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Variational Inequalities and Their Finite Element Discretization

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Abstract

Contact problems are quite frequently phenomena occurring in mechanical applications. It quickly turned out that variational inequalities are an appropriate tool to precisely characterize contact problems. Unfortunately, variational inequalities involve a lot of computational difficulties when solving them numerically. Over the past years, a large amount of methods for the numerical computation of variational inequalities has been developed. Depending on the type and complexity of the contact problem, different kinds of variational inequalities have been evolved. The Signorini problem is a famous contact problem describing the touching between an elastic body and a rigid frictionless foundation. However, frictional contact exhibit in realistic problems. Thus, the investigations of the classical Signorini problem expanded to the examination of the Signorini problem with so-called Coulomb friction. This thesis introduces the classical frictionless Signorini problem, and we will derive its variational inequality. We deeply analyze abstract variational inequalities, distinguishing between inequalities derived from minimization problems and general inequalities, where the question about the existence of a unique solution will be answered for both cases. As an extension, we analyze hemi-variational inequalities, which are associated with frictional contact problems. The Finite Element discretization is subsequently considered for both types of inequalities, variational and hemi-variational, where the convergence of the approximate solutions is investigated. Additionally, we obtain theoretical error estimates, which are finally verified with the outcomes of numerical examples for the simplified Signorini problem and the obstacle problem.

Zusammenfassung

Kontaktprobleme sind häufig auftretende Phänomene in mechanischen Anwendungen. Es stellte sich schnell heraus, dass variationelle Ungleichungen ein geeignetes Hilfsmittel für die genaue Charakterisierung von Kontaktproblemen darstellen. Unglücklicherweise treten bei der numerischen Berechnung von variationallen Ungleichungen viele Schwierigkeiten auf. In den letzten Jahren hat sich eine große Anzahl an Methoden für die numerische Berechnung von variationellen Ungleichungen entwickelt. Unterschiedliche Arten von variationellen Ungleichungen entstanden in Abhängigkeit vom Typ und der Komplexität des Kontaktproblems. Das Signorini Problem ist ein bekanntes Kontaktproblem, dass den Kontakt zwischen einem elastischen Körper und einem staaren reibungslosen Fundament beschreibt. Allerdings tritt Reibung in realistischen Problemen häufig auf, sodass die Forschung des Singorini Problems sich auf die Untersuchung des Signorini Problems mit sogenannter Coulombscher Reibung erweiterte. In dieser Arbeit wird das klassische reibungslose Signorini Problem eingeführt und dessen variationelle Ungleichung hergeleitet. Es werden abstrakte variationelle Ungleichungen vertieft analysiert. Dabei unterscheidet man variationelle Ungleichungen, die von Minimierungsproblemen abgeleitet werden und allgemeine Ungleichungen. In beiden Fällen wird die Frage über die Existenz einer eindeutigen Lösung beantwortet. Desweiteren werden hemi-variationelle Ungleichungen analysiert, die mit reibungsbehafteten Kontaktproblemen in Verbindnung gebracht werden. Die Finite Elemente Diskretisierung von beiden Ungleichungstypen, sowohl variationel als auch hemi-variationel, wird betrachtet und die Konvergenz von Näherungslösungen wird untersucht. Zusätzlich werden theoretische Fehlerschätzer hergeleitet, die schließlich anhand von Ergebnissen numerischer Beispiele für das vereinfachte Signorini Problem und das Hindernis Problem verifiziert werden.

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Contents

Chapter 1

Introduction

In the early nineteenth century, many engineers, physicists and mathematicians have started a rigorous investigation about contact problems of solid bodies after the establishment of the fundamental principles of continuum mechanics. In fact, the first treatment with deformable bodies has been considered within the framework of continuum mechanics and was later transferred to problems involving contact. Surprisingly, there is no consistent definition of the contact. However, many people describe the contact of two bodies as the touching of the surface of the two bodies at a certain time. The first successful contribution about contact problems in elasticity, where a linear elastic body comes in touch with a rigid and frictionless foundation, was achieved by A. Signorini towards the end of the 1950s. Some years later, his student G. Fichera continued the examination on Signorini's problem and represented the first result about the existence and uniqueness of a solution for variational inequalities arising from minimization of functionals on convex subsets of Banach spaces. He named this type of contact problem after his teacher as an honor, calling it the Signorini problem. Fichera's work was the starting point for many scientific researchers to indulge in deep analysis of abstract variational inequalities, which are not necessarily related to minimization problems. However, variational inequalities are often connected to problems defined on convex sets. Hence, the convexity is an essential property for deducing some special results. In 1967, J. Lions and G. Stampacchia [44] published astonishing results about the theory of variational inequalities, followed by G. Duvaut and J. Lions [9] in 1972, associating variational inequalities with mechanical applications. After these contributions, many other works from prestigious authors of this area as K. Atkinson and W. Han [1], D. Kinderlehrer and G. Stampacchia [19], R. Glowinski [13], R. Glowinski, J. Lions and R. Tremolieres [14] and N. Kikuchi and T. Oden [18], to name just a few, have been presented.

Contact problems are inherently nonlinear, such that many difficulties must be dealt with when solving the mathematical problem numerically. Over the past years it turned out that variational inequalities are the most effective mathematical models to describe and solve contact problems, since they appropriately handle the conditions and restrictions on the contact boundary. Nowadays, contact problems are immediately associated with variational inequalities and conversely. Additionally, a wide range of numerical methods for computing approximate solutions of variational inequalities have been developed and many different techniques are provided depending on the structure and level of difficulty of the variational form.

Although the Signorini problem covers various applications of contact problems, there is still a large amount of problems, which require different descriptions. The classical Signorini problem involves the contact of an elastic body with a rigid frictionless foundation. However, frictional contact problems are a frequently discussed topic in mechanical applications, but cause severe difficulties during the computation of solutions. The first investigations of contact problem with dry friction are due to G. Duvaut and J. Lions [9], where the contact of an elastic body with a rigid foundation was considered. In the literature, such problems are usually called

Signorini problem with Coulomb friction, since the physical friction laws are derived from the physician C. Coulomb. Frictional problems require more general observations of variational inequalities, so-called quasi- or hemi-variational inequalities. These type of inequalities involve enormous mathematical complications, where some of them have not been resolved until now. Important first results in this direction have been published by J. Nečas, J. Jarusek, J. Haslinger [46]. This release opened the door for many other researchers to deeply study the theory of frictional contact problems and their related quasi- or hemi-variational inequalities. Some relevant publications are published by J.T. Oden and E. Pires [48], L. Demkowicz and J.T. Oden [33] and M. Cocu [32], to name just a few. The contributions by K. Atkinson and W. Han [1], R. Glowinski [13], R. Glowinski, J.L. Lions, R. Tremolieres [14], N. Kikuchi and T. Oden [18], I. Hlaváček, J. Haslinger, J.Nečas, J. Lovíšek [16], P.D. Panagiotopoulos [24], A.R. Capatina, M. Cocu [31] and A. Capatina, M. Cocou, M. Raous [30] about the numerical approximations of variational and hemi-variational inequalities highly enriched the theory of frictional contact problems. Recent results about the approximation of variational inequalities were obtained by A. Capatina [6], W. Han [34], S. Migorski, S. Zeng [45], R. Krause [41], R. Krause and B. Wohlmuth [43] and W. Han, M. Sofonea [35].

The motivation of this thesis is the detailed examination of the Signorini problem and its derived variational inequality. We consider an abstract form of variational inequalities and examine the relation of variational inequalities and minimization problems. As an extension, we consider more general variational inequalities, not necessarily related to minimization problems, and tackle the question about the existence of a unique solution. We extend our investigations to hemi-variational inequalities, which can be connected to frictional contact, and present a result about the existence and uniqueness of a solution as well. Moreover, we analyze the Finite Element discretization of both types of inequalities, variational and hemi-variational. The convergence of the discrete solution is of special importance. We derive error estimates dependent on the mesh size for certain simplified contact problems, which will be verified with numerical results. A more realistic frictional contact problem will be introduced and we investigate the connection between the deformation of the elastic body and the friction. The structure of the remainder of this work is the following:

Chapter 2 introduces the basic notations and important mathematical concepts and theorems which will be used throughout this thesis. In Chapter 3, we give a short description about the well known linear elasticity theory, which serves as a fundamental foundation for our contact problems. The classical Signorini problem is pictured in Chapter 4, where the (linearized) contact conditions are precisely described. Additionally, we derive its variational form and formulate a simplified version of the Signorini problem and another contact problem, called the obstacle problem. We observe an abstract form of variational inequalities, given in Chapter 5, and differentiate between inequalities coming from two different natures. On the one hand, we can derive variational inequalities arising from minimization problems, which inherently have good properties that are used for the numerical computation. On the other hand, there are general variational inequalities, not necessarily related to minimization problems, where sufficiently good properties need to be forced in order to have a profit in numerical approximations. For both types, we answer the question about the existence of a unique solution. The end of Chapter 5 is devoted to hemi-variational inequalities and to an application of the Signorini problem with Coulomb friction. In Chapter 6, we describe the numerical approximation of variational and hemi-variational inequalities and obtain mesh size dependent error estimates for the simplified Signorini problem and obstacle problem. These theoretical error estimates are verified with numerical examples using different methods in Chapter 7. Furthermore, we introduce a more realistic frictional contact problem, where we investigate the connection between the displacement of an elastic body and the friction coefficients. This work ends with Chapter 8 giving some conclusion and providing a possible outlook to future work.

Chapter 2

Preliminaries

This chapter introduces the basic notations and definitions, which will be used throughout this work. Furthermore, we define appropriate spaces, such that the weak formulations of our problems are well defined. The contribution of this chapter are based on [3, Chapter 2], [4, Chapter 1, Chapter 2], [12, Chapter 5], [15, Chapter 3] and [18, Chapter 5].

2.1 Notations

In this section, the basic notations are introduced, which will be used in the sequel of this thesis.

Notation. The inner product in the Euclidean space $\mathbb{R}^d$ is denoted by

$$x \cdot y = \sum_{i=1}^{d} x_i y_i,$$

for $x, y \in \mathbb{R}^d$. The Euclidean norm is defined by

$$|x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2},$$

for $x \in \mathbb{R}^d$.

For our purpose, we define a tensor as a linear transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$. If we fix a orthonormal basis $\{e_i\}_{i=1}^d$, then for any tensor σ, a matrix $\sigma = (\sigma_{ij})_{i,j=1}^d \in \mathbb{R}^{d \times d}$ is associated, i.e. the linear transformation $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is a tensor if and only if σ is written as a matrix $\sigma \in \mathbb{R}^{d \times d}$ for a fixed orthonormal basis $\{e_i\}_{i=1}^d$, which is denoted by the same symbol. Note that changing the basis does not affect the tensor while a matrix will modify its entries.

Notation. The tensor-vector product is defined by

$$(\sigma \cdot x)_i = \sum_{j=1}^{d} \sigma_{ij} x_j \quad \text{for all } i = 1, ..., d,$$

where $\sigma \in \mathbb{R}^{d \times d}$ and $x \in \mathbb{R}^d$.

Notation. We define the scalar product for tensors by

$$\sigma : \epsilon = \sum_{i,j=1}^{d} \sigma_{ij} \epsilon_{ij},$$

for $\sigma, \epsilon \in \mathbb{R}^{d \times d}$.

Notation. The partial derivative of a function $u : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$D^\alpha u(X) = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} X_1 \cdots \partial^{\alpha_d} X_d},$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is the so-called multiindex and integers $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for $\alpha_i \geq 0$, $i = 1, \dots, d$.

Furthermore, we need the definition of the gradient for scalar functions and vector fields. We denote the space of non-negative real numbers as $\mathbb{R}^+$ and associate $\mathbb{R}^+$ with the time.

Definition 2.1. *The gradient for a sufficiently smooth scalar function $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$$\nabla_X u(X,t) = \left(\frac{\partial u}{\partial X_1}, \ \cdots, \frac{\partial u}{\partial X_d} \right)^T.$$

If $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is a sufficiently smooth vector field, then the gradient of u is given by the tensor field

$$\nabla_X u(X,t) = \begin{pmatrix} \frac{\partial u_1}{\partial X_1} & \cdots & \frac{\partial u_1}{\partial X_d} \\ \frac{\partial u_2}{\partial X_1} & \cdots & \frac{\partial u_2}{\partial X_d} \\ \vdots & \cdots & \vdots \\ \frac{\partial u_d}{\partial X_1} & \cdots & \frac{\partial u_d}{\partial X_d} \end{pmatrix}.$$

The divergence of the vector field u is defined by

$$\text{div } u(X,t) = \sum_{i=1}^{d} \frac{\partial u_i}{\partial X_i}.$$

The divergence of a tensor field $\sigma : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is defined by

$$\text{div } \sigma(X,t) = \nabla_X \cdot \sigma = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial X_1} + \cdots + \frac{\partial \sigma_{1d}}{\partial X_d} \\ \frac{\partial \sigma_{21}}{\partial X_1} + \cdots + \frac{\partial \sigma_{2d}}{\partial X_d} \\ \vdots \\ \frac{\partial \sigma_{d1}}{\partial X_1} + \cdots + \frac{\partial \sigma_{dd}}{\partial X_d} \end{pmatrix}.$$

In addition, we use the following notation for the spaces of continuous functions.

Notation.

$$\begin{aligned} C(\overline{\Omega}) &= \{v : \Omega \to \mathbb{R} \mid v \text{ is continuous on } \overline{\Omega}\}, \\ C^k(\Omega) &= \{v \in C(\Omega) \mid D^\alpha v \in C(\Omega) \text{ for all } |\alpha| \leq k\}, \\ C^\infty(\Omega) &= \{v \in C(\Omega) \mid v \text{ is infinitely differentiable}\}, \\ C_0^\infty(\Omega) &= \{v \in C^\infty(\Omega) \mid v \text{ has compact support}\}, \end{aligned}$$

where the support of a function is $\text{supp}(v) = \overline{\{X \in \Omega \mid v(X) \neq 0\}}$. The space of continuous functions for vector fields is

$$[C(\Omega)]^d = \{v : \Omega \to \mathbb{R}^d \mid v_i \in C(\Omega) \text{ for } i = 1, \dots, d\}.$$

2.2 Function spaces

In the upcoming chapters, we will formulate a mathematical model for the Signorini problem and derive its variational or weak form. For this purpose, we introduce Sobolev spaces, such that the variational formulations of our problems are well defined. We need the definition of the weak or generalized derivative in order to define the Sobolev spaces.

Definition 2.2. *Let $w : \mathbb{R}^d \to \mathbb{R}$ be a integrable function. We say u is the weak derivative of w if*

$$\int_\Omega u D^\alpha \varphi \; dx = (-1)^{|\alpha|} \int\limits_\Omega w\varphi \; dx \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2.1}$$

We often use the notation $w = D^\alpha u$ for the weak derivative, but consider it as in (2.1).

Remark 2.3. *Applying the weak derivative (2.1) to the gradient of Definition 2.1, we obtain the (spatial) weak or generalized gradient. For a function $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, the weak gradient $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is defined by*

$$\int\limits_\Omega u(X,t) \cdot \varphi(X,t) \; dX = - \int\limits_\Omega w(X,t) \; \text{div} \; \varphi(X,t) \; dX \quad \text{for all } \varphi \in [C_0^\infty(\Omega)]^{d+1},$$

also denoted by $\nabla w = u$.

We are now in the position to introduce the Lebesgue-measurable spaces $L_p(\Omega)$. From now on, we assume $\Omega \subset \mathbb{R}^d$ to be a bounded Lipschitz domain, i.e. the boundary can be described by a Lipschitz continuous parametrization, c.f. [12, Chapter 5].

Definition 2.4. *Let $\Omega \subset \mathbb{R}^d$ and $1 \leq p < \infty$. The Lebesgue space $L_p(\Omega)$ of measurable functions is defined by*

$$L_p(\Omega) = \left\{ v : \int_\Omega |v(x)|^p \; dx < +\infty \right\},$$

with the norm

$$\|v\|_{L_p(\Omega)} = \left(\int_\Omega |v(x)|^p \; dx \right)^{1/p}.$$

The space of essentially bounded, i.e. bounded up to a set of measure zero, functions is denoted by

$$L_\infty(\Omega) = \left\{ v : \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| < +\infty \right\},$$

with norm

$$\|v\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

Theorem 2.5. *Let $1 \leq p \leq \infty$. The Lebesgue space $(L_p(\Omega), \|\cdot\|_{L_p(\Omega)})$ is a Banach space, i.e. complete normed vector space. In addition, $L_2(\Omega)$ equipped with the inner product*

$$(u,v)_{L_2(\Omega)} = \int\limits_\Omega u(x)v(x) \; dx,$$

for $u, v \in L_2(\Omega)$, is a Hilbert space, i.e. a Banach space with a norm induced by a inner product (c.f. [12, Chapter 5]).

Proof. See [8, Chapter 4]. □

Remark 2.6. *Note that the two functions $u, v \in L_p(\Omega)$ are equal in $L_p(\Omega)$ if $\|u - v\|_{L_p(\Omega)} = 0$. This means, if u and v have different values only on sets with measure zero, then they would be treated equally in the L_p-sense and we would say, that u and v are identical almost everywhere. The boundary $\partial\Omega$ of Ω is a set of measure zero, hence the boundary values of functions in the $L_p(\Omega)$ space are not well defined. We will later overcome this problem with the concept of traces.*

With the help of Lebesgue spaces we can define the so-called Sobolev spaces $W_p^k(\Omega)$.

Definition 2.7. *The Sobolev space $W_p^k(\Omega)$ is the space of all functions v, whose weak derivatives up to order k belong to $L_p(\Omega)$, i.e.*

$$W_p^k(\Omega) = \{v \in L_p(\Omega) \mid D^\alpha v \in L_p(\Omega) \text{ for all } \alpha \leq k\},$$

with norm

$$\|v\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |[D^\alpha v](x)|^p \; dx \right)^{1/p},$$

for $1 \leq p \leq \infty$.

Theorem 2.8. *Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then the Sobolev space $(W_p^k(\Omega), \|\cdot\|_{W_p^k(\Omega)})$ is a Banach space.*

Proof. See [8, Chapter 4]. □

Remark 2.9. *If $p = 2$, we will make use of the notation*

$$H^k(\Omega) = W_2^k(\Omega).$$

The Sobolev space $H^k(\Omega)$ for every $k \in \mathbb{N}$ is a Hilbert space with norm denoted by

$$\|v\|_k = \|v\|_{W_2^k(\Omega)},$$

and inner product

$$(u, v)_k = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L_2(\Omega)}.$$

Note that for $k = 0$, we consider the space $L_2(\Omega)$ with norm denoted by

$$\|v\|_0 = \|v\|_{L_2(\Omega)}.$$

In addition, we define the semi-norm in $H^k(\Omega)$ by

$$|v|_k = \left(\sum_{|\alpha| = k} \int_\Omega |[D^\alpha v](x)|^p \; dx \right)^{1/p}.$$

Additionally, we will need the concept of the dual space of $W_p^k(\Omega)$, which is the denoted by $[W_p^k(\Omega)]^*$. It is defined as the set of all linear and bounded mappings $l : W_p^k(\Omega) \to \mathbb{R}$ (usually called functionals), which forms a normed space with norm

$$\|l\|_* = \sup_{0 \neq v \in W_p^k(\Omega)} \frac{|l(v)|}{\|v\|_{W_p^k(\Omega)}},$$

where $l(v)$ shall denote the duality pairing $\langle l, v \rangle$, with $\langle \cdot, \cdot \rangle : [W_p^k(\Omega)]^* \times W_p^k(\Omega) \to \mathbb{R}$.

As we have already remarked, see Remark 2.6, it is not possible to distinguish functions in Lebesgue spaces on zero sets like the boundary $\partial\Omega$. A remedy for this problem is the introduction of so-called traces.

Theorem 2.10 (Trace theorem). *Let Ω be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Then there exists a bounded linear operator*

$$T : W_p^1(\Omega) \to L_p(\Gamma)$$

and a constant $c > 0$ not depended on u, such that

i) $Tu = u|_\Gamma$ for $u \in W_p^1(\Omega) \cap C(\overline{\Omega})$, and

ii)

$$\|Tu\|_{L_p(\Gamma)} \leq c\|u\|_{W_p^1(\Omega)},$$

for all $u \in W_p^1(\Omega)$.

Proof. See [12, Chapter 5]. □

The operator T is called trace operator and Tu is the trace of $u \in W_p^1(\Omega)$ on $\Gamma = \partial\Omega$. The image of this mapping defines as new function space on the boundary Γ. For our purpose, it is enough to consider the case $p = 2$,

$$T(H^1(\Omega)) = H^{1/2}(\Gamma) \subset L_2(\Gamma). \tag{2.2}$$

Thus, we can define $H^{1/2}(\Gamma)$ as the following space

$$H^{1/2}(\Gamma) = \{w \in L_2(\Gamma) \mid \exists v \in H^1(\Omega) : w = Tv\}.$$

The norm in $H^{1/2}(\Gamma)$ is defined by

$$\|w\|_{H^{1/2}(\Gamma)} = \inf\{\|v\|_1 \mid v \in H^1(\Omega),\ w = Tv\}.$$

The dual space of $H^{1/2}(\Gamma)$ is given by $H^{-1/2}(\Gamma)$ and its norm is

$$\|g\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma)} \frac{|g(w)|}{\|w\|_{H^{1/2}(\Gamma)}}.$$

A very important space for our framework is $H^1(\Omega)$, whose elements have zero trace on the boundary. We denote this space by

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid Tv = v|_\Gamma = 0 \text{ on } \Gamma = \partial\Omega\}.$$

Remark 2.11. *For the special case $p = 2$ and zero trace zero trace on the boundary, i.e. $W_{2,0}^k(\Omega) = H_0^k(\Omega)$, the dual space is denoted by $H^{-k}(\Omega)$.*

2.3 Important theorems

We close this chapter with important theorems, which we frequently use throughout this work. Considering the fundamental theorem of calculus, we can state the Gauss theorem.

