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Abstract

We propose and analyse preconditioners for the Schur complement of a mixed finite element discretisation of the biharmonic Dirichlet boundary value problem. Since the system matrix is spectrally equivalent to the piecewise defined Sobolev space $\widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma)$ we may use either an appropriate boundary element approximation of local single layer boundary integral operators, or we consider a multilevel preconditioner where the resulting spectral condition number is optimal up to a logarithmic factor. Numerical experiments illustrate the obtained theoretical results.

1 Introduction

In this paper we consider preconditioners for the Schur complement matrix of a mixed finite element discretization of the biharmonic Dirichlet boundary value problem, which has numerous applications in solid and fluid mechanics. The mixed formulation of the biharmonic Dirichlet problem was first given in [3], see also [7] for a further discussion of the stability and error analysis.

First results on iterative methods for the biharmonic Dirichlet problem in a mixed formulation were given for Uzawa–type methods in [11], and for a multilevel algorithm in [19] where convergence was shown when assuming H^3–regularity for the solution. A variable V–cycled multigrid approach was considered for piecewise quadratic or higher order shape functions in [8]. Further, a W–cycled multigrid method was analyzed where the number of smoothing steps has to be sufficiently large. An arbitrary black box multigrid approach for the biharmonic equation was considered in [14]. In [1] the authors consider a preconditioned conjugate gradient method for solving the Schur complement system when eliminating the dual variable. Afterwards, in [20] it is shown that the Schur complement matrix is spectrally equivalent to a mesh depending norm, and the related preconditioner is realized by a special factorization.

A different approach for the iterative solution of the mixed finite element formulation is based on the elimination of all interior degrees of freedom. This requires the solution of two Dirichlet boundary value problems for the Poisson equation, and results in a Schur complement system to find the Dirichlet datum of the dual variable [3, 7]. Since the Schur complement matrix implies an equivalent norm in the piecewise defined fractional Sobolev space $\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)$, several choices of suitable preconditioners are available. First we consider a representation of this particular Sobolev norm by using locally defined single layer boundary integral operators. In fact, this approach corresponds to an additive Schwarz method [9, 15]. Although this method results in a constant bound of the spectral condition number, its realization requires the inversion of a block diagonal matrix including a coarse problem. Instead one may use multilevel representations of fractional Sobolev norms [2, 5, 16, 17], i.e. of $H^{-1/2}(\Gamma)$, which results in an almost optimal preconditioner, where the bound of the spectral condition number is constant up to a logarithmic term. A different choice is the use of the Galerkin discretization of the stabilized hypersingular boundary integral operator [24].

This paper is organized as follows: In Sect. 2 we recall the mixed finite element discretization of the biharmonic Dirichlet boundary value problem and derive the Schur complement system for which we aim to construct a preconditioner. In Theorem 2.4 we present the basic spectral equivalence inequalities for the biharmonic Schur complement. In Sect. 3 we discuss the construction of different preconditioners. First we consider the representation of the piecewise defined fractional Sobolev norm in $\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)$ by using locally defined single layer boundary integral operators and provide related bounds for the spectral condition number. As an alternative strategy we consider a multilevel preconditioner which is spectrally equivalent to the norm in $H^{-1/2}(\Gamma)$, i.e. we prove related norm equivalence inequalities which correspond to an additive Schwarz method. In Sect. 4 we present some numerical experiments for two– and three–dimensional model problems which illustrate the obtained theoretical results. We end the paper by some conclusions and final remarks in Sect. 5.

2 Biharmonic Dirichlet boundary value problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded Lipschitz domain with a piecewise smooth boundary $\Gamma = \partial\Omega$, and let $f \in L_2(\Omega)$ be given. The first boundary value problem of the biharmonic equation is then given by

$$\Delta^2 p(x) = f(x) \quad \text{for } x \in \Omega, \quad p(x) = \frac{\partial}{\partial n_x} p(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.1}$$

For a mixed formulation of (2.1) we introduce an additional unknown $u = -\Delta p$. Integration by parts leads, for all test functions $v \in H^1(\Omega)$, and by using the boundary condition $\partial_n p = 0$ on Γ, to

$$0 = \int_\Omega u(x)v(x)\,dx + \int_\Omega [\Delta p(x)]v(x)\,dx = \int_\Omega u(x)v(x)\,dx - \int_\Omega \nabla p(x) \cdot \nabla v(x)\,dx\,.$$

For the remaining equation, i.e. for $-\Delta u = f$, we obtain analogously

$$\int_\Omega f(x)q(x)\,dx = \int_\Omega [-\Delta u(x)]q(x)\,dx = \int_\Omega \nabla u(x)\cdot\nabla q(x)\,dx \quad \text{for all } q \in H_0^1(\Omega).$$

Hence we conclude the following mixed variational formulation of the boundary value problem (2.1): Find $(u,p) \in H^1(\Omega)\times H_0^1(\Omega)$ such that

$$\begin{array}{rcrcl} \displaystyle\int_\Omega u(x)v(x)\,dx & - & \displaystyle\int_\Omega \nabla p(x)\cdot\nabla v(x)\,dx & = & 0, \\ \displaystyle\int_\Omega \nabla u(x)\cdot\nabla q(x)\,dx & & & = & \displaystyle\int_\Omega f(x)q(x)\,dx \end{array} \tag{2.2}$$

is satisfied for all $(v,q) \in H^1(\Omega)\times H_0^1(\Omega)$. The existence and uniqueness of the solution of problem (2.2) is discussed, e.g., in [3].

