})$. More realistic flow behavior is obtained by the control approach for the energy space $\widetilde{H}^{1/2}(\Gamma_\mathrm{c})$.

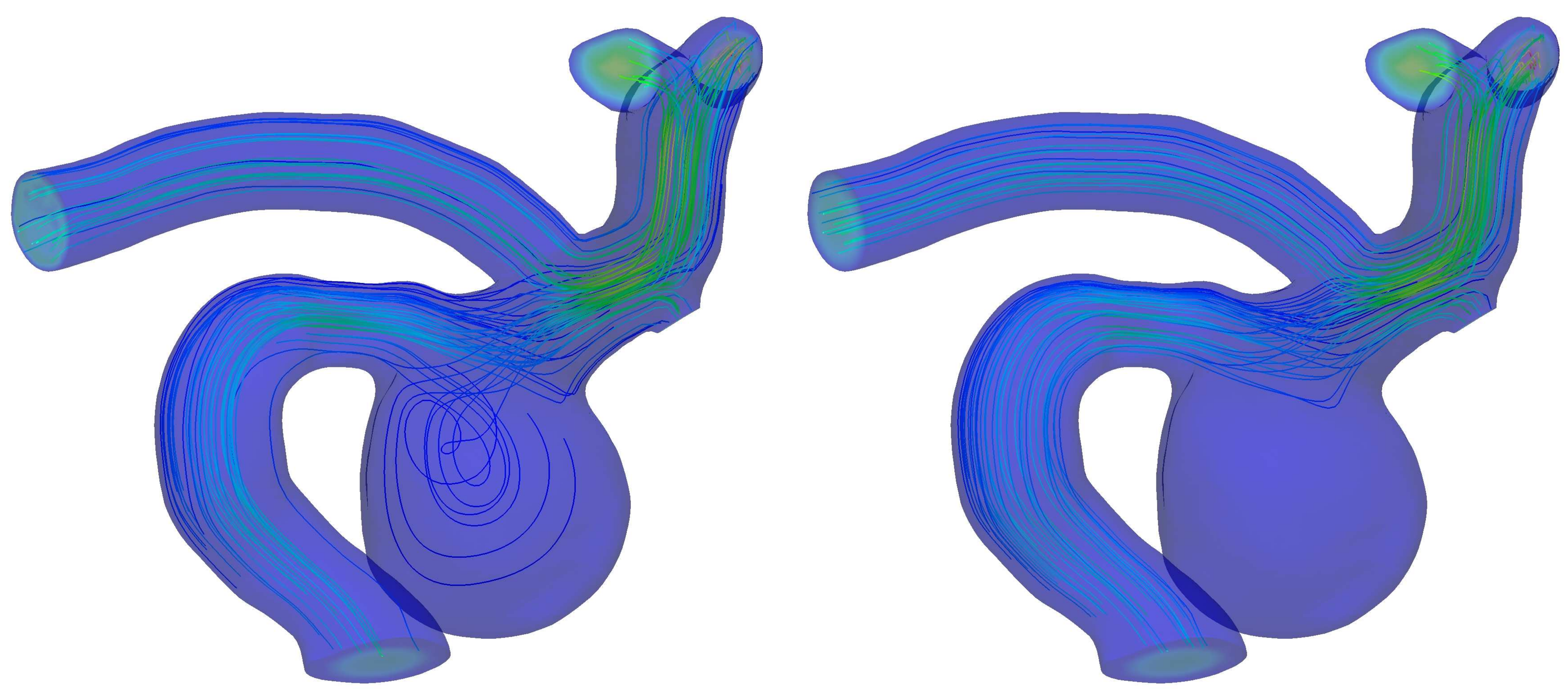

Figure 1: Streamlines for $L_2(\Gamma_\mathrm{c})$ (left), and $\widetilde{H}^{1/2}(\Gamma_\mathrm{c})$ (right).

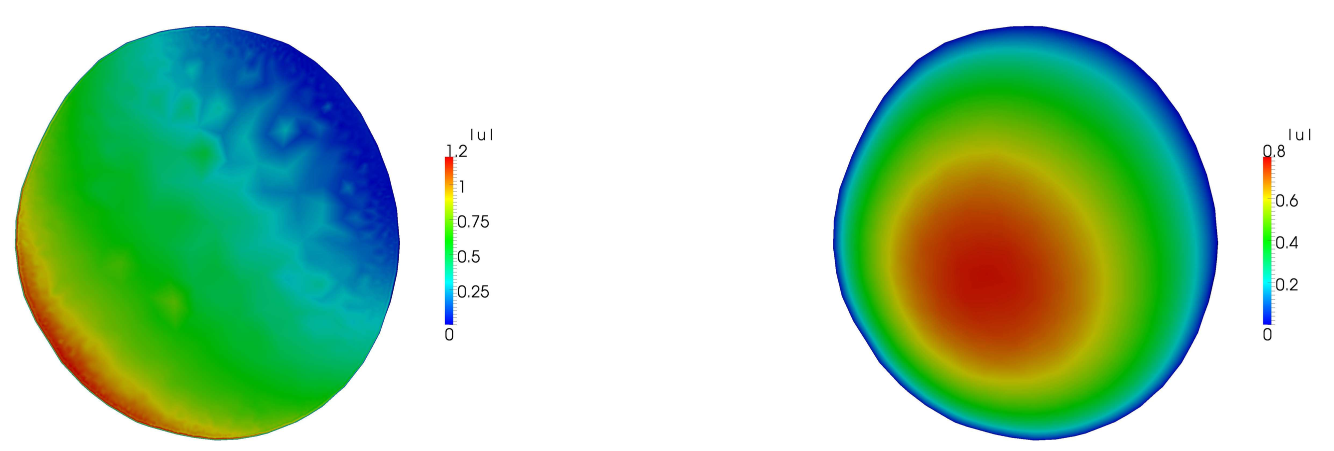

Figure 2: Controlled inflow $\underline{z}$ for $L_2(\Gamma_\mathrm{c})$ (left), and $\widetilde{H}^{1/2}(\Gamma_\mathrm{c})$ (right).

References

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[4] G. Of, T. X. Than, and O. Steinbach. *An energy space finite element approach for elliptic Dirichlet boundary control problems*. Report 2009/13, Institute of Computational Mathematics, Graz University of Technology, 2009.