Theorem 2.12 (Gauss theorem). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $v : \overline{\Omega} \to \mathbb{R}$ be a scalar field, $\varphi : \overline{\Omega} \to \mathbb{R}^d$ a vector field and $\sigma : \overline{\Omega} \to \mathbb{R}^{d\times d}$ a tensor field, all of them sufficiently smooth. Then we have the following identities:*

$$\int_{\partial\Omega} v(x)n(x)\ ds = \int_{\Omega} \nabla v(x)\ dx, \tag{2.3}$$

$$\int_{\partial\Omega} \varphi(x) \cdot n(x)\ ds = \int_{\Omega} \text{div}\ \varphi(x)\ dx, \tag{2.4}$$

$$\int_{\partial\Omega} \sigma(x) \cdot n(x)\ ds = \int_{\Omega} \text{div}\ \sigma(x)\ dx, \tag{2.5}$$

where $n(x)$ is the unit outer normal on the boundary $\partial\Omega$.

Immediately, we can deduce the following integration by parts formula from Theorem 2.12.

Corollary 2.13. *Under the assumptions of Theorem 2.12 the subsequent identities are valid:*

$$\int_\Omega v(x)\ \mathrm{div}\ \varphi(x)\ dx = -\int_\Omega \nabla v(x)\cdot\varphi(x)\ dx + \int_{\partial\Omega} v(x)(\varphi(x)\cdot n(x))\ ds, \tag{2.6}$$

$$\int_\Omega \varphi(x)\ \mathrm{div}\ \sigma(x)\ dx = -\int_\Omega \nabla\varphi(x) : \sigma(x)\ dx + \int_{\partial\Omega} \varphi(x)(\sigma(x)\cdot n(x))\ ds. \tag{2.7}$$

Lastly, an important theorem is Banach's fixed point theorem, which will be used in the sequel to prove existence results of a solution for variational inequalities.

Theorem 2.14 (Banach's fixed point theorem)**.** *Let $\mathbb{K}$ be a nonempty closed set in a Banach space V and let $B : \mathbb{K} \to \mathbb{K}$ be a contraction, i.e. a mapping such that*

$$\|B(u) - B(v)\|_V \leq \alpha\|u - v\|_V \quad \textit{for all } u, v \in \mathbb{K},$$

with a constant $\alpha \in [0, 1)$. Then there exists a unique $u \in \mathbb{K}$, such that

$$B(u) = u,$$

called the fixed point of B. In addition, for any $u_0 \in \mathbb{K}$, the sequence $\{u_n\}_{n\geq 0} \subset \mathbb{K}$, defined by $B(u_n) = u_{n+1}$, converges to the fixed point u, i.e.

$$\|u_n - u\|_V \to 0 \qquad \textit{as } n \to \infty.$$

Moreover, the following bounds are valid:

$$\|u_n - u\|_V \leq \alpha\|u_{n-1} - u\|_V \leq \cdots \leq \alpha^n\|u_0 - u\|_V,$$
$$\|u_n - u\|_V \leq \frac{\alpha^n}{1-\alpha}\|u_0 - u_1\|_V.$$

Proof. See [1, Chapter 5]. □

Chapter 3

Linear Elasticity

The aim of this chapter is to derive the well known model of linear elasticity and to associate the equations of the model with their physical meaning. For our purpose, we consider solid elastic materials, which have the essential characteristic that on every part of the body the same physical properties can be obtained. Usually, external forces are applied to the surface of the material, which lead to a deformation of the material body. In the framework of elasticity, the body returns to its original shape if the external loads are removed, otherwise we fall into the theory of plasticity. We call the original state of the body the reference configuration and denote it by $\Omega_0 \subset \mathbb{R}^d$, $d = 1, 2, 3$, throughout this chapter. Applying forces to the surface (boundary) of the material, the body changes its shape to $\Omega_t \subset \mathbb{R}^d$ after some time $t \in \mathbb{R}^+$, which is called the deformed configuration. Indeed, a broad range of problems for solids are described by linear elasticity realistically. We only consider macroscopic, i.e. large scale, behavior of materials for the elasticity problem since it is appropriate for almost all engineering purposes and we ignore the microscopic, or small scale, behavior. The goal is now to derive the mathematical model describing the displacement of a so-called elastic St.Venant-Kirchhoff material after the deformation based on the work of [2, Chapter 3], [3, Chapter 9], [4, Chapter 6], [7, Chapter 4], [40] and [49]. The equations of the model will be derived from general physical principles such as conservation laws, on one hand and from constitutive laws, which describe the material properties, on the other hand.

3.1 Eulerian and Lagrangian coordinates and deformation

In general, it is important to differentiate between the material points of the body in the reference configuration and the deformed configuration. For this purpose, we define the Eulerian and Lagrangian coordinates, respectively.

Definition 3.1. *The Lagrangian coordinates describe the position of the material points in the reference configuration (at time $t = 0$), which is given by*

$$X = \sum_{i=1}^{d} X_i e_i, \tag{3.1}$$

where X_i are the coefficients of the position vector in the reference configuration and e_i are the unit vectors of $\mathbb{R}^d$.

Definition 3.2. *The spatial or Eulerian coordinates describe the position of the material points* *in the deformed configuration, which is given by*

$$x = \sum_{i=1}^{d} x_i e_i, \tag{3.2}$$

where x_i are the coefficients of the position vector in the reference configuration and e_i are the unit vectors of $\mathbb{R}^d$.

Since the Lagrangian and Eulerian coordinates describe the same points in different configurations, there must exist a connection between these two types, which we want to call the motion or deformation of the body.

Definition 3.3. *The mapping $\varphi : \Omega_0 \times \mathbb{R}^+ \to \Omega_t$, which maps the reference configuration to the deformed configuration at time t, i.e.*

$$x = \varphi(X, t), \tag{3.3}$$

is called the motion or deformation of the body Ω_0. It describes the position of the point $X \in \Omega_0$ after the deformation $\varphi(X, t)$ at time t. The reference configuration is the original configuration of the body at time $t = 0$, so

$$X = \varphi(X, 0).$$

We want to point out, that not every mapping $\varphi : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^d$ is a valid deformation. Some requirements must be admitted to keep the deformation physically meaningful. The following three conditions must be fulfilled to satisfy a smooth deformation.

1. The function $\varphi(X, t)$ is continuously differentiable with respect to X and t.
2. The function $\varphi(X, t)$ is injective, i.e. whenever $\varphi(X_1, t) = \varphi(X_2, t)$, then $X_1 = X_2$ for a fixed time $t \in \mathbb{R}^+$.
3. The determinant of the deformation gradient $\nabla_X \varphi$, i.e. Jacobian matrix of the deformation φ, satisfies $\det(\nabla_X \varphi) > 0$. Usually the deformation gradient $\nabla_X \varphi$ is denoted by F, i.e. $\nabla_X \varphi(X, t) = F(X, t)$ for all $X \in \Omega_0$ and $t \in \mathbb{R}^+$.

The mapping $\varphi(X, t)$ is assumed to satisfy the above mentioned conditions except for sets of measure zero. The first assumption is needed for the ability to compute derivatives for the location, i.e. the deformation gradient, and the velocity, i.e. the time derivative with respect to t. The injectivity requirement ensures, that the body does not penetrate itself. So every point in the reference configuration Ω_0 has a unique point in the deformed configuration Ω_t, and vice versa. This is ensured if the Jacoby matrix (Jacobian) $\nabla_X \varphi$ is uniformly regular. Since the deformation gradient is regular, i.e. the inverse exists, its determinant cannot be equal to zero, which gives a relation between the second and third assumption. In the third assumption we even propose the determinant of the deformation gradient to be greater 0. It describes the local orientation of the body, which shall not be changed, as we obtain later in the upcoming section about the mass conservation. Some examples for deformations are rigid rotations, which preserve distances, simple shears, or compressions and tensions. Rigid deformations are very simple and do not change the length, the surface and the volume of the body.

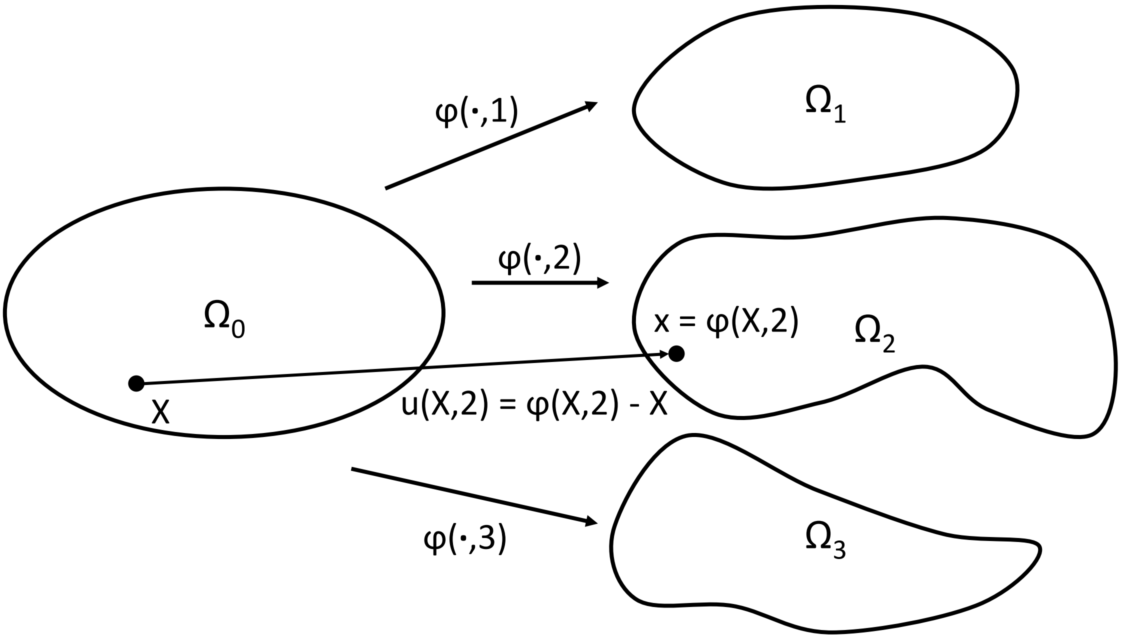


Figure 3.1: Deformation of body Ω_0 at time $t = 1, 2, 3$ and the displacement u of a point X.

3.2 Displacement, velocity and acceleration

Next, we describe the displacement, which plays an important role in the equations of the mathematical model of elasticity. Shortly, we give the definitions of the velocity and the acceleration, but we recommend [21, Chapter 2] for detailed description and their physical properties. We want to start with the definition of the displacement.

Definition 3.4. *The displacement $u \in \mathbb{R}^d$ of a material point $X \in \Omega_0$ is the difference between the current position $x \in \Omega_t$ and the original position of the point X and is defined by*

$$u(X,t) = x - X = \varphi(X,t) - X = \varphi(X,t) - \varphi(X,0). \tag{3.4}$$

Figure 3.1 illustrates the deformation of a body Ω_0 and the displacement of a point X. The velocity and the acceleration in the Lagrangian coordinates are the following.

Definition 3.5. *The velocity $V : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^d$ in Lagrangian coordinates is defined by*

$$V(X,t) = \frac{\partial}{\partial t}\varphi(X,t) = \frac{\partial}{\partial t}u(X,t), \tag{3.5}$$

which is the rate of change of the position for a fixed material point $X \in \Omega$.

The acceleration $A : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^d$ in Lagrangian coordinates is defined by

$$A(X,t) = \frac{\partial}{\partial t}v(X,t) = \frac{\partial^2}{\partial t^2}u(X,t), \tag{3.6}$$

which is the rate of change of the velocity for a fixed material point $X \in \Omega$.

It is possible to obtain the velocity and the acceleration in terms of the spatial coordinates. For this reason we will make use of the chain rule.

Definition 3.6. *The velocity* $v : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^d$ *in spatial coordinates is defined by*

$$v(x,t) = V(\varphi^{-1}(x,t),t). \tag{3.7}$$

The acceleration $a : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^d$ *in spatial coordinates is given by*

$$a(x,t) = A(\varphi^{-1}(x,t),t). \tag{3.8}$$

Remark 3.7. *Note that the time derivatives are usually called material derivatives. Furthermore, the following relation in spatial coordinates holds.*

$$a(x,t) = \frac{D}{Dt}v(x,t) = \frac{D}{Dt}v(\varphi(X,t),t) = \frac{\partial}{\partial t}v(x,t) + v \cdot \nabla_X v, \tag{3.9}$$

where the last term is called the transport term and is defined by $v \cdot \nabla_X v = \sum_{i=1}^{d} v_i \frac{\partial v(x,t)}{\partial x_i}$. *The last equality in (3.9) follows from the chain rule.*

3.3 Strain tensor

A very important quantity in Continuum mechanics is the so-called strain measure. It is the measure of deformation representing the displacement between two particles in the deformed body relative to a reference length in the reference configuration. Figure 3.2 illustrates this incident, where the reference length is denoted by dX in the reference configuration and the relative length in deformed configuration is denoted by dx. The mostly common strain measure in connection with Continuum mechanics is the so-called Green strain E.

Definition 3.8. *The Green strain tensor* E *is defined by*

$$dx^2 - dX^2 = 2\ dX \cdot E \cdot dX, \tag{3.10}$$

where dx^2 *is the squared length of a line segment after the deformation with original squared length* dX^2.

The tensor E measures the strain in an element dX as the original coordinates of a point X are moved to new coordinates x by

$$x = X + u. \tag{3.11}$$

Considering the line segments in the infinitesimal case and differentiating (3.11) gives

$$dx = \frac{\partial x}{\partial X}dX = \nabla_X \varphi\ dX = \frac{\partial(X+u)}{\partial X}dX. \tag{3.12}$$

Substitution of (3.12) into the definition of the Green strain tensor (3.10) gives

$$dX \cdot (\nabla_X \varphi(X,t)^T \nabla_X \varphi(X,t) - I - 2E) \cdot dX = 0,$$

which can be rewritten as

$$E = \frac{1}{2}(\nabla_X \varphi(X,t)^T \nabla_X \varphi(X,t) - I) = \frac{1}{2}(F^T F - I), \tag{3.13}$$

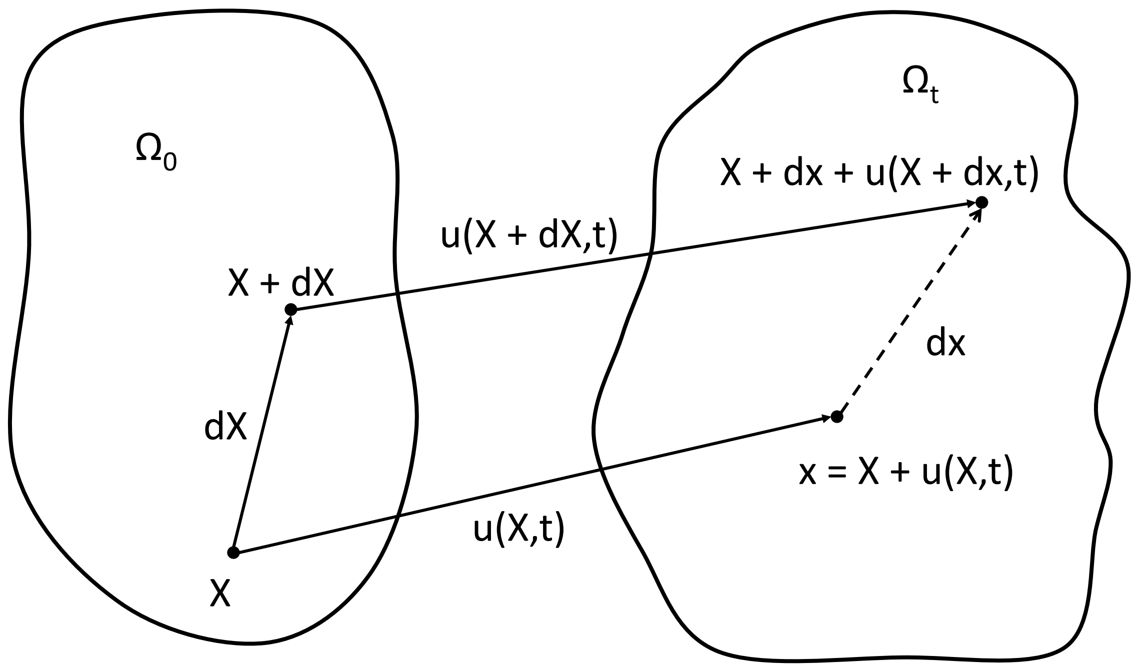


Figure 3.2: Line segments in reference and deformed configuration.

where F denotes the deformation gradient as previously mentioned. In the literature, the strain tensor (3.13) is called Green-St.Venant strain tensor. Moreover, the Green-St.Venant strain tensor can be described in terms of the displacement gradients, since

$$\begin{aligned} F^T F &= \nabla_X (X + u(X,t))^T \nabla_X (X + u(X,t)) \\ &= \nabla_X u(X,t)^T + \nabla_X u(X,t) + \nabla u(X,t) \nabla u(X,t)^T + I. \end{aligned}$$

This property allows us to rewrite the Green-St.Venant strain tensor as

$$E = \frac{1}{2} \left((\nabla_X u(X,t))^T + \nabla_X u(X,t) + \nabla_X u(X,t) \nabla_X u(X,t)^T \right), \tag{3.14}$$

or in a short way

$$E(u) = \frac{1}{2} \left(\nabla u^T + \nabla u + \nabla u \nabla u^T \right).$$

In this work we want to focus on linear elasticity, which allows us to drop the second order terms of the strain. This assumption leads to a modification of the Green-St.Venant strain tensor, usually called the linearized Green-St.Venant strain tensor

$$\epsilon(u) = \frac{1}{2} \left(\nabla u^T + \nabla u \right). \tag{3.15}$$

3.4 Forces and stresses

In this section we want to introduce different types of forces and stresses, which both have an impact on the body Ω_t. Forces are the fundamental sources of mechanical deformations. In general, we can distinguish between two types of forces, the body forces and surface forces or tractions.

Body forces act on the particles or atoms of the body and are defined as volume integrals.

Definition 3.9. *The body forces* $F_b \in \mathbb{R}^d$ *acting on a part of a body* $B \subset \Omega_t$ *are defined as*

$$F_b(B) = \int_B f(x,t)\ dx, \tag{3.16}$$

where $f : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^d$ *is called the body force density.*

The counterpart of body forces are surface forces, which act only on the surface of the body or parts of it. These forces depend not only on the points on the surface, i.e. location, but also on the normal vector of the points on the surface.

Definition 3.10. *Let* $\Sigma = \{y \in \mathbb{R}^d \mid \|y\| = 1\}$ *be the set of all directions. The surface forces* $F_c \in \mathbb{R}^d$ *acting on a part of a body* $B \subset \Omega_t$ *are defined as*

$$F_c(B) = \int_{\partial B} \vec{t}(x,t,n(x))\ ds, \tag{3.17}$$

where $\vec{t} : \Omega_t \times \mathbb{R}^+ \times \Sigma \to \mathbb{R}^d$ *is called the surface force density and* $n(x) \in \Sigma$ *is the unit outward normal vector at* $x \in \partial B$.

The total forces $F_g \in \mathbb{R}^d$ *acting on a part of a body* $B \subset \Omega_t$ *are the sum of the body forces and surface forces,*

$$F_g(B) = F_b(B) + F_c(B). \tag{3.18}$$

The stress is a measure for the force on a surface of a body or parts of it on which external forces (body or surface forces) are applied. In this work we introduce the Cauchy stress σ. The existence of the Cauchy stress is the result of Cauchy's Theorem.