For the finite element discretization of the variational formulation (2.2) we consider a globally quasi–uniform admissible finite element mesh $\mathcal{T}_h$, and introduce the finite dimensional subspaces

$$\mathcal{V}_h = \mathrm{span}\{\varphi_i\}_{i=1}^{n_I+n_C} \subset H^1(\Omega), \quad \mathcal{Q}_h = \mathrm{span}\{\varphi_i\}_{i=1}^{n_I} \subset H_0^1(\Omega), \tag{2.3}$$

both of piecewise linear and globally continuous shape functions φ_i. Note that $n_I = \dim \mathcal{Q}_h$ denotes the number of interior degrees of freedom, while n_C is the number of degrees of freedom on the boundary with $\dim \mathcal{V}_h = n_I + n_C$. We introduce the mass matrix $M_h \in \mathbb{R}^{(n_I+n_C)\times(n_I+n_C)}$, the stiffness matrix $K_h \in \mathbb{R}^{n_I\times(n_I+n_C)}$, and the load vector $\underline{f} \in \mathbb{R}^{n_I}$, given by

$$M_h[j,i] = \int_\Omega \varphi_i(x)\varphi_j(x)\,dx, \quad K_h[\ell,i] = \int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_\ell(x)\,dx, \quad f[\ell] = \int_\Omega f(x)\varphi_\ell(x)\,dx$$

for all $i,j = 1,\ldots,n_I+n_C$, and $\ell = 1,\ldots,n_I$. Further, we make use of the isomorphism $(u_h,p_h) \in \mathcal{V}_h\times\mathcal{Q}_h \leftrightarrow (\underline{u},\underline{p}) \in \mathbb{R}^{n_I+n_C}\times\mathbb{R}^{n_I}$. The linear system of algebraic equations, which is equivalent to the Galerkin variational formulation of (2.2), is then given by

$$\begin{pmatrix} M_h & -K_h^\top \\ K_h & \end{pmatrix}\begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix}.$$

According to the interior and boundary degrees of freedom we can decompose the vector $\underline{u} = (\underline{u}_I, \underline{u}_C)^\top$ and thus rewrite the linear system, by using $K_{CI} = K_{IC}^\top \in \mathbb{R}^{n_I\times n_C}$, as

$$\begin{pmatrix} M_{II} & M_{CI} & -K_{II} \\ M_{IC} & M_{CC} & -K_{IC} \\ K_{II} & K_{CI} & \end{pmatrix}\begin{pmatrix} \underline{u}_I \\ \underline{u}_C \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{0} \\ \underline{f} \end{pmatrix},$$

or, by a simple reordering of the variables, as

$$\begin{pmatrix} M_{II} & -K_{II} & M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} \end{pmatrix}\begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{u}_C \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \\ \underline{0} \end{pmatrix}. \tag{2.4}$$

By using
$$\underline{u}_I = K_{II}^{-1}[\underline{f} - K_{CI}\underline{u}_C]$$
and
$$\underline{p} = K_{II}^{-1}[M_{II}\underline{u}_I + M_{CI}\underline{u}_C] = K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f} - K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI}\underline{u}_C + K_{II}^{-1}M_{CI}\underline{u}_C$$
we obtain the Schur complement system
$$\begin{aligned}\Big[M_{CC} - M_{IC}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI}\Big]\underline{u}_C& \\ = \Big[K_{IC}K_{II}^{-1}M_{II} - M_{IC}\Big]K_{II}^{-1}\underline{f}.&\end{aligned} \tag{2.5}$$
In order to use a preconditioned conjugate gradient scheme for an iterative solution of the linear system (2.5) we need to have a preconditioner C_{T_h} for the Schur complement matrix
$$T_h = M_{CC} - M_{IC}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI}\,. \tag{2.6}$$
For all $\underline{v}_C \in \mathbb{R}^{n_C}$ we rewrite the induced bilinear form as
$$\begin{aligned}(T_h\underline{v}_C, \underline{v}_C) &= (M_{CC}\underline{v}_C, \underline{v}_C) - 2\,(M_{IC}K_{II}^{-1}K_{CI}\underline{v}_C, \underline{v}_C) + (M_{II}K_{II}^{-1}K_{CI}\underline{v}_C, K_{II}^{-1}K_{CI}\underline{v}_C) \\ &= (M_{CC}\underline{v}_C, \underline{v}_C) + 2\,(M_{IC}\underline{v}_I, \underline{v}_C) + (M_{II}\underline{v}_I, \underline{v}_I)\end{aligned}$$
with
$$\underline{v}_I := -K_{II}^{-1}K_{CI}\underline{v}_C \in \mathbb{R}^{n_I}.$$
By using the isomorphism $\underline{v} = (\underline{v}_I, \underline{v}_C)^\top \in \mathbb{R}^{n_I+n_C} \leftrightarrow v_h \in \mathcal{V}_h$ we finally obtain
$$(T_h\underline{v}_C, \underline{v}_C) = (M_h\underline{v}, \underline{v}) = \int_\Omega [v_h(x)]^2\,dx = \|v_h\|^2_{L_2(\Omega)}. \tag{2.7}$$
Note that $v_h \in \mathcal{V}_h$ is the discrete harmonic extension of $v_{h|\Gamma} \leftrightarrow \underline{v}_C \in \mathbb{R}^{n_C}$ which is the unique solution of the variational problem
$$\int_\Omega \nabla v_h(x)\cdot\nabla q_h(x)\,dx = 0 \quad \text{for all } q_h \in \mathcal{Q}_h.$$
Since the boundary $\Gamma = \partial\Omega$ was assumed to be piecewise smooth, and thus is decomposable into $J \in \mathbb{N}$ smooth parts, we have
$$\Gamma = \bigcup_{i=1}^J \overline{\Gamma}_i, \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for all } i \neq j,\ i,j \in \{1,\ldots,J\}.$$
Further, we define the Sobolev space of piecewise smooth functions
$$H^{1/2}_{\mathrm{pw}}(\Gamma) = \Big\{v \in L_2(\Gamma)\ :\ v|_{\Gamma_i} \in H^{1/2}(\Gamma_i),\ i = 1,\ldots,J\Big\}, \quad H^{1/2}(\Gamma) \subset H^{1/2}_{\mathrm{pw}}(\Gamma), \tag{2.8}$$

with the corresponding norm

$$\|v\|_{H^{1/2}_{\mathrm{pw}}(\Gamma)} = \left(\sum_{i=1}^{J} \|v_{|\Gamma_i}\|^2_{H^{1/2}(\Gamma_i)} \right)^{1/2}.$$