Theorem 3.11 (Cauchy's Theorem). *Let* $\vec{t}$ *be as in Definition 3.10 and continuously differentiable in every component and variable. Then there exists a tensor field* σ*, continuously differentiable in every component and variable, called the Cauchy tensor, such that*

$$\sigma(x,t)^T \cdot n(x) = \vec{t}(x,t,n(x)) \quad \text{for all } x \in \Omega_t,\ t \in \mathbb{R}^+ \text{ and } n \in \Sigma. \tag{3.19}$$

Remark 3.12. *We will see that the Cauchy stress tensor is symmetric which follows from the conservation of angular momentum in the next section. In the literature, another stress measure can be found which we refer to as the (second) Piola-Kirchhoff stress tensor. It is defined in terms of the Cauchy stress tensor by*

$$\Sigma_2^T(X,t) = \det(F(X,t))F^{-1}(X,t)\ \sigma(\varphi(X,t))\ F^{-T}(X,t) \quad \text{for all } X \in \Omega_0,$$

where $F = \nabla_X \varphi(X,t)$*. By this representation the symmetry of the Piola-Kirchhoff stress tensor follows inherently from the symmetry of the Cauchy stress tensor. For detailed description of these two stress measures we give [2, Chapter 3] as a reference.*

Due to the symmetry of Cauchy stress tensor σ*, (3.19) can be rewritten as*

$$\sigma \cdot n = \vec{t}. \tag{3.20}$$

For the remainder of this work we will only consider the Cauchy stress tensor σ*.*

3.5 Conservation laws

The purpose of this section is to present the conservation laws, which give a part of the fundamental equations of continuum mechanics. These laws must be always satisfied in physical systems. The relevant laws are the conservation of mass, conservation of linear momentum and conservation for angular momentum. For a detailed description of the conservation laws we refer to [2, Chapter 3], [4, Chapter 6] and [40]. In order to derive these laws, fundamental theorems are needed. The first one is the already introduces Gauss Theorem 2.12. Another fundamental theorem is Reynolds Transport Theorem. It describes the rate of change for a material domain.

Theorem 3.13 (Reynolds Transport Theorem)**.** *Let $f : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^d$ be a piece-wise continuously differentiable function. Then the following relation holds:*

$$\frac{D}{Dt} \int_{\Omega_t} f(x,t)\ dx = \int_{\Omega_t} \frac{\partial f(x,t)}{\partial t} + \operatorname{div}\ (v(x,t) f(x,t))\ dx, \tag{3.21}$$

where v is the velocity of the body.

Proof. See [10, Chapter 5]. □

With the help of these two fundamental theorems we can derive the mathematical equations for the conservation laws of interest. We want to start with the mass conservation.

3.5.1 Conservation of mass

The mass $m(B)$ for any part B of continuum mechanical body Ω_t is given by

$$m(B) = \int_B \rho(x,t)\ dx, \tag{3.22}$$

where $\rho : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^+$ is the mass density of the body. The conservation of mass demands that the mass of any part of the body is constant, i.e. the mass does not change, since no mass can be added or removed from the boundary. Mathematically, the conservation of mass can be described as

$$\frac{Dm}{Dt} = \frac{D}{Dt} \int_B \rho(x,t)\ dx = 0. \tag{3.23}$$

Using Reynolds Transport Theorem 3.13, we deduce from (3.23) that

$$\int_B \left(\frac{\partial \rho(x,t)}{\partial t} + \operatorname{div}\ (\rho(x,t) v(x,t)) \right)\ dx = 0. \tag{3.24}$$

Since the mass conservation holds for any part B of the body Ω_t, it follows from (3.24) that

$$\frac{\partial \rho(x,t)}{\partial t} + \operatorname{div}\ (\rho(x,t) v(x,t)) = 0 \quad \text{in } \Omega_t. \tag{3.25}$$

Equation (3.25) is the mass conservation equation, also called continuity equation.

3.5.2 Conservation of linear momentum

In this subsection we derive the equations of linear momentum, also known as momentum conservation principle or balance of momentum. The linear momentum $p : \Omega_t \times \mathbb{R}^+ \to \mathbb{R}^d$ characterizes the status of motion of a body and is defined as the product of the mass and velocity of the body Ω_t, i.e.

$$p(x,t) = \int_{\Omega_t} \rho(x,t)v(x,t)\ dx, \tag{3.26}$$

where ρ is the density of Ω_t, v the velocity and ρv is the linear momentum per unit volume.

The conservation of linear momentum states, that the material time derivative of the linear momentum is equal to the total force (3.18), which can be written as

$$\frac{D}{Dt}p(x,t) = F_g(\Omega_t),$$

or equivalently

$$\frac{D}{Dt}\int_{\Omega_t} \rho(x,t)v(x,t)\ dx = \int_{\Omega_t} f(x,t)\ dx + \int_{\partial\Omega_t} \vec{t}(x,t,n(x))\ ds. \tag{3.27}$$

Using now Reynolds Transport Theorem 3.13 for the material derivative in (3.27), we get that

$$\begin{aligned}\frac{D}{Dt}\int_{\Omega_t} \rho v\ dx &= \int_{\Omega_t} \frac{\partial(\rho v)}{\partial t} + \operatorname{div}\left(v(\rho v)^T\right)\ dx \\ &= \int_{\Omega_t} \rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + \operatorname{div}\left(v(\rho v)^T\right)\ dx,\end{aligned} \tag{3.28}$$

where the product rule has been used in the last step. Furthermore, we can use the obvious property that, $\operatorname{div}\left(v(\rho v)^T\right) = (\nabla v)(\rho v)^T + v\ \operatorname{div}\ (\rho v)$ to obtain

$$\begin{aligned}&\int_{\Omega_t} \rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + \operatorname{div}\left(v(\rho v)^T\right)\ dx \\ =&\int_{\Omega_t} \rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + (\nabla v)(\rho v)^T + v\ \operatorname{div}\ (\rho v)\ dx,\end{aligned}$$

such that (3.28) reduces due to the mass conservation (3.25) to

$$\frac{D}{Dt}\int_{\Omega_t} \rho v\ dx = \int_{\Omega_t} \rho\frac{\partial v}{\partial t} + (\nabla v)(\rho v)^T\ dx. \tag{3.29}$$

With the help of (3.29) we can rewrite (3.27) as

$$\int_{\Omega_t} \rho\frac{\partial v}{\partial t} + (\nabla v)(\rho v)^T\ dx = \int_{\Omega_t} f\ dx + \int_{\partial\Omega_t} \vec{t}\ ds. \tag{3.30}$$

Considering now Cauchy's Theorem 3.11 and the Gauss Theorem 2.12 for the boundary expression, we deduce from (3.30) the conservation of linear momentum in the integral form

$$\int_{\Omega_t} \rho\frac{\partial v}{\partial t} + (\nabla v)(\rho v)^T\ dx = \int_{\Omega_t} f\ dx + \int_{\Omega_t} \operatorname{div}\ \sigma\ dx,$$

which is equivalent to

$$\int_{\Omega_t} \left(\rho \frac{\partial v}{\partial t} + (\nabla v)(\rho v)^T - f - \text{div } \sigma \right) \, dx = 0,$$

and therefore we get the equation of the conservation of linear momentum

$$\rho \frac{D}{Dt} v = \rho \left(\frac{\partial v}{\partial t} + (\nabla v) v \right) = f + \text{div } \sigma. \tag{3.31}$$

Remark 3.14. *In fact, the conservation of linear momentum is equivalent to Newton's second law, which states that the product of mass and acceleration equals the force of a body.*

3.5.3 Equilibrium equation

The equilibrium equation is a consequence of the conservation of linear momentum equation (3.31), since in many problems the loads are applied slowly and hence, the acceleration can be neglected. The equilibrium equation reads as

$$f + \text{div } \sigma = 0 \qquad \text{or} \qquad -\text{div } \sigma = f. \tag{3.32}$$

3.5.4 Conservation of angular momentum

The angular momentum can be obtained by taking the cross product of each term in (3.27) with the position vector x. Its integral from is therefore

$$\frac{D}{Dt} \int_{\Omega_t} x \times \rho v \, dx = \int_{\Omega_t} x \times f \, dx + \int_{\partial \Omega_t} x \times \vec{t} \, ds. \tag{3.33}$$

As it is mentioned before, the conservation of angular momentum yields the symmetry of the stress tensor. We want to state this property as a theorem.

Theorem 3.15. *Let ρ, v, f and $\vec{t}$ be sufficiently smooth and let the conservation of linear momentum (3.31) and the conservation of angular momentum (3.33) hold. Then the stress tensor σ is symmetric.*

Proof. [10, Chapter 5]. □

3.6 Constitutive laws

The section before has given a quick overview about the physical laws that any material must satisfy in continuum mechanics. In this section we want to specify the material properties, which are described by so-called constitutive or material laws. The constitutive equations characterize the material and give a relation between the stress and the deformation history of the body in terms of the so-called response function. We briefly give the material equation for an elastic, St.Venant-Kirchhoff material. For a more detailed description we refer to [2, Chapter 5], [10, Chapter 5] and [40].

We start with the definition of an elastic body.

Definition 3.16 (Elastic material). *A material is said to be elastic if the stress σ depends on the material point and the deformation gradient, i.e. there exists a function $\hat{T}$, called the response function, such that*

$$\sigma(x,t) = \hat{T}(X, \nabla_X \varphi(X,t)). \tag{3.34}$$

The elastic material is called homogeneous if the stress only depends on the deformation gradient and not on the reference points, i.e. $\sigma = \hat{T}(\nabla_X \varphi(X,t))$. Otherwise, the material is said to be heterogeneous.

Not every response function of an elastic material describes a real physical body. In order to handle this incidence we require that the constitutive equation shall not depend on the choice of coordinate system. This kind of requirement is also known as the material frame indifference or objectivity. We always assume that the response function fulfills this requirement and refer to [2, Chapter 3] for the complete description.

Finally, we need a relation between the stress and the strain, which falls into the framework of the constitutive laws. A linear strain-stress relation is the so-called Hooke's law.

Definition 3.17 (Hooke's law). *The stress-strain relation of an elastic material can be described by Hooke's law, which is*

$$\sigma = CE(u) \qquad or \qquad \sigma_{ij}(u) = \sum_{k,l=1}^{d} C_{ijkl} E_{kl}(u) \quad for\ i,j = 1,...,d. \tag{3.35}$$

where $C = (C_{ijkl})_{i,j,k,l=1}^{d} \in \mathbb{R}^{d\times d\times d\times d}$ is a tensor of forth order with the symmetry property

$$C_{ijkl} = C_{jikl} = C_{klij}.$$

Remark 3.18. *Note, that Hooke's law for the linearized strain tensor reads as*

$$\sigma = C\epsilon(u) \qquad or \qquad \sigma_{ij}(u) = \sum_{k,l=1}^{d} C_{ijkl} \epsilon_{kl}(u) \quad for\ i,j = 1,...,d. \tag{3.36}$$

Summarizing all the conservation laws and taking the constitutive laws into account we are now able to give the mathematical model for the deformation of an elastic St.Venant-Kirchhoff material. We assume that the body is clamped on a certain part of the surface Γ_D and we only apply the tractions on the rest of the surface Γ_F, i.e. $|\Gamma_D| > 0$, $\Gamma_D \cup \Gamma_F = \partial\Omega$ and $\Gamma_D \cap \Gamma_F = \emptyset$. The problem is: Find the displacement u, such that

$$-\text{div}\,\sigma(u) = f \qquad \text{in } \Omega, \tag{3.37a}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{3.37b}$$

$$\sigma(u)\cdot n = \vec{t} \qquad \text{on } \Gamma_F, \tag{3.37c}$$

$$\sigma(u) = C\epsilon(u), \tag{3.37d}$$

$$\epsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right). \tag{3.37e}$$

Remark 3.19. *Hooke's law for an isotropic material under consideration of the frame indifference and the Rivlin-Ericksen Theorem (cf. [27]) can be written as*

$$\sigma(u) = 2\mu\epsilon(u) + \lambda \text{tr}(\epsilon(u)) I,$$

where μ and λ are called the Lamé parameters. The Lamé coefficients can be also described in terms of Young's Modulus E and Poisson's ration ν by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{3.38}$$

Chapter 4

Signorini and Obstacle Problem

The aim of this chapter is to formulate the classical Signorini problem and give the mathematical model with its corresponding physical background. The Signorini problem is the basis for contact problems in solid mechanics and describes the contact of a linearly elastic body with a frictionless rigid foundation. We kick of this chapter with the introduction of the contact conditions and the raising of classical mathematical PDE model following the ideas of [16, Chapter 2], [18, Chapter 2] and [42, Chapter 2]. The variational formulation of this problem leads to a variational inequality, whose analysis will be the topic of the next chapter. The end of this chapter addresses a simplified version of the Signorini problem, also called simplified Signorini problem, and a special obstacle problem. Just as the classical Signorini problem, each of these problems involves the contact between a linearly elastic body and a rigid foundation. We refer to [13, Chapter 2], [14, Chapter 4], [28, Chapter 1] and [39] for their detailed description.

We assume small deformations throughout this chapter, such that the previously described linear elasticity theory will be a part of the model for contact problems. In fact, the rigid foundation plays the role of a constraint on the boundary of the body, which can be seen in the upcoming section.

4.1 Classical Signorini problem

The Signorini problem investigates the deformation of an elastic body when it comes into contact with a rigid foundation. The contact boundary depends on the displacement of the elastic body, which makes it to an unknown variable in the model. Due to fact that the contact surface is a-priori unknown, the Signorini problem is highly nonlinear and non-differentiable with respect to the displacement. We start this section with the description of the contact conditions, where the focus lies on the linearized contact. With the help of the contact conditions, we can formulate the classical model of the Signorini problem and conclude this section with its variational form. We will see that the variational form leads to variational inequalities by reason of the contact conditions. The contribution of this section is based on [16, Chapter 2], [18, Chapter 2] and [42, Chapter 2].

As we have introduced in the chapter before, we describe the particles of the body $\Omega_0 \subset \mathbb{R}^3$ in reference configuration by Lagrangian coordinates $X = (X_1, X_2, X_3)$. The body is set in motion and changes its location and shape in every time step t, where the position $x = (x_1, x_2, x_3)$ of the particles of the deformed body $\Omega_t \subset \mathbb{R}^3$ can be described by the deformation φ, i.e. $x = \varphi(X, t)$. The deformation φ satisfies the conditions in Definition 3.3 as well as the below requirements for a smooth deformation. The displacement of a point $X \in \Omega_0$ is given as in (3.4), i.e. $u(X, t) = \varphi(X, t) - X$.

From now on, we consider the deformation of the body only for one time step or rather for a fixed time t_1. This consideration allows us to ignore the dependence on the time t.

Consequently, the reference configuration will be denoted by Ω and the deformation by $\varphi(X)$. The displacement u is then given as

$$u(X) = \varphi(X) - X. \tag{4.1}$$

The Signorini problem considers a body $\Omega \in \mathbb{R}^3$ in reference configuration and a rigid and fixed foundation $\mathcal{F} \subset \mathbb{R}^3$ as illustrated in Figure 4.1. The boundary of the body $\partial\Omega$ is divided into three different boundary types, the Dirichlet boundary $\partial\Omega_D = \Gamma_D$ at where the body is fixed, the Neumann part of the boundary $\partial\Omega_F = \Gamma_F$, where tractions $\vec{t}$ are applied to and the contact boundary $\partial\Omega_C = \Gamma_C$, where the body may come in touch with the foundation. If the body does not touch the foundation, then we are able to measure the initial gap in terms of the gap function g. We are interested in the deformed body caused by the applied surface forces, where some portion of the material surface of the body comes in contact with the foundation $\mathcal{F}$. Since the foundation is rigid and fixed, the body cannot penetrate the foundation, which restricts the displacement on the contact boundary Γ_C. Thus, the aim of the following subsection is to derive these non-penetration contact conditions. Figure 4.1 illustrates the Signorini problem.

4.1.1 Linearized contact conditions

For simplicity, we assume that the boundary of the foundation $\partial\mathcal{F}$ is Lipschitz continuous, i.e. the boundary may be presented locally by a Lipschitz continuous parametrization. Hence, we can assume that for each point $y = (y_1, y_2, y_3) \in \partial\mathcal{F}$ the following parametrization exists,

$$y_3 = \eta_y(y_1, y_2), \tag{4.2}$$

where $\eta_y : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Incorporating the same assumption for the contact boundary Γ_C of the body in the current configuration, we can represent the points $X = (X_1, X_2, X_3) \in \Gamma_C$ by

$$X_3 = \eta_X(X_1, X_2), \tag{4.3}$$

where $\eta_X : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Thus, an admissible displacement must satisfy the following conditions for any point $X \in \Gamma_C$,

$$u_i\left(X_1, X_2, \eta_X(X_1, X_2)\right) = \varphi_i\left(X_1, X_2, \eta_X(X_1, X_2)\right) - X_i \quad \text{for } i = 1, 2, \tag{4.4}$$

and

$$\begin{aligned}\eta_X(X_1, X_2) + u_3\left(X_1, X_2, \eta_X(X_1, X_2)\right) \leq \eta_y(X_1 + &u_1\left(X_1, X_2, \eta_X(X_1, X_2)\right), \\ &X_2 + u_2\left(X_1, X_2, \eta_X(X_1, X_2)\right)).\end{aligned} \tag{4.5}$$

We refer to inequality (4.5) as the kinematical contact condition for finite displacements of a body constrained by a fixed rigid foundation with respect to the non-penetration assumption. Physically (4.5) means that the elastic body cannot penetrate the rigid foundation, therefore the displacement on Γ_C is restricted.

We want to obtain a more clearer version of the inequality (4.5) by rewriting (4.4) into

$$x_i = X_i + u_i(X_1, X_2, \eta_X(X_1, X_2)) \quad \text{for } i = 1, 2. \tag{4.6}$$

Moreover, we suppose that there exists a function $\zeta : \mathbb{R}^2 \to \mathbb{R}$, which describes the reference coordinates in terms of the spatial coordinates by

$$X_i = \zeta_i(x_1, x_2) \quad \text{for } i = 1, 2. \tag{4.7}$$

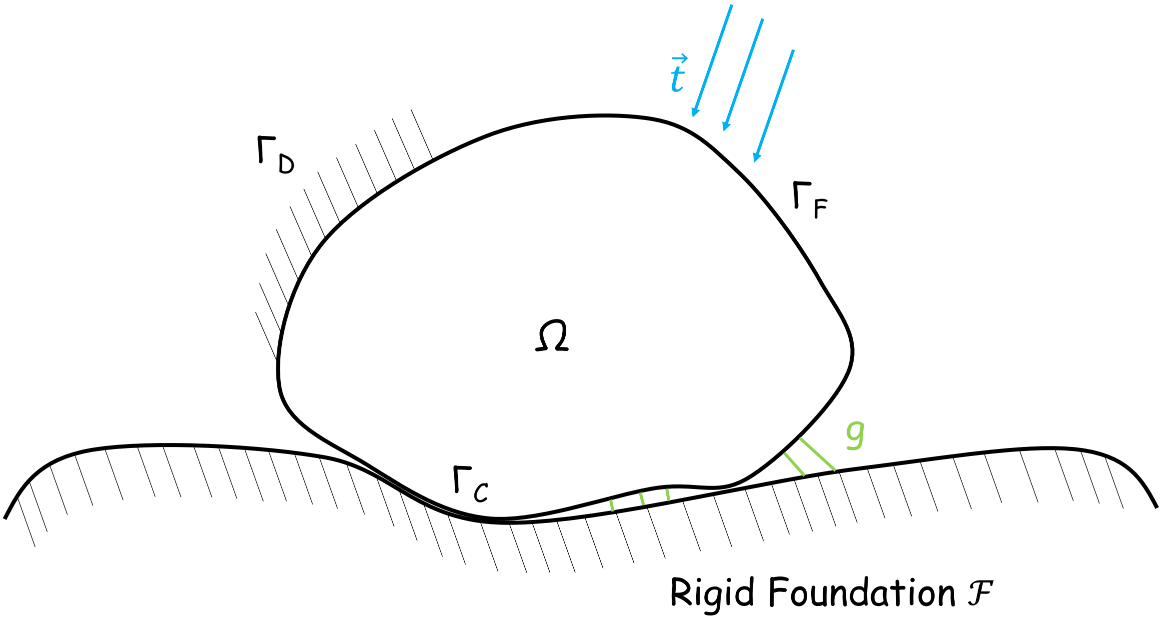


Figure 4.1: Contact between the body Ω and a rigid foundation $\mathcal{F}$.

Then the kinematical contact condition (4.5) can be rewritten as

$$\hat{\eta}_X(x_1, x_2) + \hat{u}_3(x_1, x_2) \leq \eta_y(x_1, x_2), \tag{4.8}$$

where

$$\begin{aligned}\hat{\eta}_X(x_1, x_2) &= \eta_X\left(\zeta_1(x_1, x_2), \zeta_2(x_1, x_2)\right), \\ \hat{u}_3(x_1, x_2) &= u_3\left(\zeta_1(x_1, x_2), \zeta_2(x_1, x_2), \eta_X(\zeta_1(x_1, x_2), \zeta_2(x_1, x_2))\right), \\ \eta_y(x_1, x_2) &= \eta_y(X_1 + u_1, X_2 + u_2).\end{aligned}$$

Condition (4.8) is derived just as (4.5) from kinematical observations. However, it also must be compatible with the stress condition on the contact surface Γ_C. We emphasize at this point, that no external tractions are applied on Γ_C. Nevertheless stress is developed if the elastic body touches the rigid foundation. For this purpose, let $\sigma = \sigma(x_1, x_2, x_3)$ be the Cauchy stress tensor of particle $X \in \Omega$ whose position is $x = (x_1, x_2, x_3)$ and let $n = (n_1, n_2, n_3)$ be the unit outer normal of Γ_C. The normal and tangential components of the stress vector $\sigma \cdot n$ at the boundary of the body $\partial\Omega = \Gamma$ are $(\sigma \cdot n)_n$ and $(\sigma \cdot n)_T$ as illustrated in Figure 4.2, respectively, where

$$\begin{aligned}(\sigma \cdot n)_n &= ((\sigma \cdot n) \cdot n)n, \\ (\sigma \cdot n)_T &= \sigma \cdot n - (\sigma \cdot n)_n.\end{aligned} \tag{4.9}$$

Usually, the quantities

$$\begin{aligned}\sigma_n(x) &= \sigma_{ij}(x) n_i(x) n_j(x), \\ \sigma_{T_i}(x) &= \sigma_{ij}(x) n_j(x) - \sigma_n(x) n_i(x),\end{aligned} \qquad 1 \leq i, j \leq 3, \tag{4.10}$$

are used in the literature, where $x = \varphi^{-1}(X) \in \Gamma_C$ and σ_n denotes the length of $(\sigma \cdot n)_n$. In the sequel we will call σ_n the normal component of the stress and σ_{T_i} the $i - th$ coordinate of tangential component of the stress.