Recall that for any smooth and open part $\Gamma_i \subset \Gamma$ we have

$$\widetilde{H}^{1/2}(\Gamma_i) = \left\{ v = \widetilde{v}_{|\Gamma_i} \ : \ \widetilde{v} \in H^{1/2}(\Gamma), \ \mathrm{supp}\ \widetilde{v} \subset \overline{\Gamma}_i \right\}$$

with the norm

$$\|v\|_{\widetilde{H}^{1/2}(\Gamma_i)} = \|\widetilde{v}\|_{H^{1/2}(\Gamma)},$$

and for the dual spaces

$$H^{-1/2}(\Gamma_i) = [\widetilde{H}^{1/2}(\Gamma_i)]^*, \quad \widetilde{H}^{-1/2}(\Gamma_i) = [H^{1/2}(\Gamma_i)]^*.$$

Moreover, the dual space of (2.8) is then given by

$$\widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma) = \prod_{i=1}^{J} \widetilde{H}^{-1/2}(\Gamma_i)$$

with the norm

$$\|\psi\|_{\widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma)} = \left(\sum_{i=1}^{J} \|\psi_{|\Gamma_i}\|^2_{\widetilde{H}^{-1/2}(\Gamma_i)} \right)^{1/2}.$$

In fact we have $\widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma) \subset H^{-1/2}(\Gamma)$, i.e. for all $\psi \in \widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma)$ there holds

$$\|\psi\|_{H^{-1/2}(\Gamma)} = \sup_{0 \neq v \in H^{1/2}(\Gamma)} \frac{\langle \psi, v \rangle_\Gamma}{\|v\|_{H^{1/2}(\Gamma)}} \leq \sup_{0 \neq v \in H^{1/2}_{\mathrm{pw}}(\Gamma)} \frac{\sum_{i=1}^{J} \langle \psi_{|\Gamma_i}, v_{|\Gamma_i} \rangle_{\Gamma_i}}{\|v\|_{H^{1/2}_{\mathrm{pw}}(\Gamma)}} \leq \|\psi\|_{\widetilde{H}^{-1/2}_{\mathrm{pw}}(\Gamma)}. \tag{2.9}$$

For a further discussion and generalization for $s \in \mathbb{R}$ of the above Sobolev spaces we refer to [6, 12, 23], and the references therein. The following statement is a consequence of the Neumann trace theorem, see, e.g., [10, Theorem 4.2.1, p. 178] in the case of $C^{1,1}$ domains; or [18, p. 53].

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain with piecewise smooth boundary $\Gamma = \partial\Omega$. For all $v \in H^2(\Omega)$ we have $\partial_n v \in H^{1/2}_{\mathrm{pw}}(\Gamma)$ and*

$$\|\partial_n v\|_{H^{1/2}_{\mathrm{pw}}(\Gamma)} \leq c_T \, \|v\|_{H^2(\Omega)}.$$

Vice versa, for each $\lambda \in H^{1/2}_{\mathrm{pw}}(\Gamma)$ there exists a $v \in H^2(\Omega)$ with $\partial_n v = \lambda$ on Γ and

$$\|v\|_{H^2(\Omega)} \leq c_{IT} \, \|\lambda\|_{H^{1/2}_{\mathrm{pw}}(\Gamma)}.$$

An extended version, see [6, Theorem I.1.6, p. 9], of the second statement of Lemma 2.1 is given as follows, where we enforce zero Dirichlet boundary conditions.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain with piecewise smooth boundary Γ. For each $\lambda \in H^{1/2}_{\text{pw}}(\Gamma)$ there exists a $v \in H^2(\Omega) \cap H^1_0(\Omega)$ with $\partial_n v = \lambda$ and*

$$\|v\|_{H^2(\Omega)} \leq \widetilde{c}_{IT} \, \|\lambda\|_{H^{1/2}_{\text{pw}}(\Gamma)}.$$

The above results are required to prove the following theorem.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a convex and bounded Lipschitz domain with piecewise smooth boundary Γ. For $z \in H^{1/2}(\Gamma)$ let $u_z \in H^1(\Omega)$ be the harmonic extension satisfying*

$$-\Delta u_z(x) = 0 \quad \textit{for } x \in \Omega, \quad u_z(x) = z(x) \quad \textit{for } x \in \Gamma.$$

Then there hold the spectral equivalence inequalities

$$c_1 \, \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)} \leq \|u_z\|_{L_2(\Omega)} \leq c_2 \, \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. We will first prove the upper estimate $\|u_z\|_{L_2(\Omega)} \leq c_2 \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)}$. For a test function $v \in H^2(\Omega) \cap H^1_0(\Omega)$ we have, since $u_z \in H^1(\Omega)$ is the harmonic extension of $z \in H^{1/2}(\Gamma)$, by applying duality and the Hölder inequality,

$$\begin{aligned}
\int_\Omega [-\Delta v(x)] u_z(x) dx &= \int_\Omega \nabla v(x) \cdot \nabla u_z(x) dx - \int_\Gamma \frac{\partial}{\partial n_x} v(x) u_z(x) ds_x \\
&= -\int_\Gamma \frac{\partial}{\partial n_x} v(x) z(x) ds_x = -\sum_{i=1}^{J} \langle \partial_n v_{|\Gamma_i}, z_{|\Gamma_i} \rangle_{\Gamma_i} \\
&\leq \sum_{i=1}^{J} \|\partial_n v_{|\Gamma_i}\|_{H^{1/2}(\Gamma_i)} \|z_{|\Gamma_i}\|_{\widetilde{H}^{-1/2}(\Gamma_i)} \leq \|\partial_n v\|_{H^{1/2}_{\text{pw}}(\Gamma)} \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)}.
\end{aligned}$$

Let $w_z \in H^1_0(\Omega)$ be the unique solution of the Dirichlet problem

$$-\Delta w_z(x) = u_z(x) \quad \text{for } x \in \Omega, \quad w_z(x) = 0 \quad \text{for } x \in \Gamma.$$

Since Ω is assumed to be convex, we have $w_z \in H^2(\Omega) \cap H^1_0(\Omega)$, which satisfies

$$\|w_z\|_{H^2(\Omega)} \leq c \, \|u_z\|_{L_2(\Omega)}.$$

Thus, with Lemma 2.1 we conclude

$$\begin{aligned}
\|u_z\|^2_{L_2(\Omega)} &= \int_\Omega [-\Delta w_z(x)] u_z(x) dx = -\int_\Gamma \frac{\partial}{\partial n_x} w_z(x) z(x) ds_x \\
&\leq \|\partial_n w_z\|_{H^{1/2}_{\text{pw}}(\Gamma)} \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)} \leq c \, \|w_z\|_{H^2(\Omega)} \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)} \\
&\leq c_2 \, \|u_z\|_{L_2(\Omega)} \|z\|_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)}.
\end{aligned}$$

This proves the upper estimate and it remains to show the inequality

$$c_1 \, \|z\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} \le \|u_z\|_{L_2(\Omega)} \quad \text{for all } z \in H^{1/2}(\Gamma).$$

From Lemma 2.2 we conclude that for any $\lambda \in H^{1/2}_{\rm pw}(\Gamma)$ there exists a $v \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $\partial_n v = \lambda$ and

$$\|v\|_{H^2(\Omega)} \le \widetilde{c}_{\rm IT} \, \|\lambda\|_{H^{1/2}_{\rm pw}(\Gamma)}.$$

Hence we find

$$\begin{aligned} \|z\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} &= \sup_{0 \ne \lambda \in H^{1/2}_{\rm pw}(\Gamma)} \frac{\langle z, \lambda \rangle_\Gamma}{\|\lambda\|_{H^{1/2}_{\rm pw}(\Gamma)}} = \sup_{0 \ne \lambda \in H^{1/2}_{\rm pw}(\Gamma)} \frac{\displaystyle\int_\Omega [\Delta v(x)] u_z(x) dx}{\|\lambda\|_{H^{1/2}_{\rm pw}(\Gamma)}} \\ &\le \sup_{0 \ne \lambda \in H^{1/2}_{\rm pw}(\Gamma)} \frac{\|\Delta v\|_{L_2(\Omega)} \|u_z\|_{L_2(\Omega)}}{\|\lambda\|_{H^{1/2}_{\rm pw}(\Gamma)}} \le \widetilde{c}_{\rm IT} \, \|u_z\|_{L_2(\Omega)}, \end{aligned}$$

which completes the proof. ■

Now we are in a position to state the required spectral equivalence inequalities for the Schur complement T_h. For this we define the finite element trace space

$$Z_h = \text{span}\{\phi_k\}_{k=1}^{n_C} := \mathcal{V}_{h|\Gamma} = \text{span}\{\varphi_{n_I+k|\Gamma}\}_{k=1}^{n_C} \subset H^{1/2}(\Gamma).$$

Theorem 2.4 *For all $\underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h$ there hold the spectral equivalence inequalities*

$$(T_h \underline{z}_C, \underline{z}_C) \simeq \|z_h\|^2_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)}.$$

Proof. For $z_h \in Z_h \leftrightarrow \underline{z}_C \in \mathbb{R}^{n_C}$ let $u_{z_h} \in H^1(\Omega)$ be the harmonic extension for which we have, by Theorem 2.3,

$$c_1 \, \|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} \le \|u_{z_h}\|_{L_2(\Omega)} \le c_2 \, \|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)}. \tag{2.10}$$

On the other hand, by defining $\underline{u}_I = -K_{II}^{-1} K_{CI} \underline{z}_C$ and by setting $\underline{u} = (\underline{u}_I, \underline{z}_C)^\top \leftrightarrow u_{z_h,h} \in \mathcal{V}_h$, which is the discrete harmonic extension of z_h, we obtain by using (2.7),

$$(T_h \underline{z}_C, \underline{z}_C) = \|u_{z_h,h}\|^2_{L_2(\Omega)}.$$

Since $u_{z_h,h} \in \mathcal{V}_h$ is the standard finite element approximation of $u_{z_h} \in H^1(\Omega)$, we have, by applying the spectral equivalence (2.10), the standard finite element error estimate in $L_2(\Omega)$, the continuity of the Dirichlet trace of the harmonic extension $u_{z_h} \in H^1(\Omega)$, and an inverse inequality

$$\begin{aligned} \|u_{z_h,h}\|_{L_2(\Omega)} &\le \|u_{z_h}\|_{L_2(\Omega)} + \|u_{z_h,h} - u_{z_h}\|_{L_2(\Omega)} \\ &\le c_2 \, \|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} + c_3 \, h \, \|u_{z_h}\|_{H^1(\Omega)} \\ &\le c_2 \, \|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} + c_4 \, h \, \|z_h\|_{H^{1/2}(\Gamma)} \\ &\le c_2 \, \|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} + c_5 \, \|z_h\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