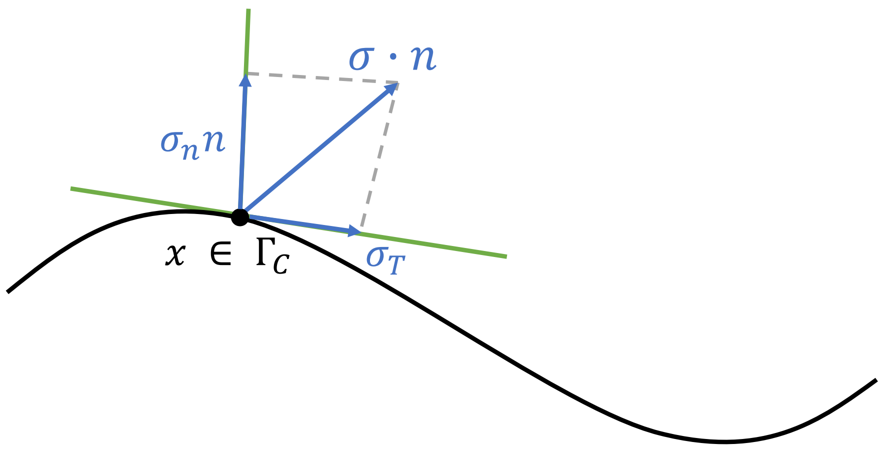


Figure 4.2: Normal and tangential component of stress $\sigma \cdot n$ at point $x \in \Gamma_C$.

We can make physical observations analyzing these components of the stress tensor. Firstly, if there is no contact between the body and the foundation, then there is no stress at all, therefore $\sigma_n = 0$. On the other hand, if the body is in touch with the foundation on Γ_C, then normal stress σ_n must be developed on Γ_C. Secondly, the tangential stress σ_{T_i} must be zero on Γ_C, since the foundation surface $\partial\mathcal{F}$ is frictionless. The mathematical interpretation of these physical observations is

$$\begin{aligned} \sigma_n(x) &= 0 \quad \text{if } \hat{\eta}_X(x_1,x_2) + \hat{u}_3(x_1,x_2) < \eta_y(x_1,x_2), \\ \sigma_n(x) &\leq 0 \quad \text{if } \hat{\eta}_X(x_1,x_2) + \hat{u}_3(x_1,x_2) = \eta_y(x_1,x_2), \\ \sigma_{T_i}(x) &= 0, \end{aligned} \tag{4.11}$$

for $1 \leq i \leq 3$ and for all $x = \varphi^{-1}(X) \in \Gamma_C$. Considering condition (4.8) and the first two conditions in (4.11), it results the following relation for the normal vector,

$$\sigma_n(x)\left(\hat{\eta}_X(x_1,x_2) + \hat{u}_3(x_1,x_2) - \eta_y(x_1,x_2)\right) = 0, \tag{4.12}$$

for all $x \in \Gamma_C$. Gathering all the considerations for the contact between the body and the rigid foundation, we deduce the general contact conditions in the frictionless case

$$\begin{aligned} \hat{\eta}_X(x_1,x_2) + \hat{u}_3(x_1,x_2) &\leq \eta_y(x_1,x_2), \\ \sigma_n(x) &\leq 0, \\ \sigma_{T_i}(x) &= 0, \\ \sigma_n(x)\left(\hat{\eta}_X(x_1,x_2) + \hat{u}_3(x_1,x_2) - \eta_y(x_1,x_2)\right) &= 0, \end{aligned} \tag{4.13}$$

for all $x \in \Gamma_C$ and $1 \leq i \leq 3$.

The goal is now to derive the linearized contact conditions from (4.13) in terms of the reference configuration Ω. For this reason, we assume that the body makes "small" displacements relative to its initial position. Consequently, the distance between the body and the rigid foundation is rather short, which allows us to consider them to be essentially parallel in the sense that the unit normal vector n and the normalized gap function g can be obtained in terms of the body surface Γ_C. In addition, we assume that the parametrizations η_X and η_y have at least continuous first partial derivatives and bounded second partial derivatives everywhere in their respective domains. Then we receive that

$$\begin{aligned} x_i &= X_i + u_i(X_1, X_2, X_3) = X_i + u_i(X_1, X_2, \eta_X(X_1, X_2)) \\ &= X_i + u_i(x_1 + X_1 - x_1, x_2 + X_2 - x_2, \eta_X(x_1, x_2) + \eta_X(X_1, X_2) - \eta_X(x_1, x_2)) \\ &= X_i + u_i(x_1 - u_1, x_2 - u_2, \eta_X(x_1, x_2) - u_3). \end{aligned} \tag{4.14}$$

Making use of the Taylor expansion (c.f. [23, Chapter 4]) yields

$$x_i = X_i + u_i(x_1, x_2, \eta_X(x_1, x_2)) + \mathcal{O}(|u_i|^2, |u_{i,j}|^2) \quad \text{for } i = 1, 2, 3, \tag{4.15}$$

where $\mathcal{O}(\cdot)$ denotes the higher order terms of u and corresponding partial derivatives. Since we want to linearize condition (4.15), we drop the higher order terms and keep the linear expressions, which lead to

$$x_i = X_i + u_i(x_1, x_2, \eta_X(x_1, x_2)), \tag{4.16}$$

for $x \in \Gamma_C$. Similarly, we conclude that

$$\eta_X(X_1, X_2) = \eta_X(x_1 + X_1 - x_1, x_2 + X_2 - x_2) = \eta_X(x_1 - u_1, x_2 - u_2). \tag{4.17}$$

Using again the Taylor expansion and dropping the higher order terms yields

$$\eta_X(X_1, X_2) = \eta_X(x_1, x_2) - \frac{\partial \eta_X(x_1, x_2)}{\partial x_1} u_1 - \frac{\partial \eta_X(x_1, x_2)}{\partial x_2} u_2. \tag{4.18}$$

Likewise, we obtain by retaining the linear terms that

$$u_3(X_1, X_2, \eta_X(X_1, X_2)) = u_3(x_1, x_2, \eta_X(x_1, x_2)). \tag{4.19}$$

We can insert now the linearized conditions (4.16), (4.18) and (4.19) into the kinematical contact condition (4.5) to deduce the linearized kinematical contact condition

$$\eta_X(x_1, x_2) - \frac{\partial \eta_X(x_1, x_2)}{\partial x_1} u_1 - \frac{\partial \eta_X(x_1, x_2)}{\partial x_2} u_2 + u_3(x_1, x_2, \eta_X(x_1, x_2)) \le \eta_y(x_1, x_2). \tag{4.20}$$

Defining now the direction $\tilde{n}(x) = \left(-\frac{\partial \eta_X(x_1,x_2)}{\partial x_1}, -\frac{\partial \eta_X(x_1,x_2)}{\partial x_2}, 1\right)$, we can equivalently rewrite (4.20) as

$$\left(-\frac{\partial \eta_X(x_1, x_2)}{\partial x_1}, -\frac{\partial \eta_X(x_1, x_2)}{\partial x_2}, 1\right) \cdot (u_1, u_2, u_3)^T \le \eta_y(x_1, x_2) - \eta_X(x_1, x_2), \tag{4.21}$$

for all $x \in \Gamma_C$. The direction vector $\tilde{n}(x)$ is defined as the outward normal vector for the point x on the contact boundary Γ_C. Dividing (4.21) by $\|\tilde{n}\| = \sqrt{1^2 + \left(\frac{\partial \eta_X(x_1,x_2)}{\partial x_1}\right)^2 + \left(\frac{\partial \eta_X(x_1,x_2)}{\partial x_2}\right)^2}$ we obtain the normalized and linearized contact condition

$$u(x) \cdot n(x) \le g(x) \quad \text{for all } x \in \Gamma_C, \tag{4.22}$$

where

$$n(x) = \frac{\tilde{n}(x)}{\|\tilde{n}(x)\|},$$
$$g(x) = \frac{\eta_y(x_1, x_2) - \eta_X(x_1, x_2)}{\|\tilde{n}(x)\|}.$$

The function $g : \mathbb{R}^3 \to \mathbb{R}$ denotes the normalized initial gap between the surface of the foundation $\partial\mathcal{F}$ and Γ_C. Due to the assumptions of small displacement and the close initial positions of the surfaces of the body and the rigid foundation, i.e.

$$\eta_y - \eta_X = \mathcal{O}(|u_3|),$$
$$u(X) \cdot n(X) = u(x) \cdot n(x) + \mathcal{O}(|u_{n,i}| \cdot |u_n|),$$
$$g(X) = g(x) + \mathcal{O}(|u_3| \cdot |u_{3,i}|),$$

we can write condition (4.22) in terms of particle $X \in \Gamma_C$ in the reference configuration with respect to its outer normal $n(X)$, which is

$$u_n(X) - g(X) \leq 0, \tag{4.23}$$

where $u_n(X) = u(X) \cdot n(X)$. Finally, we can derive from (4.13) the fully linearized contact conditions in the frictionless case,

$$\begin{aligned} u_n(X) - g(X) &\leq 0, \\ T_n(X) &\leq 0, \\ T_{T_i}(X) &= 0, \\ T_n(X)\,(u_n(X) - g(X)) &= 0, \end{aligned} \tag{4.24}$$

for all $X \in \Gamma_C$, where T shall to be understood as the Cauchy stress tensor measured in the reference configuration, i.e. $T_n(X) = \sigma_n(\varphi(X))$ and $T_{T_i}(X) = \sigma_{T_i}(\varphi(X))$, and T_n and T_{T_i} are defined as in (4.10).

4.1.2 Classical form

With the help of the linearized contact conditions, we are now able to formulate the Signorini problem in its classical form by terms of differential equations. For the rest of this chapter, we always consider the framework of linear elasticity. For this reason we act on the assumption of the reference configuration Ω and denote its particles by $x \in \Omega$. The Cauchy stress T measured in the reference configuration as in (4.24) will be denoted as σ, i.e. $T = \sigma$. Note that the linearized contact conditions (4.24) are described with respect to the particles $X = x$. If we take Hook's law (3.36) for the linear strain tensor into account, then the stress can be written as $\sigma(x) = \sigma(x, u) = \sigma(u)$. Thus, (4.24) is still valid for the stress tensor $\sigma(u)$ described in terms of the displacement u.

As already mentioned, we assume the body Ω to be clamped along a part of the boundary Γ_D and surface tractions $\vec{t}$ are applied to a certain part of the body Γ_F. The contact surface, where the body may come in touch with the rigid foundation $\mathcal{F}$, is denoted by Γ_C. The actual contact surface is not known in advance but it is assumed to be a subset of Γ_C. The initial gap g between the body and the rigid foundation is known. Recalling the equilibrium equation (3.32) and taking the boundary conditions into account, we are able formulate the component-wise classical frictionless Signorini problem in linear elasticity for elastic and homogeneous materials given by Hooke's law (3.36): Find the displacement u, such that

$$-\frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i \quad \text{in } \Omega, \tag{4.25a}$$

$$u_i = 0 \quad \text{on } \Gamma_D, \tag{4.25b}$$

$$\sigma_{ij}(u)n_j = t_i \quad \text{on } \Gamma_F, \tag{4.25c}$$

$$\left.\begin{aligned} &\sigma_{T_i}(u) &&= 0,\\ &\sigma_n(u)(u_n - g) &&= 0,\\ &u_n - g &&\leq 0,\\ &\sigma_n(u) &&\leq 0, \end{aligned}\right\} \quad \text{on } \Gamma_C, \tag{4.25d}$$

$$\sigma_{ij}(u) = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{ij}(u), \tag{4.25e}$$

$$\epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.25f}$$

where

$$\begin{aligned} \sigma_n(u) &= \sigma_{ij}(u)n_i n_j,\\ \sigma_{T_i}(u) &= \sigma_{ij}(u)n_j - \sigma_n(u)n_i, \qquad \text{for } 1 \leq i,j \leq 3. \end{aligned}$$

4.1.3 Variational form

In this section, we want to derive the variational form of the classical Signorini problem. Fur this purpose, let $V = \{v \in [H^1(\Omega)]^3 \mid v = 0 \text{ on } \Gamma_D\}$ be the vector-valued Hilbert space, which denotes the set of virtual displacements. We assume, that the functions $v \in V$ are sufficiently smooth in the sense that every operation we want to do is well defined. Particularly, this means that the virtual work $\int_\Omega \sigma(u) : \epsilon(u)$ is well defined for all $u, v \in V$. For any given positive gap function $g : \Gamma_C \to \mathbb{R}$, the contact conditions can be incorporated by the convex subset $\mathbb{K} \subset V$ defined as the set of admissible displacements satisfying the kinematic contact conditions,

$$\mathbb{K} = \{v \in V \mid v_n - g \leq 0 \text{ on } \Gamma_C\}. \tag{4.26}$$

It turns out that the variational formulation of the Signorini problem is indeed a variational inequality. It can be formulated as: Find $u \in \mathbb{K}$, such that

$$a(u, v-u) \geq \langle F, v-u\rangle \quad \text{for all } v \in \mathbb{K}, \tag{4.27}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \sigma_{ij}(u)\frac{\partial v_i}{\partial x_j}\,dx,\\ \langle F, v\rangle &= \int_\Omega f_i v_i\,dx + \int_{\Gamma_F} t_i v_i\,ds. \end{aligned}$$

The next theorem confirms that a solution of the classical form the Signorini problem can be characterized by the variational inequality and conversely.

Theorem 4.1. *Let $u \in \mathbb{K}$ be a sufficiently smooth solution of the classical form of the Signorini problem (4.25). Then u solves the variational inequality (4.27). On the other hand, if $u \in \mathbb{K}$ solves (4.27) and is sufficiently smooth, then it is also a solution of (4.25).*

Proof. Firstly, we prove that a solution $u \in \mathbb{K}$ of (4.25) solves (4.27). For this purpose, let $v \in \mathbb{K}$ be an arbitrary element. We multiply (4.25a) with the test function $v - u \in \mathbb{K}$ and integrate over the domain Ω to obtain

$$-\int_{\Omega} \frac{\partial \sigma_{ij}(u)}{\partial x_j}(v_i - u_i)\ dx = \int_{\Omega} f_i(v_i - u_i)\ dx. \tag{4.28}$$

Using integration by parts on the left hand side of (4.28) , we receive

$$-\int_{\Omega} \frac{\partial \sigma_{ij}(u)}{\partial x_j}(v_i - u_i)\ dx = \int_{\Omega} \sigma_{ij}(u)\frac{\partial}{\partial x_j}(v_i - u_i)\ dx - \int_{\partial\Omega} \sigma_{ij}(u)n_j(v_i - u_i)\ ds. \tag{4.29}$$

The boundary $\partial\Omega$ of the body is decomposed into the three disjoint parts $\Gamma_D, \Gamma_F, \Gamma_C$, which gives

$$\begin{aligned}\int_{\partial\Omega} \sigma_{ij}(u)n_j(v_i - u_i)\ ds = \int_{\Gamma_D} \sigma_{ij}(u)n_j(v_i - u_i)\ ds + \int_{\Gamma_F} \sigma_{ij}(u)n_j(v_i - u_i)\ ds \\ + \int_{\Gamma_C} \sigma_{ij}(u)n_j(v_i - u_i)\ ds.\end{aligned} \tag{4.30}$$

Since $v - u$ vanishes on Γ_D and (4.25c) holds on Γ_F, we can conclude from (4.28) with the help of (4.29) and (4.30) that

$$\int_{\Omega} \sigma_{ij}(u)\frac{\partial}{\partial x_j}(v_i - u_i)\ dx = \int_{\Omega} f_i(v_i - u_i)\ dx + \int_{\Gamma_F} t_i(v_i - u_i)\ ds + \int_{\Gamma_C} \sigma_{ij}(u)n_j(v_i - u_i)\ ds. \tag{4.31}$$

Now, splitting the stress on Γ_C up into the normal and tangential component alike (4.10) and considering the frictionless case and the contact conditions (4.25d), we get

$$\begin{aligned}\sigma_{ij}(u)n_j(v_i - u_i) &= (\sigma_{T_i}(u) + \sigma_n(u)n_i)(v_i - u_i) \\ &= 0 + \sigma_n(u)(v_n - u_n) \\ &= \sigma_n(u)(v_n - u_n + g - g) \\ &= \sigma_n(u)(v_n - g).\end{aligned}$$

Since $\sigma_n(u) \leq 0$ and $(v_n - g) \leq 0$ due to (4.25d) and (4.26), respectively, $\sigma_n(u)(v_n - g) \geq 0$ and finally (4.31) changes to

$$\int_{\Omega} \sigma_{ij}(u)\frac{\partial}{\partial x_j}(v_i - u_i)\ dx \geq \int_{\Omega} f_i(v_i - u_i)\ dx + \int_{\Gamma_F} t_i(v_i - u_i)\ ds. \tag{4.32}$$

Thus, $u \in \mathbb{K}$ solves (4.27).

Conversely, let $u \in \mathbb{K}$ be the solution of (4.27) and sufficiently smooth. We want to show that u solves (4.25). Since $C_0^\infty(\Omega) \subset \mathbb{K}$, we can choose $v = u \pm w$, such that $w_i \in C_0^\infty(\Omega)$ and $v \in \mathbb{K}$. Inserting $v = u + w$ in (4.27) gives $a(u, w) - \langle F, w\rangle \geq 0$, or equivalently

$$\int_{\Omega} \sigma_{ij}(u)\frac{\partial w_i}{\partial x_j}\ dx - \int_{\Omega} f_i w_i\ dx - \int_{\Gamma_F} t_i w_i\ ds \geq 0. \tag{4.33}$$

Integrating the first term of (4.33) by parts yields

$$-\int_{\Omega} \frac{\sigma_{ij}(u)}{\partial x_j} w_i \, dx - \int_{\Omega} f_i w_i \, dx - \int_{\Gamma_F} t_i w_i \, ds + \int_{\partial\Omega} \sigma_{ij}(u) n_j w_i \, ds \geq 0. \tag{4.34}$$

Since $w_i \in C_0^\infty(\Omega)$ the integrals vanish on $\partial\Omega$ and $\Gamma_F \subset \partial\Omega$, which gives

$$-\int_{\Omega} \frac{\sigma_{ij}(u)}{\partial x_j} w_i \, dx - \int_{\Omega} f_i w_i \, dx \geq 0,$$

or rather

$$\int_{\Omega} \left(\frac{\sigma_{ij}(u)}{\partial x_j} + f_i \right) w_i \, dx \leq 0. \tag{4.35}$$

Choosing now $v = u - w$ and insert it into (4.27), we obtain by repeating the same steps as before

$$\int_{\Omega} \left(\frac{\sigma_{ij}(u)}{\partial x_j} + f_i \right) w_i \, dx \geq 0,$$

which gives together with (4.35) that

$$\int_{\Omega} \left(\frac{\sigma_{ij}(u)}{\partial x_j} + f_i \right) w_i \, dx = 0. \tag{4.36}$$

Since (4.36) is valid for all $v \in \mathbb{K}$, hence for every $w_i \in C_0^\infty(\Omega)$, we deduce the equilibrium equation (4.25a).

Condition (4.25b) is naturally satisfied, due to the fact that the definition of the set of admissible displacements $\mathbb{K}$ incorporates the Dirichlet boundary condition.

To derive (4.25c), we consider again (4.34) for the choice $v = u \pm w \in \mathbb{K}$. Using now (4.36) and the fact that $\Gamma_F \subset \partial\Omega$, it follows that

$$-\int_{\Gamma_F} t_i w_i \, ds + \int_{\Gamma_F} \sigma_{ij}(u) n_j w_i \, ds \geq 0$$

for $v = u + w$, and

$$-\int_{\Gamma_F} t_i w_i \, ds + \int_{\Gamma_F} \sigma_{ij}(u) n_j w_i \, ds \leq 0$$

for $v = u - w$, which gives together

$$-\int_{\Gamma_F} t_i w_i \, ds + \int_{\Gamma_F} \sigma_{ij}(u) n_j w_i \, ds = 0. \tag{4.37}$$

Since (4.37) is valid for all $v \in \mathbb{K}$, hence for every $w_i \in C_0^\infty$, (4.25c) follows.