Now the upper estimate follows by using (2.9). To prove the reverse estimate we first have, by using an inverse inequality, and the bound of the Dirichlet trace of the discrete harmonic extension $u_{z_h,h} \in \mathcal{V}_h \subset H^1(\Omega)$,

$$\|u_{z_h,h}\|_{L_2(\Omega)} \geq c_6 h \, \|u_{z_h,h}\|_{H^1(\Omega)} \geq c_7 h \, \|z_h\|_{H^{1/2}(\Gamma)} = c_7 \, \underbrace{\frac{h\|z_h\|_{H^{1/2}(\Gamma)}}{\|z_h\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)}}}_{=:\alpha} \, \|z_h\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)}.$$

On the other hand we conclude, as above,

$$\begin{aligned} \|u_{z_h,h}\|_{L_2(\Omega)} &\geq \|u_{z_h}\|_{L_2(\Omega)} - \|u_{z_h,h} - u_{z_h}\|_{L_2(\Omega)} \\ &\geq c_1 \, \|z_h\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)} - c_4 h \, \|z_h\|_{H^{1/2}(\Gamma)} = (c_1 - c_4\alpha) \, \|z_h\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)}. \end{aligned}$$

In particular we have

$$\|u_{z_h,h}\|_{L_2(\Omega)} \geq \max\{c_7\alpha, c_1 - \alpha c_4\} \, \|z_h\|_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)},$$

and by using

$$\min_{\alpha>0} \max\{c_7\alpha, c_1 - \alpha c_4\} = \frac{c_1 c_7}{c_7 + c_4} > 0$$

this concludes the proof. ∎

3 Preconditioning strategies

In this section we discuss two different approaches to construct a preconditioner for the Schur complement matrix T_h as defined in (2.6). The first approach is based on the spectral equivalence inequalities in $\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)$, see Theorem 2.4, while the second approach makes use of additional spectral equivalence inequalities which relates T_h to a Sobolev norm in $H^{-1/2}(\Gamma)$.

3.1 Boundary integral operator preconditioning

By using Theorem 2.4 the Schur complement matrix T_h satisfies the spectral equivalence inequalities

$$(T_h \underline{z}_C, \underline{z}_C) \simeq \|z_h\|^2_{\widetilde{H}_{\mathrm{pw}}^{-1/2}(\Gamma)} = \sum_{i=1}^{J} \|z_{h|\Gamma_i}\|^2_{\widetilde{H}^{-1/2}(\Gamma_i)} \quad \text{for all } \underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h. \tag{3.1}$$

For the construction of a preconditioning matrix C_{T_h} it is therefore sufficient to find a computable representation of the local Sobolev norms $\|\cdot\|^2_{\widetilde{H}^{-1/2}(\Gamma_i)}$, $i = 1, \ldots, J$, which can be done by using local boundary integral operators, see, e.g., [10, 23].

For a given $\psi_i \in \widetilde{H}^{-1/2}(\Gamma_i)$, $i = 1, \ldots, J$, we define the local single layer boundary integral operator as

$$(V_i\psi_i)(x) = \int_{\Gamma_i} U^*(x,y)\psi_i(y)ds_y \quad \text{for } x \in \Gamma_i, \tag{3.2}$$

where

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n = 3 \end{cases}$$

is the fundamental solution of the Laplace operator. It turns out that the local single layer boundary integral operator $V_i : \widetilde{H}^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is bounded and $\widetilde{H}^{-1/2}(\Gamma_i)$–elliptic, see, e.g., [13, Theorem 2.4], and hence

$$\|\psi_i\|^2_{V_i} := \langle V_i\psi_i, \psi_i\rangle_{\Gamma_i} \simeq \|\psi_i\|^2_{\widetilde{H}^{-1/2}(\Gamma_i)} \tag{3.3}$$

defines an equivalent norm in $\widetilde{H}^{-1/2}(\Gamma_i)$, $i = 1, \ldots, J$. Note that for the two–dimensional case we assume that the length of all Γ_i is less than 4. By combining the spectral equivalence inequalities (3.1) and (3.3) we therefore conclude the spectral equivalence inequalities

$$(T_h\underline{z}_C, \underline{z}_C) \simeq \sum_{i=1}^{J}\langle V_i z_{h|\Gamma_i}, z_{h|\Gamma_i}\rangle_{\Gamma_i} = \sum_{i=1}^{J}(A_i^\top V_{i,h}A_i\underline{z}_C, \underline{z}_C) \quad \text{for all } \underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h. \tag{3.4}$$

In (3.4), $V_{i,h} \in \mathbb{R}^{n_{C,i}\times n_{C,i}}$ with $n_{C,i} = \dim Z_{h|\Gamma_i}$ is the Galerkin boundary element matrix of the local single layer boundary integral operator given as

$$V_{i,h}[\ell,k] = \langle V_i\phi_{k,i}, \phi_{\ell,i}\rangle_{\Gamma_i} \quad \text{for all } k,\ell = 1,\ldots,n_{C,i},$$

and

$$Z_{h|\Gamma_i} = \operatorname{span}\{\phi_{k,i}\}_{k=1}^{n_{C,i}}$$

is the localized finite element space, for $i = 1, \ldots, J$. The relation between the global and local degrees of freedom is described by connectivity matrices $A_i \in \mathbb{R}^{n_{C,i}\times n_C}$. The spectral equivalence inequalities (3.4) imply the definition of the preconditioning matrix

$$C_{\mathrm{SLP}} := \sum_{i=1}^{J} A_i^\top V_{i,h}A_i\,, \tag{3.5}$$

where the spectral condition number of the preconditioned system,

$$\kappa(C_{\mathrm{SLP}}^{-1}T_h) \le c, \tag{3.6}$$

is bounded by a constant which is independent of the discretization.