It remains to deduce kinematical contact conditions (4.25d) on the contact boundary Γ_C. Since (4.25a) is already valid, we multiply it with the test function $v - u \in \mathbb{K}$, integrate over the domain Ω and use integration by parts to obtain

$$\int_{\Omega} \sigma_{ij}(u) \frac{\partial}{\partial x_j}(v_i - u_i) \, dx = \int_{\Omega} f_i (v_i - u_i) \, dx + \int_{\partial\Omega} \sigma_{ij}(u) n_j (v_i - u_i) \, ds. \tag{4.38}$$

Since $u \in \mathbb{K}$ is the solution of the variational inequality (4.27), it follows from (4.38) that

$$a(u, v-u) = \int_{\Omega} f_i(v_i - u_i)\ dx + \int_{\partial\Omega} \sigma_{ij}(u) n_j (v_i - u_i)\ ds \geq \int_{\Omega} f_i(v_i - u_i)\ dx + \int_{\Gamma_F} t_i(v_i - u_i)\ ds,$$

therefore

$$\int_{\partial\Omega} \sigma_{ij}(u) n_j (v_i - u_i)\ ds \geq \int_{\Gamma_F} t_i(v_i - u_i)\ ds. \tag{4.39}$$

Due to the definition of $\mathbb{K}$, $v - u \in \mathbb{K}$ vanishes on the Dirichlet boundary Γ_D and due to (4.37), the inequality (4.39) reduces to

$$\int_{\Gamma_C} \sigma_{ij}(u) n_j (v_i - u_i)\ ds \geq 0. \tag{4.40}$$

Dividing the stress into its normal and tangential component as in (4.10), we can rewrite (4.40) equivalently as

$$\int_{\Gamma_C} \left(\sigma_{T_i}(u) + \sigma_n(u) n_i\right)(v_i - u_i)\ ds \geq 0. \tag{4.41}$$

Let again $v = u \pm w \in \mathbb{K}$ for $w \in C_0^\infty(\Omega)$ and we choose w, such that $w_n = w_i n_i = 0$ on Γ_C. Then (4.41) becomes to

$$\int_{\Gamma_C} \sigma_{T_i}(u)(\pm w_i) + \sigma_n(u) n_i (\pm w_i)\ ds = \int_{\Gamma_C} \sigma_{T_i}(u)(\pm w_i) \geq 0,$$

which implies

$$\sigma_{T_i}(u) = 0 \quad \text{on } \Gamma_C, \tag{4.42}$$

in (4.25d). The condition $u_n - g \leq 0$ is naturally satisfied, since u is an element in $\mathbb{K}$. To verify $\sigma_n(u) \leq 0$, we choose for $v = u + w$ the element w, such that $w_n = \psi \leq 0$. Hence, (4.41) together with (4.42) gives

$$0 \leq \int_{\Gamma_C} \sigma_n(u) w_n\ ds = \int_{\Gamma_C} \sigma_n(u)\psi\ ds \quad \text{for all } \psi \leq 0.$$

Since the integral is positive, the integrand must be positive, therefore it must hold $\sigma_n(u) \leq 0$. The last contact condition, which is $\sigma_n(u)(u_n - g) = 0$, can be derived as follows. Let $u_n - g < 0$ at a point $x \in \Gamma_C$. Then there exists a smooth function $\psi \geq 0$ on Γ_C, such that $\psi(x) > 0$ and $u_n - g + \psi \leq 0$ on Γ_C. Also, an element $w \in V$ exists, such that $w_n = \psi$ on Γ_C, hence $v = u + w \in \mathbb{K}$. Condition (4.41) together with $\sigma_n(u) \leq 0$ and (4.42) implies

$$0 \leq \int_{\Gamma_C} \sigma_n(u) w_n\ ds = \int_{\Gamma_C} \sigma_n(u)\psi\ ds \quad \text{for } \psi > 0.$$

Hence, it follows

$$\sigma_n(u) = 0,$$

and, as a consequence, $\sigma_n(u)(u_n - g) = 0$. Thus, (4.25d) follows. □

Remark 4.2. *At this point we want to emphasize that we have used many fundamental properties about the theory of Sobolev spaces, weak formulations and trace theory in the latest proof. A small introduction has been done in Chapter 2 and we refer the reader to [4, Chapter 2], [12, Chapter 5], [15, Chapter 3] and [18, Chapter 5] for more detailed description.*

Remark 4.3. *The variational inequality (4.27) can be also written in the form of*

$$a(u, v - u) + j(v) - j(u) \geq \langle F, v - u \rangle,$$

where the functional $j : V \to \mathbb{R}$ is defined as

$$j(v) = \begin{cases} 0 & \text{if } v \in \mathbb{K}, \\ +\infty & \text{if } v \in V \setminus \mathbb{K}. \end{cases}$$

In the case of contact with friction, j turns to a non-differentiable functional as we will see in the next chapter.

The question about the existence of a unique solution for the Signorini problem (4.25) or rather (4.27) is left for the next chapter. However, we may forestall, that two approaches are presented to handle this question. Firstly, we will investigate how variational inequalities are related to minimization problems and answer the question about the existence of a unique solution in terms of results from known applications of minimization problems. The second approach is to observe a general variational inequality and prove the existence and uniqueness of a solution under specific assumptions.

As a next step, we turn our attention to a simplified version of the Signorini problem, which will be later considered for numerical experiments.

4.2 Simplified Signorini problem

This section gives a simplified model of the Signorini problem, also known as the simplified Signorini problem. We refer to [9], [13, Chapter 2] and [14] for a detailed analysis of the simplified version of the Signorini problem. The aim is to examine the characterization between the classical model and the variational formulation. The simplified Signorini problem can be described in the following classical form: Find $u \in C^2(\Omega) \cup C^1(\Omega \cup \Gamma) \cup C(\bar{\Omega})$, such that

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{4.43a}$$

$$u \geq 0 \quad \text{on } \Gamma, \tag{4.43b}$$

$$\frac{\partial u}{\partial n} \geq g \quad \text{on } \Gamma, \tag{4.43c}$$

$$u \left(\frac{\partial u}{\partial n} - g \right) = 0 \quad \text{on } \Gamma, \tag{4.43d}$$

where f and g shall be sufficiently smooth and $\Gamma = \partial\Omega$. The corresponding variational form of this problem is described as: Find $u \in \mathbb{K} = \{v \in V \mid v \geq 0 \text{ on } \Gamma = \partial\Omega\} \subset V = H^1(\Omega)$, such that

$$a(u, v - u) \geq \langle F, v - u \rangle \quad \text{for all } v \in \mathbb{K}, \tag{4.44}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v + uv \; dx,$$

$$\langle F, v \rangle = \int_\Omega fv \; dx + \int_\Gamma gv \; ds.$$

As in the section before, we can show that a solution of the classical form is indeed a solution of the variational inequality and conversely. In order to show this characterization, we need definition of a convex cone and the help of an auxiliary Lemma.

Definition 4.4. *Let X be a vector space, and let $C \subset X$ be a subset and $x \in C$. Then C is a cone with vertex at x if for all $y \in C$ and $t \geq 0$, also $x + t(y - x) \in C$ holds. Moreover, we call a cone C convex if for all $u, v \in C$ and $\lambda \in [0, 1]$, the relation $\lambda x + (1 - \lambda)y \in C$ holds.*

Lemma 4.5. *Let V be a real Hilbert space, $a : V \times V \to \mathbb{R}$ a bilinear form, $l \in V^*$ a linear and bounded functional and $C \subset V$ a convex cone in V with vertex at 0. Then every solution of: Find $u \in C$, such that*

$$a(u, v - u) \geq l(v - u) \quad \textit{for all } v \in C, \tag{4.45}$$

is also a solution of: Find $u \in C$, such that

$$a(u, v) \geq l(v) \quad \textit{for all } v \in C, \tag{4.46}$$

$$a(u, u) = l(u), \tag{4.47}$$

and conversely.

Proof. We first assume that $u \in C$ is a solution of (4.45). Due to the linearity of the bilinear form a and the functional l, we can rewrite (4.45) as

$$a(u, v) - a(u, u) \geq l(v) - l(u) \quad \text{for all } v \in C. \tag{4.48}$$

Now, (4.47) is valid, since for $v = 0 \in C$ in (4.45),

$$a(u, 0 - u) \geq l(0 - u),$$

or rather

$$a(u, u) \leq l(u) \tag{4.49}$$

holds, and for $v = 2u \in C$ in (4.45),

$$a(u, u) \geq l(u) \tag{4.50}$$

holds. (4.49) and (4.50) together imply (4.47). Using (4.47) in (4.48) finally gives (4.46).

On the other hand, we assume that $u \in C$ is a solution of (4.46) and (4.47). We subtract $l(u)$ on both sides in (4.46) and obtain

$$a(u, v) - l(u) \geq l(v) - l(u) \quad \text{for all } v \in C.$$

Since (4.47) holds, we get

$$a(u, v) - a(u, u) \geq l(v) - l(u) \quad \text{for all } v \in C.$$

Using the linearity of the bilinear form a and the linear functional l, (4.45) follows. □

With this auxiliary Lemma we are now able to prove the characterization of the solution between the classical and the variational form of the simplified Signorini problem.

Theorem 4.6. *Let $V = H^1(\Omega)$, $\mathbb{K} = \{v \in V \mid v \geq 0 \text{ on } \Gamma\} \subset V$ and let f, g be sufficiently smooth. Then $\mathbb{K}$ is a nonempty, closed and convex cone with vertex at 0 and if $u \in \mathbb{K}$ solves the classical problem of the simplified Signorini problem (4.43), then it is a solution of its variational inequality (4.44) and conversely.*

Proof. We can easily obtain that $\mathbb{K}$ is a nonempty, closed and convex cone with vertex at 0. Indeed $\mathbb{K}$ is nonempty, since $0 \in \mathbb{K}$ (actually $H_0^1(\Omega) \subset \mathbb{K}$). Let $y \in \mathbb{K}$ and $t \geq 0$, then

$$ty \in \mathbb{K},$$

since $ty \geq 0$ on Γ. In order to prove the convexity of $\mathbb{K}$, we assume that $x, y \in \mathbb{K}$ and let $\lambda \in [0,1]$. Then

$$\lambda x + (1-\lambda)y \in \mathbb{K},$$

since $\lambda, 1-\lambda \geq 0$ and $\lambda x, (1-\lambda)y \in V$, such that $\lambda x, (1-\lambda)y \geq 0$ on Γ. In order to prove that $\mathbb{K}$ is closed, we assume that the sequence $(v_n)_n \subset \mathbb{K}$ converges to $v \in H^1(\Omega)$, i.e. $v_n \to v \in H^1(\Omega)$. Since the trace operator T in Theorem 2.10 is continuous, we obtain $Tv_n \to Tv$. Now $v_n \geq 0$ on Γ, since $v_n \in \mathbb{K}$. Thus, $v \geq 0$ on Γ, therefore $v \in \mathbb{K}$, which shows that $\mathbb{K}$ is closed.

Firstly, we assume that $u \in \mathbb{K}$ is a solution of the variational form (4.44). With the help of Lemma 4.5 we can rewrite the variational inequality as

$$\begin{aligned} a(u,v) &\geq \langle F, v\rangle \quad \text{for all } v \in \mathbb{K}, \\ a(u,u) &= \langle F, u\rangle. \end{aligned}$$

Since $C_0^\infty(\Omega) \subset \mathbb{K}$, we can choose $v = \pm w$ for $w \in C_0^\infty(\Omega)$ to obtain

$$a(u,w) = \int_\Omega (\nabla u \cdot \nabla w + uw)\, dx \geq \int_\Omega fw\, dx + \underbrace{\int_\Gamma gw\, ds}_{=0} = \langle F, w\rangle \quad \text{for all } w \in C_0^\infty, \tag{4.51}$$

and

$$a(u,w) \leq \langle F, w\rangle \quad \text{for all } w \in C_0^\infty. \tag{4.52}$$

(4.51) and (4.52) together gives

$$a(u,w) = \langle F, w\rangle \quad \text{for all } w \in C_0^\infty. \tag{4.53}$$

Applying integration by parts to the main part of the bilinear form a gives

$$\int_\Omega \nabla u \cdot \nabla w\, dx = -\int_\Omega \Delta u w\, dx + \underbrace{\int_\Gamma uw\, ds}_{=0},$$

which changes (4.53) to

$$\int_\Omega -\Delta u w + uw\, dx = \int_\Omega fw\, dx \quad \text{for all } w \in C_0^\infty. \tag{4.54}$$

Since (4.54) is valid for every $w \in C_0^\infty$, (4.43a) follows. (4.43b) is naturally satisfied due to the definition of the subset $\mathbb{K}$.
To verify (4.43c), let $v \in \mathbb{K}$. We multiply (4.43a) with v and use integration of parts to obtain

$$a(u,v) = \int_\Omega fv\, dx + \int_\Gamma \frac{\partial u}{\partial n} v\, ds \quad \text{for all } v \in \mathbb{K}. \tag{4.55}$$

Using now (4.46) we deduce from (4.55) that

$$\int_\Omega fv\,dx + \int_\Gamma \frac{\partial u}{\partial n} v\,ds = a(u,v) \geq \int_\Omega fv\,dx + \int_\Gamma gv\,ds \quad \text{for all } v \in \mathbb{K},$$

which implies

$$\int_\Gamma \left(\frac{\partial u}{\partial n} - g\right) v\,ds \geq 0 \quad \text{for all } v \in \mathbb{K}. \tag{4.56}$$

Since the integral in (4.56) is non-negative, also the integrand must be non-negative. Because $v \geq 0$ on Γ, it follows that

$$\frac{\partial u}{\partial n} - g \geq 0 \quad \text{on } \Gamma,$$

therefore (4.43c) holds.
For the last boundary condition (4.43d), we consider (4.55) with the choice $v = u$ and use property (4.47) to get

$$\int_\Omega fu\,dx + \int_\Gamma \frac{\partial u}{\partial n} u\,ds = a(u,u) = \int_\Omega fu\,dx + \int_\Gamma gu\,ds,$$

which leads to

$$\int_\Gamma \left(\frac{\partial u}{\partial n} - g\right) u\,ds = 0. \tag{4.57}$$

Since $u \geq 0$ on Γ and $\frac{\partial u}{\partial n} - g \geq 0$ on Γ, it follows from (4.57) that

$$\left(\frac{\partial u}{\partial n} - g\right) u = 0 \quad \text{on } \Gamma.$$

This shows, that a solution $u \in \mathbb{K}$ of (4.44) is also a solution of (4.43).

On the other hand, we assume that $u \in \mathbb{K}$ is a solution of the classical problem (4.43) and we want to prove that u solves the variational form (4.44). Starting from (4.43a), we multiply the equation with a test function $v - u \in \mathbb{K}$, integrate over the domain and use Green's formula to obtain

$$\int_\Omega \nabla u \cdot \nabla(v-u) + u(v-u)\,dx = \int_\Omega f(v-u)\,dx + \int_\Gamma \frac{\partial u}{\partial n}(v-u)\,ds \quad \text{for all } v \in \mathbb{K}.$$

Using the boundary condition (4.43c), we get

$$\int_\Omega \nabla u \cdot \nabla(v-u) + u(v-u)\,dx \geq \int_\Omega f(v-u)\,dx + \int_\Gamma g(v-u)\,ds \quad \text{for all } v \in \mathbb{K},$$

and therefore (4.44) holds. □

For completeness we want to give the minimization problem of the simplified Signorini problem. It can be easily proven that the solution of variational form of the simplified Signorini problem (4.44) may be characterized by a minimization problem and conversely. The minimization problem is: Find $u \in \mathbb{K}$, such that

$$E(u) = \inf_{v \in \mathbb{K}} E(v), \tag{4.58}$$

where

$$E(v) = \int_\Omega \left(\frac{1}{2}\left(|\nabla v| + v\right)^2 - fv\right)\,dx - \int_\Gamma gv\,ds.$$

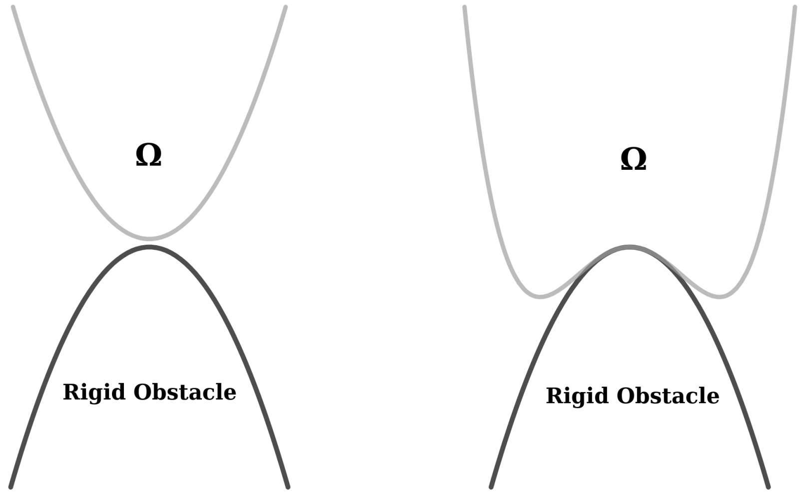


Figure 4.3: The obstacle problem in initial configuration (left) and after applying certain body forces f (right).

4.3 Obstacle problem

Another very famous type of contact problems is the so-called obstacle problem. It involves the contact between an elastic solid Ω and a rigid obstacle ψ as illustrated in Figure 4.3. Contrary to the simplified Signorini problem, the set of admissible displacements $\mathbb{K}$ is restricted on the whole body Ω and not only on the boundary Γ. Furthermore, if the elastic body is in touch with the obstacle, there are no gaps in between. In the Signorini problem gaps may occur which are measured by the gap function and makes the problem significantly more difficult. However, the aim of this section is to present the classical and variational form of the obstacle problem without physical description and further analysis. We refer to [28, Chapter 1] and [39] for the detailed problem description. The obstacle problem will be used to generate numerical experiments viewed as simplified contact problems.

The classical form of the obstacle problem is the following:
Find the displacement u, such that

$$-\Delta u \geq f \qquad \text{in } \Omega, \tag{4.59a}$$

$$u \geq \psi \qquad \text{in } \Omega, \tag{4.59b}$$

$$(u-\psi)(-\Delta u - f) = 0 \qquad \text{in } \Omega, \tag{4.59c}$$

where the obstacle $\psi \in H^1(\Omega)$ with non-positive trace on Γ and $f \in L_2(\Omega)$.

The variational formulation of the obstacle problem is:
Find $u \in \mathbb{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ in } \Omega\}$, such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)\; dx \geq \int_\Omega f(v-u)\; dx \quad \text{for all } v \in \mathbb{K}. \tag{4.60}$$

For completeness, we want to give the minimization problem. Find $u \in \mathbb{K}$, such that

$$E(u) = \inf_{v \in \mathbb{K}} E(v), \tag{4.61}$$

with

$$E(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right)\; dx.$$

Chapter 5

Analysis of Variational Inequalities

The intent of this chapter is to present an analysis of variational inequalities in an abstract and general form. As a motivation, we described the Signorini problem in the previous chapter, which led to a variational inequality. As a matter of fact, most contact problems in continuum mechanics can be characterized by variational inequalities as well. The goal is to answer the question about the existence of a unique solution for a wide range of different contact problems obtaining general variational inequalities. Since many variational forms are related to minimization problems, we will investigate the characterization by minimization problems as a first step. We begin with introductory examples in the finite dimensional case following the ideas of [19, Chapter 1]. Consequentially, we extend the observations of the finite dimensional case to an analysis on Hilbert spaces motivated by [1, Chapter 11], [13, Chapter 1], [18, Chapter 3], [19, Chapter 3] and [26]. Using some results from the theory about minimization problems, a first statement about the existence and uniqueness of a solution can be derived. We extend our investigation to so-called elliptic variational inequalities, which are not necessarily related to minimization problems and answer the question about the unique solution as in [1, Chapter 11] and [13, Chapter 1]. However, elliptic variational inequalities do not describe by far all kinds of contact problems, which gives us the opportunity to examine elliptic hemi-variational inequalities. For this kind of problems, we refer to [6, Chapter 4], [11], [16, Chapter 2] and [34]. The end of this chapter is left for an application of a contact problem with friction using the ideas of hemi-variational inequalities, motivated by [6, Chapter 8] and [18, Chapter 10].

5.1 Introductory examples

We want to start with introductory examples in the finite dimensional case and we verify how minimization problems are related to variational inequalities. We refer to [19, Chapter 1] for a very clear introduction of variational inequalities.

Example 5.1. We consider the problem: Find $x_0 \in I = [a, b]$, such that

$$f(x_0) = \min_{x \in I} f(x),$$

where f is a real-valued smooth function. In order to find this minimum, three possible cases can occur:

1. If $a < x_0 < b$, then $f'(x_0) = 0$.
2. If $x_0 = a$, then $f'(x_0) \geq 0$.
3. If $x_0 = b$, then $f'(x_0) \leq 0$.

These three possible cases can be easily summarized to

$$f'(x_0)(x - x_0) \geq 0 \quad \text{for all } x \in I. \tag{5.1}$$

We may call (5.1) a variational inequality in the finite dimensional case.

A further example for the d-dimensional case is the following.

Example 5.2. Let f be a smooth and real-valued function defined on the nonempty, closed and convex set $\mathbb{K}$ of the Euclidean d-dimensional space $\mathbb{R}^d$. Again, we seek for $x_0 \in \mathbb{K}$, such that

$$f(x_0) = \min_{x \in \mathbb{K}} f(x).$$

We assume that $x_0 \in \mathbb{K}$ is the minimum point and let $x \in \mathbb{K}$. Since $\mathbb{K}$ is convex, the segment $(1-t)x_0 + tx = x_0 + t(x - x_0)$ for $t \in [0,1]$, is also in $\mathbb{K}$. Defining the function

$$\psi(t) = f(x_0 + t(x - x_0)), \qquad 0 \leq t \leq 1,$$

we can see, that the minimum is reached at $t = 0$. Therefore,

$$\psi'(0) = \nabla f(x_0) \cdot (x - x_0) \geq 0 \qquad \text{for any } x \in \mathbb{K}.$$

As a consequence, the point x_0 satisfies the variational inequality: Find $x_0 \in \mathbb{K}$, such that

$$\nabla f(x_0) \cdot (x - x_0) \geq 0 \qquad \text{for any } x \in \mathbb{K}. \tag{5.2}$$

These examples were presented to give the first idea how to derive variational inequalities from minimization problems in the finite dimensional case.

5.2 Variational inequalities in $\mathbb{R}^d$

In this section, we want to examine variational inequalities in the d-dimensional case. A bunch of nonlinear problems arising from variational inequalities can be solved by means of fixed point applications.

Definition 5.3. *Let S be an arbitrary set and let $F : S \to S$ be a mapping on S. A point $x \in S$ is called fixed point of F, if*

$$F(x) = x \tag{5.3}$$

Usually, fixed point problems as (5.3) are solved in terms of contractions.

Definition 5.4. *Let (S, m) be a metric space with metric function m. A mapping $F : S \to S$ is a contraction mapping, if*

$$m(F(x), F(y)) \leq \alpha \; m(x, y) \qquad \textit{for all } x, y \in S, \tag{5.4}$$

and for $\alpha \in [0, 1)$. If $\alpha = 1$, the mapping is called non-expansive.

Only in terms of the contraction property we are able to guarantee a fixed point in complete metric spaces.

Theorem 5.5 (Contraction mapping theorem)**.** *Let S be a complete metric space and let $F : S \to S$ be a contraction mapping. Then there exists a unique fixed point of F.*

Proof. See [22]. □

Remark 5.6. *If F is non-expansive, Theorem 5.5 is generally not true. For example, rigid body rotations do not have any fixed points, where $S = [0,1] \times [0,1] \times [0,1]$ and $F(x) = Qx + c$ for all $x \in S$, with a constant rotation matrix Q (i.e. $Q^T Q = I$, $\det Q > 0$) and a constant vector c.*

Another fundamental theorem is Brouwer's theorem, which ensures the existence of a fixed point on a closed ball.

Theorem 5.7 (Brouwer)**.** *Let $F : B \to B$ be a continuous mapping on a closed ball $B \subset \mathbb{R}^d$. Then F admits at least one fixed point.*

Proof. See [22]. □

Recalling Example 5.2, we receive from (5.2) that the minimum x_0 fulfills

$$\langle \nabla f(x_0), x - x_0 \rangle \geq 0 \quad \text{for all } x \in \mathbb{K},$$

where $\mathbb{K}$ is a nonempty, closed and convex subset in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle : (\mathbb{R}^d)^* \times \mathbb{R}^d \to \mathbb{R}$ is the duality pairing in $\mathbb{R}^d$. In fact, the problem that we want to solve in the finite dimensional case is the following.

Problem 5.8. Find $x \in \mathbb{K}$, such that

$$\langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in \mathbb{K}, \tag{5.5}$$

where $\mathbb{K} \subset \mathbb{R}^d$ is a nonempty, closed and convex subset and $F : \mathbb{K} \to (\mathbb{R}^d)^*$ is continuous.

Remark 5.9. *The dual space of any finite dimensional space R^d is the space itself, i.e. $(\mathbb{R}^d)^* = \mathbb{R}^d$.*

At this point, we are able to formulate a first statement about the uniqueness of variational inequalities of the from (5.5). In general, the solution of a variational inequality need not to be unique.

Theorem 5.10. *Let x and x' be two different solutions of (5.5). If the condition*

$$\langle F(x) - F(x'), x - x' \rangle > 0, \tag{5.6}$$

is satisfied, then the solution of the variational inequality (5.5) *in $\mathbb{R}^d$ is unique.*

Proof. Let $x, x' \in \mathbb{K}$ be two distinct solutions of (5.5), i.e.

$$\begin{aligned} \langle F(x), y - x \rangle &\geq 0, \qquad \text{for all } y \in \mathbb{K}, \\ \langle F(x'), \tilde{y} - x' \rangle &\geq 0, \qquad \text{for all } \tilde{y} \in \mathbb{K}. \end{aligned}$$

Choosing $y = x'$ and $\tilde{y} = x$ and adding these two inequalities, we obtain that

$$\langle F(x) - F(x'), x - x' \rangle \leq 0.$$

Since condition (5.6) holds with $x \neq x'$, we deduce that, there cannot be two different solutions. □

In fact, condition (5.6) is used to define a strictly monotone mapping.

Definition 5.11. *A mapping $F : \mathbb{K} \to (\mathbb{R}^d)^*$ is monotone if*

$$\langle F(x) - F(x'), x - x' \rangle \geq 0 \qquad \text{for all } x, x' \in \mathbb{K}.$$

The mapping is called strictly monotone, if equality holds only for $x = x'$, i.e.

$$\langle F(x) - F(x'), x - x' \rangle > 0 \qquad \text{for all } x, x' \in \mathbb{K} \text{ and } x \neq x'.$$

As we have seen in Theorem 5.10, the strict monotonicity guarantees the uniqueness of a solution of the variational inequality (5.5). Thus, for more general problems, e.g. variational inequalities in Hilbert spaces, we need the monotonicity as a requirement for the uniqueness and, as we will see later, even for the existence. We want to investigate now, how variational inequalities in $\mathbb{R}^d$ are related to minimization problems. As we have seen in the introductory examples, finding a minimum can be rewritten as a variational inequality.

Theorem 5.12. *Let $\mathbb{K} \subset \mathbb{R}^d$ be a nonempty, closed and convex subset, let $f \in C^1(\mathbb{K})$ and let $x \in \mathbb{K}$ be the solution of the minimization problem*

$$f(x) = \min_{y \in \mathbb{K}} f(y). \tag{5.7}$$

Then $x \in \mathbb{K}$ is also a solution of the variational inequality: Find $x \in \mathbb{K}$, such that

$$(\nabla f(x), y - x) \geq 0 \qquad \text{for all } y \in \mathbb{K}. \tag{5.8}$$

Proof. See Example 5.2. □

With the help of convexity, we can prove the converse direction.

Theorem 5.13. *Let $f \in C^1(\mathbb{K})$ be convex and let $x \in \mathbb{K}$ satisfy the inequality (5.8). Then $x \in \mathbb{K}$ solves the minimization problem (5.7).*

Proof. Let $x \in \mathbb{K}$ be a solution of (5.8). Since f is convex, we have by definition that

$$f(ty + (1-t)x) \leq tf(y) + (1+t)f(x) \quad \text{for all } y \in \mathbb{K},\ t \in [0,1]. \tag{5.9}$$

Rewriting (5.9) yields

$$f(x + t(y - x)) \leq f(x) + t(f(y) - f(x)),$$

which gives

$$f(y) \geq f(x) + \frac{f(x + t(y-x)) - f(x)}{t}, \quad \text{for all } t \in (0,1].$$

It follows that

$$f(y) \geq f(x) + (\nabla f(x), y - x), \tag{5.10}$$

as $t \to 0$. Since (5.8) holds, we deduce from (5.10) that

$$f(y) \geq f(x) \quad \text{for all } y \in \mathbb{K},$$

which concludes the proof. □

By means of the convexity it can be shown that the gradient of a function is monotone.

Theorem 5.14. *Let $S \subset \mathbb{R}^d$ be a open subset and let $f : S \to \mathbb{R}$ be a continuously differentiable and convex (or strictly convex) function. Then $F(x) = \nabla f(x)$ is monotone (or strictly monotone).*

Proof. See [19, Chapter 1]. □

We want to conclude this section with a slightly changed version of Brouwer's theorem using the property of the convexity.

Theorem 5.15 (Brouwer)**.** *Let $\mathbb{K} \subset \mathbb{R}^d$ be a nonempty, compact and convex subset and let $F : \mathbb{K} \to \mathbb{K}$ be continuous. Then F admits a fixed point.*

Proof. See [19, Chapter 1]. □

We will see later, that the fixed point theory is needed to prove the existence of a solution for variational inequalities as in (5.5). The proof will be done in the upcoming section for more general variational inequalities, which covers the finite dimensional case as a consequence.

5.3 Elliptic variational inequalities in Hilbert spaces

In this section, we consider variational inequalities defined on a Hilbert space V. Throughout this chapter, we consider V to be a real Hilbert space, e.g. $H^1(\Omega)$. As in [19, Chapter 1], we shortly want to examine projections on a nonempty, closed and convex subset $\mathbb{K} \subset V$. Afterwards, we turn our attention to elliptic variational inequalities which represent one of the main parts of this work.

Lemma 5.16. *Let $\mathbb{K}$ be a nonempty, closed and convex subset of V. Then for each $x \in V$ there is a unique $y \in \mathbb{K}$, called the projection point of x, such that*

$$\|x - y\| = \inf_{\eta \in \mathbb{K}} \|x - \eta\|. \tag{5.11}$$

The projection point can be written as

$$y = \mathrm{P}_{\mathbb{K}} x.$$

Proof. See [19, Chapter 1]. □

The next theorem gives a characterization of the projection.

Theorem 5.17. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset and let $x \in V$. Then $y = \mathrm{P}_{\mathbb{K}} x \in \mathbb{K}$ is the projection of x on $\mathbb{K}$ if and only if*

$$(y, \eta - y) \geq (x, \eta - y) \quad \textit{for all } \eta \in \mathbb{K}. \tag{5.12}$$

Proof. See [19, Chapter 1]. □

As a consequence it can be obtained, that projection mapping is non-expansive.

Corollary 5.18. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset and let $x \in V$. Then the projection operator $\mathrm{P}_{\mathbb{K}}$ is non-expansive, i.e.*

$$\|\mathrm{P}_{\mathbb{K}} x - \mathrm{P}_{\mathbb{K}} x'\| \leq \|x - x'\| \quad \textit{for all } x, x' \in V.$$

Proof. See [19, Chapter 1]. □

5.3.1 Minimization problems of elliptic variational inequalities

We turn now our attention now to elliptic variational inequalities and want to determine how they are related to minimization problems. In this section, we will follow the ideas of [1, Chapter 11], [13, Chapter 1], [18, Chapter 3] and [26]. We begin with some basic definitions, which denotes the properties of the linear functional $F : \mathbb{K} \to \mathbb{R}$, and give a statement about the connection between variational inequalities and minimization problems. As in the section before, we always consider $\mathbb{K} \subset V$ to be a nonempty, closed and convex subset of a Hilbert space V and V^* denotes the dual space of V.

Definition 5.19 (Convexity). *A functional $F : \mathbb{K} \to \mathbb{R}$ is called convex if and only if*

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v), \tag{5.13}$$

for all $u, v \in \mathbb{K}$ and for all $t \in [0,1]$. We say F is strictly convex if and only if

$$F(tu + (1-t)v) < tF(u) + (1-t)F(v), \tag{5.14}$$

for all $u, v \in \mathbb{K}$ with $u \neq v$ and for all $t \in (0,1)$.

Definition 5.20 (Gâteaux-differentiability). *Let $\mathbb{D}$ be a normed space. A mapping $F : \mathbb{K} \to \mathbb{D}$ is G-differentiable (or Gâteaux-differentiable) at a point $u \in \mathbb{K}$, if there exists a linear functional $DF(u) \in V^*$, such that*

$$\langle DF(u), v\rangle = \lim_{t\to 0} \frac{F(u+tv) - F(u)}{t}, \tag{5.15}$$

for all $v \in \mathbb{K}$, where $\langle \cdot, \cdot \rangle$ denotes the duality paring on $V^* \times V$. We call $DF(u)$ the Gâteaux derivative (G-derivative) of F at u and $\langle DF(u), v\rangle$ the Gâteaux derivative of F at point u in direction v. The second Gâteaux derivative is defined as

$$\langle D^2F(u)(v), w\rangle = \lim_{t\to 0} \frac{\langle DF(u+tv), w\rangle - \langle DF(u), w\rangle}{t},$$

for all $u, v, w \in \mathbb{K}$.

Definition 5.21 (Lower and upper semi-continuity). *A functional $F : \mathbb{K} \to \mathbb{R}$ is called lower semi-continuous on $\mathbb{K}$ if for any sequence $\{u_k\} \subset \mathbb{K}$ with the property that $\{u_k\}$ converges to $u \in \mathbb{K}$, i.e. $\|u_k - u\| \to 0$ as $k \to \infty$, we have*

$$\liminf_{k\to\infty} F(u_k) \geq F(u). \tag{5.16}$$

The function is called upper semi-continuous if

$$\limsup_{k\to\infty} F(u_k) \leq F(u), \tag{5.17}$$

for any $u_k \to u \in \mathbb{K}$. If F is lower and upper semi-continuous, then F is continuous. Replacing the strong convergence of the sequence $\{u_k\}$ by the weak convergence, i.e.

$$\lim_{k\to\infty} \langle f, u_k\rangle = \langle f, u\rangle \quad \textit{for all } f \in V^*,$$

then we call F weakly lower or weakly upper semi-continuous, respectively.

Remark 5.22. *The lower or upper semi-continuity implies the weakly lower or weakly upper semi-continuity.*

The next two theorems clarify the connection between the definitions we introduced and give some helpful properties we may use later.

Theorem 5.23. *Let $F : \mathbb{K} \to \mathbb{R}$ satisfy the following conditions:*

i) F is convex.

ii) F is G-differentiable on $\mathbb{K}$.

Then F is weakly lower semi-continuous on $\mathbb{K}$.

Proof. See [20, Chapter 3]. □

Theorem 5.24. *Let $F : \mathbb{K} \to \mathbb{R}$ be G-differentiable. Then the following statements are equivalent.*

i) F is convex.

ii) $\langle DF(u) - DF(v), u - v\rangle \geq 0, \qquad$ for all $u, v \in \mathbb{K}$.

iii) $F(v) \geq F(u) + \langle DF(u), v - u\rangle, \qquad$ for all $u, v \in \mathbb{K}$.

Moreover, if F is twice G-differentiable, then the above statements are also equivalent to

iv) $\langle D^2F(u)(v), v\rangle \geq 0, \qquad$ for all $v \in \mathbb{K}$.

Proof. See [11, Chapter 1]. □

Remark 5.25. *In fact, we have proven the equivalence of statement* i) *and* iii) *of Theorem 5.24 in the proof of Theorem 5.13 for the finite dimensional case.*

We are now able to present the relation between variational inequalities and minimization problems in terms of the above definitions and theorems.

Theorem 5.26. *Let $\mathbb{K}$ be a nonempty, closed and convex subset of V and let $F : \mathbb{K} \to \mathbb{R}$ be a G-differentiable functional. If $u \in \mathbb{K}$ is a minimizer of F in $\mathbb{K}$, i.e. $F(u) = \min_{v\in\mathbb{K}} F(v)$, then u may be characterized in one of the following ways:*

i) $u \in \mathbb{K}$ is a solution of the variational inequality

$$\langle DF(u), v - u\rangle \geq 0 \quad \textit{for all } v \in \mathbb{K}.$$

ii) If u is in the interior of $\mathbb{K}$, i.e. $u \in \operatorname{int} \mathbb{K}$, then it solves

$$\langle DF(u), v\rangle = 0 \quad \textit{for all } v \in \mathbb{K}.$$

iii) If $\mathbb{K}$ is a nonempty, closed and convex cone with vertex w, then u solves

$$\langle DF(u), u - w\rangle = 0 \quad \textit{and} \quad \langle DF(u), v - w\rangle \geq 0 \quad \textit{for all } v \in \mathbb{K}.$$

iv) If $\mathbb{K}$ is a linear subspace of V, then u solves

$$\langle DF(u), v\rangle = 0 \quad \textit{for all } v \in \mathbb{K}.$$

Proof. Follows from Theorem 5.27 with $j = 0$. □

The latter theorem can be formulated for more general variational inequalities. For this purpose, we extend the variational inequality with an functional $j : \mathbb{K} \to \mathbb{R}$.

Theorem 5.27. *Let $\mathbb{K}$ be a nonempty, closed and convex subset of V, let $F : \mathbb{K} \to \mathbb{R}$ and $j : \mathbb{K} \to \mathbb{R}$ be two convex functionals and let F be G-differentiable. Then $u \in \mathbb{K}$ solves the minimization problem*

$$F(u) + j(u) = \inf_{v \in \mathbb{K}} \left[F(v) + j(v) \right], \tag{5.18}$$

if and only if $u \in \mathbb{K}$ solves

$$\langle DF(u), v - u \rangle + j(v) - j(u) \geq 0 \quad \text{for all } v \in \mathbb{K}, \tag{5.19a}$$

$$\langle DF(v), v - u \rangle + j(v) - j(u) \geq 0 \quad \text{for all } v \in \mathbb{K}. \tag{5.19b}$$

Moreover, if $\mathbb{K}$ is a subspace of V and $j = 0$ almost everywhere, then the inequality (5.19a) reduces to an equality, i.e. $u \in \mathbb{K}$ solves

$$\langle DF(u), v \rangle = 0 \quad \text{for all } v \in \mathbb{K}. \tag{5.20}$$

Proof. First, we want to prove that a solution of (5.18) solves (5.19a). Let $u \in \mathbb{K}$ be the solution of (5.18). Due to the convexity of $\mathbb{K}$ we have that $tv + (1-t)u = u + t(v-u) \in \mathbb{K}$ for any $v \in \mathbb{K}$ and any $t \in (0,1)$. Hence,

$$\begin{aligned} F(u) + j(u) &\leq F(u + t(v-u)) + j(u + t(v-u)) \\ &= F(u + t(v-u)) + j((1-t)u + tv) \\ &\leq F(u + t(v-u)) + (1-t)j(u) + tj(v), \end{aligned}$$

where the last inequality follows from the convexity of j. Subtracting the left hand side from the right hand side gives

$$F(u + t(v-u)) - F(u) + tj(u) + tj(v) \geq 0. \tag{5.21}$$

Dividing (5.21) by t yields

$$\frac{F(u + t(v-u)) - F(u)}{t} + j(u) + j(v) \geq 0 \quad \text{for all } t \in (0,1).$$

Letting $t \to 0$ and using the fact that F is G-differentiable we obtain (5.19a).

The other way round, we assume that $u \in \mathbb{K}$ is a solution of (5.19a). Since F is convex, we can write with the help of Theorem 5.24 that

$$F(v) \geq F(u) + \langle DF(u), v - u \rangle \quad \text{for all } v \in \mathbb{K}.$$

Thus,

$$F(v) + j(v) \geq F(u) + j(v) + \langle DF(u), v - u \rangle. \tag{5.22}$$

Since u satisfies (5.19a), we can write $j(v) + \langle DF(u), v - u \rangle \geq j(u)$. Hence, it follows from (5.22) that

$$F(v) + j(v) \geq F(u) + j(u) \quad \text{for all } v \in \mathbb{K},$$

which gives (5.18).

If $j = 0$ almost everywhere, (5.19a) becomes to

$$\langle DF(u), v - u \rangle \geq 0.$$

If $\mathbb{K}$ is a subspace we can choose $v = w + u$ for any $w \in \mathbb{K}$, which gives

$$\langle DF(u), w \rangle \geq 0 \quad \text{for all } w \in \mathbb{K}. \tag{5.23}$$

Due to the fact that $\mathbb{K}$ is a subspace of V, for any $w \in \mathbb{K}$ also $-w \in \mathbb{K}$, therefore

$$\langle DF(u), w \rangle \leq 0 \quad \text{for all } w \in \mathbb{K}. \tag{5.24}$$

(5.23) and (5.24) together yield (5.20).

It remains to show the characterization of (5.19a) and (5.19b), which directly follows from the Minty Lemma 5.48 with $A = DF$. □

Remark 5.28. *As we can see in the last proof, the convexity of $\mathbb{K}$ is essential. However, the existence of a solution has been assumed. In order to formulate an existence and uniqueness statement about the solution of variational inequalities stronger assumptions must be required, which will be the topic of the next section.*

At this point we can use a result from minimization theory to give a first statement about the unique solution of the minimization problem (5.18). An adjusted version of the generalized Weierstrass theorem indicates a good result.

Theorem 5.29 (Generalized Weierstrass minimization theorem)**.** *Let $F : \mathbb{K} \to \mathbb{R}$ be a convex and lower semi-continuous functional defined on a nonempty, closed and convex subset $\mathbb{K} \subset V$. Moreover, let one of the following conditions holds:*

i) $\mathbb{K}$ is bounded.

ii) F is coercive on $\mathbb{K}$, i.e.

$$\lim_{\|v\| \to \infty} F(v) = +\infty.$$

Then the problem: Find $u \in \mathbb{K}$, such that

$$F(u) = \inf_{v \in \mathbb{K}} F(v),$$

has at least one solution. Furthermore, if F is strictly convex, then the solution is unique.