The application of the preconditioning matrix $C_{\rm SLP}^{-1}$ requires the solution of a linear system, $\underline{v} = C_{\rm SLP}^{-1}\underline{r}$. Since the preconditioner (3.5) corresponds to an additive Schwarz method for the discrete single layer boundary integral operator [9, 15], it can be realized by solving local subproblems which correspond to all the interior degrees of freedom within Γ_i, and by solving a coarse Schur complement system which corresponds to all degrees of freedom along the interfaces. In the particular case of a two–dimensional polygonal bounded domain the dimension of the coarse system coincides with the number of corner points, which is in general rather small. The situation can be quite different when considering more general three–dimensional polyhedral domains. This motivates the use of global preconditioning strategies such as a multilevel approach wich implies a spectral equivalent preconditioner in $H^{-1/2}(\Gamma)$.

3.2 Multilevel preconditioner

For the definition of a global preconditioning matrix C_{T_h} in $H^{-1/2}(\Gamma)$ we need to have, in addition to (3.1), the spectral equivalence inequalities

$$\|z_h\|^2_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} \simeq \|z_h\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h.$$

By using (2.9) we easily conclude

$$\|z_h\|^2_{H^{-1/2}(\Gamma)} \leq \|z_h\|^2_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} \quad \text{for all } \underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h. \tag{3.7}$$

The proof of the reverse inequality is more involved.

Theorem 3.1 *Let $\Gamma = \cup_{i=1}^J \overline{\Gamma}_i$ be piecewise smooth. Let $z_h \in Z_h$ be a piecewise linear and continuous function which is defined with respect to some admissible and globally quasi–uniform boundary element mesh of mesh size h, which is assumed to be sufficiently small. Then there holds*

$$\|z_h\|_{\widetilde{H}_{\rm pw}^{-1/2}(\Gamma)} \leq \overline{c}_2\, J\, [1 - \log\, h]\, \|z_h\|_{H^{-1/2}(\Gamma)}. \tag{3.8}$$

Proof. The proof of (3.8) follows the ideas as used for the analysis of the additive Schwarz method for the single layer boundary integral operator, see, e.g., [9, 15]. For $s \in (0, \frac{1}{2})$ and $i = 1, \ldots, J$ we first have, see, e.g., [13, Lemma 2.3],

$$\|\phi\|_{\widetilde{H}^s(\Gamma_i)} \leq \frac{c}{1/2 - s}\, \|\phi\|_{H^s(\Gamma_i)} \quad \text{for all } \phi \in H^s(\Gamma_i).$$

By using a duality argument and the inverse inequality we therefore conclude, for $\varepsilon \in (0, \frac{1}{2})$,

$$\begin{aligned}
\|z_h\|_{\widetilde{H}^{-1/2}(\Gamma_i)} &\leq \|z_h\|_{\widetilde{H}^{-1/2+\varepsilon}(\Gamma_i)} = \sup_{0 \neq \phi \in H^{1/2-\varepsilon}(\Gamma_i)} \frac{\langle z_h, \phi\rangle_{\Gamma_i}}{\|\phi\|_{H^{1/2-\varepsilon}(\Gamma_i)}} \\
&\leq \frac{c}{\varepsilon} \sup_{0 \neq \phi \in H^{1/2-\varepsilon}(\Gamma_i)} \frac{\langle z_h, \phi\rangle_{\Gamma_i}}{\|\phi\|_{\widetilde{H}^{1/2-\varepsilon}(\Gamma_i)}} = \frac{c}{\varepsilon}\, \|z_h\|_{H^{-1/2+\varepsilon}(\Gamma_i)} \leq \frac{\widetilde{c}}{\varepsilon}\, h^{-\varepsilon}\, \|z_h\|_{H^{-1/2}(\Gamma_i)}\,.
\end{aligned}$$

Finally, by choosing $\varepsilon = -1/\log h \in (0, \frac{1}{2})$, which is satisfied for sufficient small h, we obtain

$$\|z_h\|_{\widetilde{H}^{-1/2}(\Gamma_i)} \leq c\,[1-\log h]\,\|z_h\|_{H^{-1/2}(\Gamma_i)}.$$

Now the assertion follows by summing up, and by using again a duality argument,

$$\begin{aligned}
\|z_h\|^2_{\widetilde{H}^{-1/2}_{\text{pw}}(\Gamma)} &= \sum_{i=1}^J \|z_h\|^2_{\widetilde{H}^{-1/2}(\Gamma_i)} \leq c^2\,[1-\log h]^2 \sum_{i=1}^J \|z_h\|^2_{H^{-1/2}(\Gamma_i)} \\
&\leq c^2\,[1-\log h]^2 \left(\sum_{i=1}^J \|z_h\|_{H^{-1/2}(\Gamma_i)} \right)^2 \\
&= c^2\,[1-\log h]^2 \left(\sum_{i=1}^J \sup_{0\neq\phi_i\in\widetilde{H}^{1/2}(\Gamma_i)} \frac{\langle z_h,\phi_i\rangle_{\Gamma_i}}{\|\phi_i\|_{\widetilde{H}^{1/2}(\Gamma_i)}} \right)^2 \\
&= c^2\,[1-\log h]^2 \Big(\sup_{\phi=\sum\limits_{i=1}^J \frac{\phi_i}{\|\phi_i\|_{\widetilde{H}^{1/2}(\Gamma_i)}},\ 0\neq\phi_i\in\widetilde{H}^{1/2}(\Gamma_i)} \langle z_h,\phi\rangle_\Gamma \Big)^2 \\
&\leq c^2[1-\log h]^2 \Big(\sup_{\phi\in H^{1/2}(\Gamma),\ \|\phi\|_{H^{1/2}(\Gamma)}\leq J} \langle z_h,\phi\rangle_\Gamma \Big)^2 \\
&\leq c^2\,J^2\,[1-\log h]^2\,\|z_h\|^2_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