Proof. [47, Chapter 1] □

Remark 5.30. *For a bunch of problems, the subset $\mathbb{K}$ is unbounded and, therefore, in most situations the second property must be proven. In order to show that F is coercive on $\mathbb{K}$, an estimate of the form*

$$F(v) \geq C_0 \|v\|^{\alpha} - C_1 \|v\|^{\beta}, \tag{5.25}$$

with $C_0, C_1 > 0$ and $\alpha > \beta$ can be used.

The generalized Weierstrass minimization theorem gives the existence of a solution for the minimization problem (5.18) if we are able to guarantee one of the assumptions. In addition, it follows directly from Theorem 5.27 that the variational inequality (5.19a) admits a solution too. Considering now problems with a bilinear form a we can make use of the latest two theorems to show the following result.

Theorem 5.31. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Assume that $a : V \times V \to \mathbb{R}$ is a bounded, symmetric and V-elliptic bilinear form, $l \in V^*$ and $j : \mathbb{K} \to \mathbb{R}$ is convex and lower semi-continuous on $\mathbb{K}$. Then the minimization problem: Find $u \in \mathbb{K}$, such that*

$$E(u) = \inf_{v \in \mathbb{K}} E(v), \tag{5.26}$$

with

$$E(v) = \frac{1}{2} a(v, v) + j(v) - l(v),$$

has a unique solution. Moreover, $u \in \mathbb{K}$ is a solution of the minimization problem if and only if $u \in \mathbb{K}$ is a solution of

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \quad \text{for all } v \in \mathbb{K}. \tag{5.27}$$

Proof. Follows from Theorem 5.29, since condition *ii*) is valid for elliptic bilinear forms with the help of Lemma 5.45 and Theorem 5.27 with $\langle DF(u), v - u \rangle = a(u, v - u) - l(v - u)$. See also [1, Chapter 11]. □

Remark 5.32. *In the latest proof, the auxiliary Lemma 5.45 is needed for the functional j, in order to guarantee property ii) of Theorem 5.29. Lemma 5.45 will be introduced in the sequel.*

At this point, we want to apply Theorem 5.31 on the two examples, which we have introduced in the previous chapter. We start with the obstacle problem and move on with the simplified Signorini problem.

Example 5.33 (Obstacle problem)**.** We have already seen that the set of admissible displacements of the obstacle problem (4.59) is defined as

$$\mathbb{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ in } \Omega\} \subset V = H_0^1(\Omega),$$

which is nonempty, since the function $\max\{0, \psi\}$ is an element in $\mathbb{K}$ with $\psi \in H^1(\Omega)$ with non-positive trace, c.f. [28, Chapter 4]. This set is also closed and convex, c.f. [13, Chapter 2]. The bilinear form of the obstacle problem is defined as

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Indeed, it is bounded, symmetric and elliptic with respect to $H_0^1(\Omega)$. The right hand side of the obstacle problem is

$$l(v) = \int_\Omega f v \, dx.$$

In fact, l is bounded and $l \in V^*$. Furthermore, $j(v) = 0$ for all $v \in H_0^1(\Omega)$. Thus, we can apply Theorem 5.31 to obtain that the variational inequality: Find $u \in \mathbb{K}$, such that

$$\int_\Omega \nabla u \cdot \nabla (v - u) \, dx \geq \int_\Omega f(v - u) \, dx \quad \text{for all } v \in \mathbb{K},$$

and the corresponding minimization problem: Find $u \in \mathbb{K}$, such that

$$E(u) = \inf_{v \in \mathbb{K}} E(v),$$

with

$$E(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 - f v \right) \, dx,$$

have a unique solution.

Example 5.34 (Simplified Signorini problem)**.** We similarly conclude, that the set of admissible displacements $\mathbb{K} = \{v \in H^1(\Omega) \mid v \geq 0 \text{ on } \Gamma = \partial\Omega\} \subset V = H^1(\Omega)$ of the simplified Signorini problem (4.25) is nonempty, closed and convex. Furthermore, the bilinear form

$$a(u, v) = \int\limits_{\Omega} \nabla u \cdot \nabla v + uv \; dx$$

is bounded, symmetric and elliptic with respect to $H^1(\Omega)$, the right hand side

$$l(v) = \int\limits_{\Omega} fv \; dx + \int\limits_{\Gamma} gv \; ds,$$

is bounded and $l \in V^*$, and $j(v) = 0$ for all $v \in H^1(\Omega)$. Hence, by Theorem 5.31, the variational inequality: Find $u \in \mathbb{K}$, such that

$$\int\limits_{\Omega} \nabla u \cdot \nabla (v - u) + u(v - u) \; dx \geq \int\limits_{\Omega} f(v - u) \; dx + \int\limits_{\Gamma} g(v - u) \; ds \quad \text{for all } v \in \mathbb{K},$$

and the corresponding minimization problem: Find $u \in \mathbb{K}$, such that

$$E(u) = \inf_{v \in \mathbb{K}} E(v),$$

where

$$E(v) = \int\limits_{\Omega} \left(\frac{1}{2} \left(|\nabla v| + v \right)^2 - fv \right) \; dx - \int\limits_{\Gamma} gv \; ds,$$

have a unique solution.

5.3.2 Existence and uniqueness results of variational inequalities

In Theorem 5.31 the variational inequality (5.27) is associated with minimization problems. The aim of this section is to prove the existence and uniqueness of a solution for variational inequalities not necessarily related to minimization problems following the ideas of [1, Chapter 11] and [13, Chapter 1]. For this purpose, let V be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ and let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. We consider the following variational inequality.

Problem 5.35. Find $u \in \mathbb{K}$, such that

$$(A(u), v - u) + j(v) - j(u) \geq (f, v - u) \quad \text{for all } v \in \mathbb{K}, \tag{5.28}$$

where $A : V \to V$ is an operator, $j : \mathbb{K} \to \mathbb{R}$ is a functional and $f \in V$.

The operator needs to satisfy some conditions in order to prove the existence and uniqueness of (5.28).

Definition 5.36 (Monotonicity)**.** *Let V be a real Hilbert space. An operator $A : V \to V$ is called monotone if*

$$(A(u) - A(v), u - v) \geq 0 \quad \textit{for all } u, v \in V.$$

The operator A is strongly monotone if there is exists a constant $c_0 > 0$, such that

$$(A(u) - A(v), u - v) \geq c_0 \|u - v\|^2 \quad \textit{for all } u, v \in V.$$

Definition 5.37 (Lipschitz continuity)**.** *Let V be a real Hilbert space. An operator $A : V \to V$ is Lipschitz continuous if there exists a constant $L > 0$, such that*

$$\|A(u) - A(v)\| \leq L\|u - v\| \quad \textit{for all } u, v \in V.$$

We are now ready to formulate and prove the main theorem ensuring a unique solution of the variational inequality (5.28).

Theorem 5.38. *Let V be a real Hilbert space and $\mathbb{K} \subset V$ a nonempty, closed and convex subset. Moreover, let $A : V \to V$ be strongly monotone and Lipschitz continuous and let $j : \mathbb{K} \to \mathbb{R}$ be a convex and lower semi-continuous functional. Then the elliptic variational inequality (5.28), i.e.: Find $u \in \mathbb{K}$, such that*

$$(A(u), v - u) + j(v) - j(u) \geq (f, v - u) \quad \textit{for all } v \in \mathbb{K},$$

has a unique solution for any $f \in V$ and the solution is Lipschitz continuously depending on f.

Proof. Firstly, we prove the uniqueness of the solution. Assume that there are two solutions $u_1, u_2 \in \mathbb{K}$ for (5.28). If we choose $v = u_2$ for the first inequality and $v = u_1$ for the second inequality, then the following holds.

$$(A(u_1), u_2 - u_1) + j(u_2) - j(u_1) \geq (f, u_2 - u_1),$$
$$(A(u_2), u_1 - u_2) + j(u_1) - j(u_2) \geq (f, u_1 - u_2).$$

Adding these two inequalities yields

$$-(A(u_1) - A(u_2), u_1 - u_2) \geq 0,$$

or equivalently

$$(A(u_1) - A(u_2), u_1 - u_2) \leq 0. \tag{5.29}$$

Since A is a strongly monotone operator, we deduce from (5.29) that

$$0 \geq (A(u_1) - A(u_2), u_1 - u_2) \geq c_0\|u_1 - u_2\|^2 \geq 0, \tag{5.30}$$

for $c_0 \geq 0$. It follows from (5.30) that

$$\|u_1 - u_2\| = 0,$$

from where we deduce that $u_1 = u_2$, which proves the uniqueness.

In order to prove the existence of a solution, we convert the variational inequality into an equivalent fixed-point problem. For any $\theta > 0$, the equivalent fixed point problem is: Find $u \in \mathbb{K}$, such that

$$(u,v - u) \geq (u, v - u) + \theta\left((f, v - u) - (A(u), v - u) - j(v) + j(u)\right), \tag{5.31}$$

for all $v \in \mathbb{K}$. Problem (5.31) can be equivalently rewritten as: Find $u \in \mathbb{K}$, such that

$$(u, v - u) + \theta j(v) - \theta j(u) \geq (u, v - u) - \theta(A(u), v - u) + \theta(f, v - u), \tag{5.32}$$

for all $v \in \mathbb{K}$. Now, for any $u \in \mathbb{K}$ we consider the problem: Find $w \in \mathbb{K}$, such that

$$(w, v - w) + \theta j(v) - \theta j(w) \geq (u, v - w) - \theta(A(u), v - w) + \theta(f, v - w) \quad \text{for all } v \in \mathbb{K}. \tag{5.33}$$

(5.33) is equivalent to the minimization problem: For a given $u \in \mathbb{K}$, find $w \in \mathbb{K}$, such that

$$E(w) = \inf_{v \in \mathbb{K}} E(v), \tag{5.34}$$

where

$$E(v) = \frac{1}{2}\|v\|^2 + \theta j(v) - (u, v) + \theta(A(u), v) - \theta(f, v).$$

By the auxiliary Lemma 5.45, there exists a continuous functional $l_j \in V^*$ and a constant $c_j > 0$, such that

$$j(v) \geq l_j(v) + c_j \quad \text{for all } v \in V.$$

Applying Theorem 5.31, where

$$\begin{aligned} a(w, v - w) &= (w, v - w), \\ j(w) &= \theta j(w), \\ l(v - w) &= (u, v - w) - \theta(A(u), v - w) + \theta(f, v - w), \end{aligned}$$

we obtain that (5.34), and hence (5.33), has a unique solution $w \in \mathbb{K}$, since property $ii)$ is valid due to the coercivity of a and Lemma 5.45.
We define now for each $\theta > 0$, the mapping $P_\theta : \mathbb{K} \to \mathbb{K}$ by $w = P_\theta u$, where w is the unique solution of (5.33). Obviously, a fixed point of the mapping P_θ is a solution of our problem (5.32). We want to prove, that for sufficiently small $\theta > 0$, $P_\theta : \mathbb{K} \to \mathbb{K}$ is a contraction and, therefore, by the Banach's fixed point theorem 2.14, the inequality (5.32) has a unique solution. For any $u_1, u_2 \in \mathbb{K}$, let $w_1 = P_\theta u_1$ and $w_2 = P_\theta u_2$. Then

$$\begin{aligned} (w_1, w_2 - w_1) + \theta j(w_2) - \theta j(w_1) &\geq (u_1, w_2 - w_1) - \theta(A(u_1), w_2 - w_1) + \theta(f, w_2 - w_1), \\ (w_2, w_1 - w_2) + \theta j(w_1) - \theta j(w_2) &\geq (u_2, w_1 - w_2) - \theta(A(u_2), w_1 - w_2) + \theta(f, w_1 - w_2). \end{aligned} \tag{5.35}$$

Adding the two inequalities in (5.35) and simplifying yields

$$\|w_1 - w_2\|^2 \leq (u_1 - u_2 - \theta(A(u_1) - A(u_2)), w_1 - w_2).$$

By the Cauchy-Schwarz inequality we get

$$\|w_1 - w_2\| \leq \|u_1 - u_2 - \theta(A(u_1) - A(u_2))\|. \tag{5.36}$$

We see that

$$\begin{aligned} &\|u_1 - u_2 - \theta(A(u_1) - A(u_2))\|^2 \\ &= \|u_1 - u_2\|^2 - 2\theta(A(u_1) - A(u_2), u_1 - u_2) + \theta^2\|A(u_1) - A(u_2)\|^2 \\ &\leq (1 - 2c_0\theta + L^2\theta^2)\|u_1 - u_2\|^2, \end{aligned} \tag{5.37}$$

where the last estimate follows from the strong monotonicity with negative sign and the Lipschitz continuity of A. Thus, (5.36) yields with the help of (5.37) that

$$\|P_\theta u_1 - P_\theta u_2\| = \|w_1 - w_2\| \leq (1 - 2c_0\theta + L^2\theta^2)^{\frac{1}{2}}\|u_1 - u_2\|.$$

Hence, the mapping P_θ is a contraction for $\theta \in (0, \frac{2c_0}{L^2})$ and therefore, by Banach's fixed point theorem, P_θ has a unique fixed point, i.e. $u = P_\theta u$. Consequently, (5.31) has a unique fixed point, which means that the equivalent variational inequality (5.28) has a unique solution $u \in \mathbb{K}$.

Lastly, we want to prove the Lipschitz continuous dependency of f. Therefore, let $f_1, f_2 \in V$ and let $u_1, u_2 \in \mathbb{K}$ denote the corresponding solutions of the variational inequality (5.28). Then choosing $v = u_2$ for the first inequality and $v = u_1$ for the second inequality we get

$$\begin{aligned}(A(u_1), u_2 - u_1) + j(u_2) - j(u_1) &\geq (f_1, u_2 - u_1),\\ (A(u_2), u_1 - u_2) + j(u_1) - j(u_2) &\geq (f_2, u_1 - u_2).\end{aligned}$$

Adding these two inequalities yields

$$(A(u_1) - A(u_2), u_1 - u_2) \leq (f_1 - f_2, u_1 - u_2).$$

Using again the strong monotonicity and the Lipschitz continuity of A gives

$$\|u_1 - u_2\| \leq \frac{L}{c_0}\|f_1 - f_2\|,$$

which means that the solution depends Lipschitz continuously on f. □

Remark 5.39. *Following the proof of Theorem 5.38, it can be seen that the assumptions on the operator A can be weakened. Firstly, the strong monotonicity and the Lipschitz continuity need only to be assumed on the subset $\mathbb{K}$ and not over the whole space V. Nevertheless, there is usually a natural extension of A to an operator A_0 on V which fulfills the conditions on the whole space.*

Remark 5.40. *The results about the solution in Theorem 5.38 are valid for elliptic variational inequalities of the form (5.28), i.e. with a special right hand side (f, v) for $f \in V$. By the Riesz representation theorem, c.f. [6, Chapter 4], there exists a unique $f \in V$, such that for any $l \in V^*$*

$$l(v) = (f, v) \quad \textit{for all } v \in V.$$

This makes Theorem 5.38 valid for elliptic variational inequalities of the form

$$(A(u), v - u) + j(v) - j(u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K}. \tag{5.38}$$

In the literature, variational inequalities are grouped into two popular characterizations. We differentiate between elliptic variational inequalities of the first kind and elliptic variational inequalities of the second kind. If the functional $j(v) = 0$ for all $v \in V$, then the variational inequality reduces to a first kind inequality,

$$(A(u), v - u) \geq (f, v - u) \quad \text{for all } v \in \mathbb{K}. \tag{5.39}$$

The obstacle problem (4.60) is an example of elliptic variational inequalities of the first kind. Moreover, they are characterized by the feature that the problem is posed over a convex subset $\mathbb{K}$. In fact, if $\mathbb{K}$ is a subspace of V, then the variational inequality becomes to a variational equality, as we have similarly seen in Theorem 5.27. As a corollary of Theorem 5.38 we have the following result for elliptic variational inequalities of the first kind.

Corollary 5.41. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset and let $A : V \to V$ be strongly monotone and Lipschitz continuous. Then for any $f \in V$, the elliptic variational inequality (5.39) has a unique solution $u \in \mathbb{K}$, which depends Lipschitz continuously on f.*

Remark 5.42. *Corollary 5.41 is a generalization of the Lax-Milgram Lemma.*

The functional j in (5.28) is defined on the subset $\mathbb{K}$. However, we can extend the functional j to V, using the same symbol for its extension. The extension is

$$j(v) = \begin{cases} j(v) & \text{if } v \in \mathbb{K}, \\ +\infty & \text{if } v \in V \setminus \mathbb{K}. \end{cases} \tag{5.40}$$

Furthermore, we need the property of a proper functional.

Definition 5.43. *Let V be a real Hilbert space. We call a functional $j : V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ proper if $j(v) > -\infty$ for all $v \in V$ and $j(v) \neq +\infty$ for at least one $v \in V$.*

Remark 5.44. *The extension $j : V \to \overline{\mathbb{R}}$ is lower semi-continuous if and only if $\mathbb{K} \subset V$ is closed and $j : \mathbb{K} \to \mathbb{R}$ is lower semi-continuous.*

Theorem 5.38 can be now stated without using a convex subset $\mathbb{K}$ explicitly. If we use the extension of j (5.40), then the variational inequality (5.28) changes to an elliptic variational inequality of the second kind: Find $u \in V$, such that

$$(A(u), v - u) + j(v) - j(u) \geq (f, v - u) \quad \text{for all } v \in V. \tag{5.41}$$

We need the following assisting lemma in order to ensure a unique solution for the variational inequality of the second kind (5.41).

Lemma 5.45. *Let V be a normed space and let $j : V \to \overline{\mathbb{R}}$ be a proper, convex and lower semi-continuous functional. Then there exists a continuous linear functional $l_j \in V^*$ and a constant $c_j \in \mathbb{R}$ such that*

$$j(v) \geq l_j(v) + c_j \quad \textit{for all } v \in V.$$

Proof. See [1, Chapter 11] □

Corollary 5.46. *Let $A : V \to V$ be strongly monotone and Lipschitz continuous and let $j : V \to \overline{\mathbb{R}}$ be a proper, convex and lower semi-continuous functional on V. Then for any $f \in V$, the elliptic variational inequality of the second kind (5.41) has a unique solution.*

Remark 5.47. *Usually, the functional j is a non-differential functional and causes difficulties when it comes to solving the variational inequality numerically. As an example, variational inequalities derived from frictional contact problems contain non-differential expressions. We want to give an example about a contact problem with friction at the end of this chapter.*

An important property for variational inequalities is the so called Minty Lemma, which already occurred in the proof of Theorem 5.27. This Lemma can be seen as a symmetry statement about variational inequalities. In addition, the Minty Lemma is very useful for convergence results of numerical solutions, which we will examine in the next chapter.

Theorem 5.48 (Minty Lemma). *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Moreover, let $A : V \to V$ be strongly monotone and Lipschitz continuous and let $j : \mathbb{K} \to \mathbb{R}$ be convex and lower semi-continuous. Then $u \in \mathbb{K}$ is a solution of the variational inequality (5.28) if and only if $u \in \mathbb{K}$ is a solution of*

$$(A(v), v - u) + j(v) - j(u) \geq (f, v - u) \quad \textit{for all } v \in \mathbb{K}. \tag{5.42}$$

Proof. We first assume that $u \in \mathbb{K}$ satisfies (5.28). The monotonicity of A yields

$$(A(u) - A(v), u - v) \geq c_0 \|u - v\|^2 \geq 0,$$

which means that

$$(A(u), u - v) \geq (A(v), u - v),$$

or equivalently

$$(A(v), v - u) \geq (A(u), v - u),$$

for all $v \in \mathbb{K}$. Hence, u satisfies (5.42).

Conversely, assume that $u \in \mathbb{K}$ is a solution of (5.42). For any $v \in \mathbb{K}$ and $t \in (0, 1)$, also $u + t(v - u) \in \mathbb{K}$. Using now $u + t(v - u)$ instead of v in (5.42), we get

$$(A(u + t(v - u)), u + t(v - u) - u) + j(u + t(v - u)) - j(u) \geq (f, u + t(v - u) - u),$$

from where we get

$$t(A(u + t(v - u)), v - u) + j(tv + (1 - t)u) - j(u) \geq t(f, v - u). \tag{5.43}$$

Applying the convexity of the functional j on (5.43), we deduce that

$$t(A(u + t(v - u)), v - u) + tj(v) - tj(u) \geq t(f, v - u),$$

which is equivalent to

$$(A(u + t(v - u)), v - u) + j(v) - j(u) \geq (f, v - u).$$

As $t \to 0$, we obtain (5.28). □

In many applications the operator A corresponds to a bilinear form on the space V, which allows us to write

$$(A(u), v) = a(u, v) \quad \text{for all } u, v \in V.$$

Using the Ritz functional $l \in V^*$, we deduce the following corollary about variational inequalities with bilinear forms as a consequence of Theorem 5.38.

Corollary 5.49. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Moreover, let $a : V \times V \to \mathbb{R}$ be a continuous and V-elliptic bilinear form, let $l \in V^*$ and let $j : \mathbb{K} \to \mathbb{R}$ be a convex and lower semi-continuous functional. Then each of the following elliptic variational inequalities: Find $u \in \mathbb{K}$, such that*

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K}; \tag{5.44}$$

Find $u \in \mathbb{K}$, such that

$$a(u, v - u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K}; \tag{5.45}$$

Find $u \in V$, such that

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \quad \textit{for all } v \in V, \tag{5.46}$$

has a unique solution. In addition, the solution depends Lipschitz continuously on l.