■

By combining the spectral equivalence inequalities (3.1) with (3.7) and (3.8) we conclude the spectral equivalence inequalities

$$\widetilde{c}_1\,\|z_h\|^2_{H^{-1/2}(\Gamma)} \leq (T_h \underline{z}_C, \underline{z}_C) \leq \widetilde{c}_2\,J^2\,[1-\log h]^2\,\|z_h\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } \underline{z}_C \in \mathbb{R}^{n_C} \leftrightarrow z_h \in Z_h. \tag{3.9}$$

It remains to find preconditioners which are spectrally equivalent to the discrete norm in $H^{-1/2}(\Gamma)$. One possibility is the use of the stabilized discrete hypersingular boundary integral operator as a preconditioner of opposite order [24]. However, in what follows we will consider a geometric multilevel operator [2, 17] for piecewise linear and continuous basis functions on the boundary to represent the norm in $H^{-1/2}(\Gamma)$. Other choices involve algebraic or artificial multilevel operators as considered in, e.g., [5, 16, 22].

For the construction of the multilevel preconditioner we consider a sequence of admissible globally quasi–uniform nested finite element meshes $\{\mathcal{T}_{h_i}\}_{i\in\mathbb{N}_0}$ of mesh size $h_i \simeq 2^{-i}$. Let $\{\mathcal{V}_{h_i}\}_{i\in\mathbb{N}_0} \subset H^1(\Omega)$ denote the related sequence of finite element spaces with piecewise linear continuous basis functions. Then we consider the restrictions on the boundary,

$$Z_i := \text{span}\{\phi^i_k\}_{k=1}^{n^i_C} = \mathcal{V}_{h_i|\Gamma} = \text{span}\{\varphi^i_{n_{I,i}+k|\Gamma}\}_{k=1}^{n^i_C} \subset H^{1/2}(\Gamma), \quad i \in \mathbb{N}_0.$$

This results in a sequence of nested spaces of the form

$$Z_0 \subset Z_1 \subset \ldots \subset Z_L = Z_{h_L} \subset Z_{L+1} \subset \ldots \subset H^{1/2}(\Gamma)$$

where L denotes the current level of interest. With respect to the boundary element spaces Z_i, $i \in \mathbb{N}_0$, of piecewise linear globally continuous shape functions ϕ_k^i we introduce, for a given $z \in L_2(\Gamma)$, the L_2–projection $Q_i : L_2(\Gamma) \to Z_i$ as the unique solution $Q_i z \in Z_i$ of the variational problem

$$\langle Q_i z, v_{h_i} \rangle_\Gamma = \langle z, v_{h_i} \rangle_\Gamma \quad \text{for all } v_{h_i} \in Z_i.$$

In addition we set $Q_{-1} := 0$. It turns out, see, e.g., [2, 17, 23], that the multilevel operator

$$B_{-1/2} := \sum_{i=0}^{\infty} h_i (Q_i - Q_{i-1}) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \tag{3.10}$$

defines an equivalent norm in $H^{-1/2}(\Gamma)$, and that its inverse operator is given by

$$B_{-1/2}^{-1} = B_{1/2} = \sum_{i=0}^{\infty} h_i^{-1} (Q_i - Q_{i-1}) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

As in [23, Corollary 13.7] we finally conclude that the preconditioner C_{T_h} of the Schur complement T_h is given by

$$C_{\mathrm{BPX}} := M_{h_L} B_{h_L}^{-1} M_{h_L}, \tag{3.11}$$

where

$$M_{h_L}[\ell, k] = \langle \phi_k^L, \phi_\ell^L \rangle_\Gamma, \quad B_{h_L}[\ell, k] = \langle B_{1/2} \phi_k^L, \phi_\ell^L \rangle_\Gamma \quad \text{for } k, \ell = 1, \ldots, n_C^L$$

denote the standard mass matrix and the Galerkin matrix of the multilevel operator $B_{1/2}$. Moreover, by using the spectral equivalence inequalities (3.9) we conclude the following bound for the spectral condition number of the preconditioned system,

$$\kappa(C_{\mathrm{BPX}}^{-1} T_h) \le c\, J^2 \, [1 - \log h]^2 \, . \tag{3.12}$$

Note that J depends on the geometry of Ω, but not on the discretization.

For the application $\underline{v} = C_{\mathrm{BPX}}^{-1} \underline{r}$, e.g., within a conjugate gradient scheme, we obtain, as for the standard BPX multilevel approach [2, 23], the representation

$$\underline{v} = \sum_{i=0}^{L} \alpha_i R_i M_{h_i}^{-1} R_i^\top \underline{r}$$

with coefficients

$$\alpha_i = \begin{cases} \dfrac{1}{h_L} & \text{for } i = L, \\[2ex] \dfrac{1}{h_i} - \dfrac{1}{h_{i+1}} & \text{for } i = 0, \ldots, L-1, \end{cases}$$

where $R_i : \mathbb{R}^{n_{C,i}} \to \mathbb{R}^{n_{C,L}}$ is the prolongation operator which is related to the nested sequence of piecewise linear finite element spaces on the boundary. While for the application of multilevel preconditioners for (pseudo)differential operators of positive order one can replace the inverse mass matrices $M_{h_i}^{-1}$ by its diagonals, this is not possible in the case of the Schur complement T_h which is the Galerkin discretization of a (pseudo)differential operator of order -1, in particular we have $\alpha_i < 0$ for $i = 0, \ldots, L-1$. Hence we need to use the inverse mass matrices $M_{h_i}^{-1}$ which can be realized at low cost.

4 Numerical results

For the numerical experiments we consider the biharmonic Dirichlet boundary value problem (2.1) in the domains $\Omega = B_{1/2}(0)$ and $\Omega = (0, \frac{1}{2})^n$, both for $n = 2, 3$. The linear system (2.5) is solved by a conjugate gradient scheme without (CG) and with (PCG) preconditioning up to a relative error reduction of $\varepsilon = 10^{-8}$. In all following tables we present, for a sequence of different levels L, the number of required PCG iterations, and the related numbers $n_{I,L}$ and $n_{C,L}$ of degrees of freedom in the interior and on the boundary, respectively.

4.1 Example 1

The right–hand side f is chosen such that the exact solution is given by

$$p(x) = \begin{cases} 2^{-n} \prod\limits_{i=1}^{n} (\cos(2\pi x_i) - 1) & \text{for } \Omega = (0, \frac{1}{2})^n, \\ 2^n \exp\left(\sum\limits_{i=1}^{n} x_i^2 - \frac{1}{4} \right)^{-1} & \text{for } \Omega = B_{1/2}(0). \end{cases}$$

For the two–dimensional test problems we first consider the discrete single layer boundary integral operator preconditioner (3.5), see Table 1. As expected from the estimate (3.6) we observe a constant number of PCG iterations for both computational domains. Next we consider the multilevel preconditioner (3.11), the results are given in Table 1 too. In the case of the circular domain $\Omega = B_{1/2}(0)$ with a smooth boundary $\Gamma = \partial\Omega$ we observe a constant number of PCG iterations since the Sobolev spaces $H^{-1/2}(\Gamma) = \widetilde{H}_{\text{pw}}^{-1/2}(\Gamma)$ coincide. In contrast to the circular domain, for the polygonal bounded domain $\Omega = (0, \frac{1}{2})^2$ we observe a slightly increasing number of PCG iterations, which corresponds to the logarithmic behavior of the spectral condition number bound (3.12).

4.2 Example 2

In this example the right–hand side f is chosen by an arbitrary vector with values in $[-1, 1]$, generated by `rand()`. In Table 2 we present iteration numbers for the multilevel preconditioner (3.11) for the two–dimensional test problems. As in Example 1, we observe a constant number of PCG iterations for $\Omega = B_{1/2}(0)$, while for the square $\Omega = (0, \frac{1}{2})^2$ we obtain a logarithmic factor. This logarithmic behavior of the spectral condition number is more obvious when considering the three–dimensional test problem with $\Omega = (0, \frac{1}{2})^3$, see Table 3.

5 Conclusions

In this paper we have proposed and analyzed two different preconditioners for the solution of the Schur complement system (2.4) for the biharmonic Dirichlet boundary value problem.

			$\Omega = B_{1/2}(0)$		$\Omega = (0, \frac{1}{2})^2$	
L	$n_{I,L}$	$n_{C,L}$	C_{SLP}	C_{BPX}	C_{SLP}	C_{BPX}
0	1	4	1	1	1	1
1	5	8	5	5	5	5
2	25	16	10	10	7	6
3	113	32	13	12	5	5
4	481	64	12	11	9	9
5	1 985	128	13	13	11	12
6	8 065	256	14	14	11	14
7	32 513	512	13	15	11	14
8	130 561	1 024	13	15	11	15
9	523 265	2 048	12	15	11	15
10	2 095 105	4 096	12	15	11	16

Table 1: PCG iterations for C_{SLP} and C_{BPX} preconditioner, $n = 2$.

			$\Omega = B_{1/2}(0)$		$\Omega = (0, \frac{1}{2})^2$	
L	$n_{I,L}$	$n_{C,L}$	CG iter	PCG iter	CG iter	PCG iter
0	1	4	1	1	1	1
1	5	8	3	5	3	5
2	25	16	9	11	9	11
3	113	32	19	15	22	16
4	481	64	25	16	30	18
5	1 985	128	33	17	38	19
6	8 065	256	43	17	49	20
7	32 513	512	53	16	63	21
8	130 561	1 024	70	16	80	22
9	523 265	2 048	88	16	101	23
10	2 095 105	4 096	114	16	128	24

Table 2: Iterations for the multilevel preconditioner C_{BPX}, $n = 2$.

The application of the Schur complement T_h requires to solve two Dirchlet boundary value problems for the Poisson equation for which standard multigrid and multilevel methods can be applied. Instead of solving the Schur complement system (2.5) one may also consider the iterative solution of the coupled linear system (2.4), which in particular also requires the use of a preconditioner for the Schur complement. For a related discussion, see, e.g., [4, 21, 25].

The multilevel preconditioner (3.11) for the Schur complement matrix (2.6) of the biharmonic Dirichlet boundary value problem is also an important part when considering preconditioners for the solution of boundary control problems subject to second order

	$\Omega = B_{1/2}(0)$				$\Omega = (0, \frac{1}{2})^2$			
L	$n_{I,L}$	$n_{C,L}$	CG iter	PCG iter	$n_{I,L}$	$n_{C,L}$	CG iter	PCG iter
0	1	18	12	12	1	8	2	2
1	19	66	23	16	9	26	18	15
2	231	258	31	25	91	98	29	24
3	2 255	1 026	43	29	855	386	41	27
4	19 871	4 098	54	30	7 471	1 538	55	30
5	166 719	16 386	71	31	62 559	6 146	72	34

Table 3: Iterations for the multilevel preconditioner C_{BPX}, $n = 3$.

elliptic partial differential equations. Related results will be published elsewhere.

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