We have already seen that the obstacle problem (4.60) and the simplified Signorini problem (4.44) can be solved uniquely. With the help of Theorem 5.49 the existence and uniqueness of these problems are guaranteed without knowing the corresponding minimization problem explicitly.

5.4 Elliptic hemi-variational inequalities

We have always spoken about elliptic variational inequalities of the form (5.28) so far. In fact, many applications require more general observations of variational inequalities, where one type is called elliptic hemi-variational inequalities. For example, contact problems with friction require the usage of elliptic hemi-variational inequalities. This kind of inequalities allow the functional j to depend explicitly on the solution of the convex subset $\mathbb{K}$. As in the section before, we want to give an statement about a unique solution of elliptic hemi-variational inequalities following the ideas of [6, Chapter 4] and [34].

Remark 5.50. *In the literature, the variational inequality (5.28) is also referred to as elliptic quasi-variational inequality. However, we want to stick to our introduced designation throughout this chapter.*

As before, let V and V^* denote a Hilbert space and its dual space, e.g. $V = H^1(\Omega)$ and $V^* = H^{-1}(\Omega)$, with their respective norms $\|\cdot\|$ and $\|\cdot\|_*$. The set $\mathbb{K}$ is a nonempty, closed and convex subset of V. Furthermore, we consider an operator $A : V \to V^*$ defined on V into the dual space of V^*. Throughout this section, we consider the following elliptic hemi-variational inequality: Find $u \in \mathbb{K}$, such that

$$\langle A(u), v-u\rangle + j(u,v) - j(u,u) \geq l(v-u) \quad \text{for all } v \in \mathbb{K}, \tag{5.47}$$

where $j : V \times V \to (-\infty, +\infty]$ is a functional, $l \in V^*$ and $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$ is the duality product between V^* and V. Note that we mean the duality product $\langle l, v\rangle$ when we write $l(v)$ for $l \in V^*$.

Remark 5.51. *Note that for the variational inequality (5.28) in the previous chapter, the operator $A : V \to V$ is defined on the Hilbert space V and not on its dual space V^*. However, by the Riesz representation theorem, c.f. [6, Chapter 4], there is an isomorphism $R : V^* \to V$, called the Riesz mapping, such that*

$$A = R\tilde{A}$$

where $\tilde{A} : V \to V^*$ is defined on the dual space V^*. Thus, the statements in the previous chapter are also valid for the operator $\tilde{A}$ (with technical adaptions). In the sequel, we use for elliptic variational inequalities of the form (5.28) the operator A defined on V, i.e. $A : V \to V$. For elliptic hemi-variational inequalities (5.47), we use the operator $\tilde{A}$, denoting it by A, i.e. $A : V \to V^*$.

In order to answer the question about a unique solution of elliptic hemi-variational inequalities (5.47) we need to introduce some essential definitions.

Definition 5.52 (Hemi-continuity). *An operator $A : V \to V^*$ is called hemi-continuous if for all $u, v \in V$ and $t \in [0,1]$ the mapping*

$$t \to \langle A((1-t)u + tv), u - v\rangle \tag{5.48}$$

is continuous.

Definition 5.53. *An operator $A : V \to V^*$ is called monotone if*

$$\langle A(u) - A(v), u - v\rangle \geq 0 \quad \textit{for all } u, v \in V.$$

The operator A is called strongly monotone if there exists a constant $c_0 > 0$, such that

$$\langle A(u) - A(v), u - v\rangle \geq c_0\|u-v\|^2 \quad \textit{for all } u, v \in V.$$

Furthermore, we need some essential properties for the functional $j : V \times V \to (-\infty, +\infty]$ in order to guarantee a unique solution of (5.47). Firstly, we assume that there exists a constant $k < c_0$, such that

$$|j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1)| \leq k\|u_1 - u_2\|\|v_1 - v_2\|, \tag{5.49}$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{K}$, where c_0 is the constant coming from the strong monotonicity of A. In addition, we always consider the functional $j(u, \cdot) : V \to (-\infty, +\infty]$ to be proper, convex and lower semi-continuous for every $u \in V$.

Before we state the main theorem about the existence of a unique solution for (5.47), we recall the generalized Weierstrass Minimization Theorem 5.29 and give an adjusted version of it, called the Coercivity Theorem.

Theorem 5.54 (Coercivity Theorem)**.** *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset, let $j : \mathbb{K} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be a proper, lower semi-continuous and convex functional and let $A : V \to V^*$ be a monotone and hemi-continuous operator. If one of the following conditions is satisfied*

$$\mathbb{K} \textit{ is bounded,} \tag{5.50}$$

or, there exists a $v_0 \in \mathbb{K}$, such that

$$\lim_{\substack{\|v\| \to +\infty \\ v \in \mathbb{K}}} \frac{\langle A(v), v - v_0 \rangle + j(v) - j(v_0)}{\|v\|} = +\infty, \tag{5.51}$$

then there exists at least one solution $u \in \mathbb{K}$ of the variational inequality

$$\langle A(u), v - u \rangle + j(v) - j(u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K}. \tag{5.52}$$

In addition, if the functional j is strictly convex, i.e.

$$j(tu + (1-t)v) < tj(u) + (1-t)j(v) \quad \textit{for all } u, v \in V,\ t \in (0,1),\ \textit{and } u \neq v, \tag{5.53}$$

or A is strictly monotone, i.e.

$$\langle A(u) - A(v), v - u \rangle > 0 \quad \textit{for all } u, v \in V,\ \textit{and } u \neq v, \tag{5.54}$$

then the solution of the elliptic variational inequality (5.52) is unique.

Proof. See [6, Chapter 4]. □

As a consequence of the Coercivity Theorem 5.54 the following statement holds.

Lemma 5.55. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset, let $j : \mathbb{K} \to \overline{\mathbb{R}}$ be a proper, lower semi-continuous and convex functional and let $A : V \to V^*$ be a strongly monotone and hemi-continuous operator. Then the elliptic variational inequality (5.52) has a unique solution $u \in \mathbb{K}$.*

Proof. See [6, Chapter 4]. □

Lastly, we need an auxiliary lemma, which is comparable to Lemma 5.45.

Lemma 5.56. *Let V be a normed space and let $j : V \times V \to (-\infty, +\infty]$ be a functional, such that for every $u \in V$, the functional $j(u, \cdot) : V \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Then there exists a continuous linear functional $l_j = l_j(u) \in V^*$ and a constant $c_j \in \mathbb{R}$, such that*

$$j(u, v) \geq l_j(v) + c_j \quad \textit{for all } v \in V.$$

Proof. See [6, Chapter 4] □

Now we are able to state the existence and uniqueness theorem for hemi-variational inequalities of the form (5.47).

Theorem 5.57. *Let V be a real Hilbert space and $\mathbb{K} \subset V$ a nonempty, closed and convex subset. Moreover, let $A : V \to V^*$ be a strongly monotone and hemi-continuous operator and let $j : V \times V \to (-\infty, +\infty]$ be a functional, such that (5.49) is satisfied and for every $u \in V$, the functional $j(u, \cdot) : V \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Then the elliptic hemi-variational inequality (5.47), i.e.: Find $u \in \mathbb{K}$, such that*

$$\langle A(u), v - u\rangle + j(u, v) - j(u, u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K},$$

has a unique solution for any $l \in V^*$, which depends Lipschitz continuously on l.

Proof. Firstly, we prove the uniqueness of the solution $u \in \mathbb{K}$ for the elliptic hemi-variational inequality (5.47). Assume that there exist two different solutions $u_1, u_2 \in \mathbb{K}$ for (5.47). If we choose $v = u_2$ for the first inequality and $v = u_2$ for the second inequality, then the following holds.

$$\begin{aligned} \langle A(u_1), u_2 - u_1\rangle + j(u_1, u_2) - j(u_1, u_1) &\geq l(u_2 - u_1),\\ \langle A(u_2), u_1 - u_2\rangle + j(u_2, u_1) - j(u_2, u_2) &\geq l(u_1 - u_2). \end{aligned} \tag{5.55}$$

Adding the two inequalities in (5.55) yields

$$-\langle A(u_1) - A(u_2), u_1 - u_2\rangle - j(u_1, u_1) - j(u_2, u_2) + j(u_1, u_2) + j(u_2, u_1) \geq 0,$$

which is equivalent to

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle + j(u_1, u_1) + j(u_2, u_2) - j(u_1, u_2) - j(u_2, u_1) \leq 0. \tag{5.56}$$

Using the strong monotonicity of A and property (5.49) of j in (5.56) gives for $c_0 < k$

$$\begin{aligned} 0 &\geq \langle A(u_1) - A(u_2), u_1 - u_2\rangle + j(u_1, u_1) + j(u_2, u_2) - j(u_1, u_2) - j(u_2, u_1)\\ &\geq \langle A(u_1) - A(u_2), u_1 - u_2\rangle - |j(u_1, u_1) + j(u_2, u_2) - j(u_1, u_2) - j(u_2, u_1)|\\ &\geq c_0\|u_1 - u_2\|^2 - k\|u_1 - u_2\|^2 = (c_0 - k)\|u_1 - u_2\|^2 \geq 0. \end{aligned}$$

Hence,

$$0 \geq \|u_1 - u_2\|^2 \geq 0,$$

therefore

$$\|u_1 - u_2\| = 0,$$

holds, which proves the uniqueness of the solution.

Secondly, we prove that there exists a solution for the hemi-variational inequality (5.47) by using the Coercivity Theorem 5.54. We want to verify condition (5.51) of Theorem 5.54. For this purpose, let $v, v_0 \in \mathbb{K}$. Then

$$\begin{aligned} &\frac{\langle A(v), v - v_0\rangle}{\|v\|} = \frac{\langle A(v) - A(v_0) + A(v_0), v - v_0\rangle}{\|v\|} = \frac{\langle A(v) - A(v_0), v - v_0\rangle}{\|v\|} + \frac{\langle A(v_0), v - v_0\rangle}{\|v\|}\\ &\geq \frac{c_0\|v - v_0\|^2}{\|v\|} - \frac{\|A(v_0)\|_*\|v\| + \|A(v_0)\|_*\|v_0\|}{\|v\|}, \end{aligned} \tag{5.57}$$

where the last estimate follows from the strong monotonicity of the operator A and the Cauchy-Schwarz inequality with negative sign. Further, we have for the last expression in (5.57) that

$$\begin{aligned}&\frac{c_0\|v-v_0\|^2}{\|v\|}-\frac{\|A(v_0)\|_*\|v\|+\|A(v_0)\|_*\|v_0\|}{\|v\|}\\&=\frac{c_0\|v\|^2-2c_0\|v\|\|v_0\|+c_0\|v_0\|^2}{\|v\|}-\frac{\|A(v_0)\|_*\|v\|}{\|v\|}+\frac{\|A(v_0)\|_*\|v_0\|}{\|v\|}\\&=c_0\|v\|-2c_0\|v_0\|-\|A(v_0)\|_*+\frac{c_0\|v_0\|^2}{\|v\|}-\frac{\|A(v_0)\|_*\|v_0\|}{\|v\|}.\end{aligned} \tag{5.58}$$

(5.57) and (5.58) together yield

$$\frac{\langle A(v),v-v_0\rangle}{\|v\|}\geq c_0\|v\|-2c_0\|v_0\|-\|A(v_0)\|_*+\frac{c_0\|v_0\|^2}{\|v\|}-\frac{\|A(v_0)\|_*\|v_0\|}{\|v\|}, \tag{5.59}$$

for all $v, v_0 \in \mathbb{K}$. Hence,

$$\lim_{\|v\|\to+\infty}\frac{\langle A(v),v-v_0\rangle}{\|v\|}=+\infty. \tag{5.60}$$

Using now Lemma 5.56 we deduce that there exists a functional $l_j \in V^*$ and a constant $c_j \in \mathbb{R}$, such that for any $w \in \mathbb{K}$

$$j(w,v)\geq l_j(v)+c_j\geq -\|l_j\|_*\|v\|+c_j \quad \text{for all } v\in\mathbb{K}. \tag{5.61}$$

Thus, with the help of (5.60) and (5.61) it follows that

$$\lim_{\|v\|\to+\infty}\frac{\langle A(v),v-v_0\rangle+j(w,v)-j(w,v_0)}{\|v\|}=+\infty,$$

for all $w, v_0 \in \mathbb{K}$. Therefore, the coercivity condition (5.51) of the left hand side in (5.47) holds and by Theorem 5.54 and the fact that $j(u,\cdot)$ is proper, convex and lower semi-continuous, we deduce, that there is at least one solution of the elliptic hemi-variational inequality (5.47).

Lastly, we want to prove the dependency of the solution for any $l \in V^*$. For this reason, assume that u_1 and $u_2 \in \mathbb{K}$ are solutions of (5.47) with the respective right sides l_1 and $l_2 \in V^*$. Then choosing $v = u_2$ for the first inequality and $v = u_1$ for the second inequality we get

$$\begin{aligned}\langle A(u_1),u_2-u_1\rangle+j(u_1,u_2)-j(u_1,u_1)\geq l_1(u_2-u_1),\\ \langle A(u_2),u_1-u_2\rangle+j(u_2,u_1)-j(u_2,u_2)\geq l_2(u_1-u_2).\end{aligned}$$

Adding these two inequalities yields

$$-\langle A(u_1)-A(u_2),u_1-u_2\rangle+j(u_1,u_2)+j(u_2,u_1)-j(u_1,u_1)-j(u_2,u_2)\geq-(l_1-l_2)(u_1-u_2). \tag{5.62}$$

We reorder (5.62) to obtain

$$(l_1-l_2)(u_1-u_2)+j(u_1,u_2)+j(u_2,u_1)-j(u_1,u_1)-j(u_2,u_2)\geq\langle A(u_1)-A(u_2),u_1-u_2\rangle. \tag{5.63}$$

Using now property (5.49) of the functional j, the strong monotonicity of A and the fact that $l \in V^*$, we deduce from (5.63) that

$$\begin{aligned}&\|l_1-l_2\|_*\|u_1-u_2\|+k\|u_1-u_2\|\|u_2-u_1\|\\&\geq(l_1-l_2)(u_1-u_2)+j(u_1,u_2)+j(u_2,u_1)-j(u_1,u_1)-j(u_2,u_2)\\&\geq\langle A(u_1)-A(u_2),u_1-u_2\rangle\geq c_0\|u_1-u_2\|^2,\end{aligned}$$

which finally gives

$$\|l_1 - l_2\|_* + k\|u_1 - u_2\| \geq c_0\|u_1 - u_2\|,$$

or equivalently

$$\|u_1 - u_2\| \leq \frac{1}{c_0 - k}\|l_1 - l_2\|_*,$$

with $1/(c_0 - k) > 0$. This completes the proof. □

Remark 5.58. *Theorem 5.57 is also valid for the restriction of functional j on the convex subset $\mathbb{K}$, i.e. $j : \mathbb{K} \times \mathbb{K} \to (-\infty, +\infty]$. However, further assumptions need to be made to guarantee the existence of a unique solution which are particularly described in [6, Chapter 4].*

If the operator $A : V \to V^*$ can be related to a bilinear form $a : V \times V \to \mathbb{R}$, i.e. $\langle A(u), v\rangle = a(u, v)$ for all $u, v \in V$, then the following statement follows from Theorem 5.57 as a consequence.

Corollary 5.59. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset, let $a : V \times V \to \mathbb{R}$ be a bounded and V-elliptic bilinear form and let $j : V \times V \to (-\infty, +\infty]$ be a functional, such that (5.49) is satisfied and for every $u \in V$, the functional $j(u, \cdot) : V \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Then the elliptic hemi-variational inequality: Find $u \in \mathbb{K}$, such that*

$$a(u, v - u) + j(u, v) - j(u, u) \geq l(v - u) \quad \textit{for all } v \in \mathbb{K},$$

has a unique solution for any $l \in V^*$ which is Lipschitz continuously depended on l.

5.4.1 Application of hemi-variational inequalities: Contact problems with Coulomb friction

Hemi-variational inequalities are commonly used to set up the model for contact problems with friction. In the preceding chapter of this work we have introduced the Signorini problem in the frictionless case (4.25). We have seen that the tangential component of the stress $\sigma_T = 0$, since no friction occurred. In this section we introduce the Signorini problem with so-called Coulomb friction in the static case following the ideas of [9, Chapter 5], [16, Chapter 2], [18, Chapter 10] and [20, Chapter 4]. For this purpose, we consider the contact of two bodies, where one of the bodies is a totally rigid foundation and the other one is elastic, denoted by $\mathcal{F}$ and $\Omega \subset \mathbb{R}^3$ respectively. The boundary of the elastic body is Lipschitz continuous and is divided into three disjoint parts $\partial\Omega = \Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_F \cup \overline{\Gamma}_C$, where Γ_D is assumed to be nonempty. As well as the Signorini problem in the frictionless case, the body $\Omega \subset \mathbb{R}^3$ is fixed at the Dirichlet boundary Γ_D, i.e. $u = 0$ on Γ_D, and tractions are only applied on Γ_F, i.e. $\sigma(u) \cdot n = \vec{t}$. In addition, we impose Coulomb's friction law on the contact boundary Γ_C. However, we need to introduce the definition of friction and we must identify how friction acts on the surface between the bodies.

In a mechanical point of view, it is important to distinguish between the contact of two dry solid surfaces and the contact of other, not necessarily dry, materials. We take the first case into account, i.e. dry friction, and give the physical and mathematical definition in a short way. In general, the occurrence of friction between two bodies is a highly complicated physical phenomenon. Nevertheless, dry friction between two solid bodies can be described by Coulomb's law of friction. For a complete physical description of Coulomb friction we recommend [25, Chapter 10]. Coulomb investigated that dry frictional force F_k between two bodies which are pressed together with normal force F_n is proportional to this normal force F_n

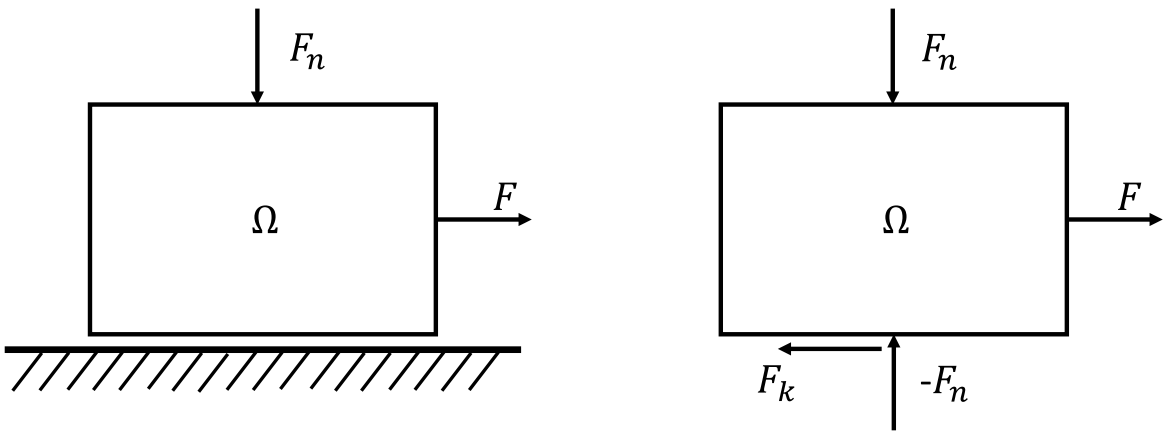


Figure 5.1: An example of a body Ω pulled with force F along a rigid plane and has normal force F_n. The friction F_k acts in the opposite direction of F.

and practically independent from the speed of slippage. Basically, Coulomb friction indicates two kinds of friction, static friction F_s on one hand and kinetic friction F_k on the other hand. Whereas the static friction must be overcome to set a body already touching another rigid solid in motion , the kinetic friction acts resistingly on the body in motion, when the static friction has been overcome. Figure 5.1 captures how the friction acts resistingly, i.e. in the opposite direction of the force with which the body is pulled. Coulomb's law states that both, the static as well as the kinetic friction, are proportional to the normal force with which the bodies are pressed together. Mathematically, this can be described as the following.

Definition 5.60. *There exist two constants $\nu_s, \nu_k > 0$, such that*

$$F_s = \nu_s F_n,$$
$$F_k = \nu_k F_n,$$

where F_s and F_k are the static and kinetic frictional forces respectively, and F_n is the normal force with which the two bodies are pressed together. The constants ν_s and ν_k are the so-called static and kinetic friction coefficients, respectively.

Additionally, Coulomb's law of friction maintains that the kinetic friction coefficient is approximately equal to the static friction coefficient, i.e.

$$\nu_s \approx \nu_k.$$

Thus, it is convenient not to distinguish between the different friction coefficients and to keep only one notation,

$$F_k = \nu_F F_n. \tag{5.64}$$

Further assumptions following from Coulomb's law of friction are that the kinetic friction has no extensive dependence on the contact area or roughness of the surface and that the kinetic friction is practically independent on the sliding velocity. Taking now Coulomb's law of friction into account, we are in the position to impose the contact conditions on Γ_C for the Signorini problem with Coulomb friction.

Contrary to the Signorini problem in the frictionless case, the tangential component of the stress σ_T does not vanish. Considering now Coulomb's friction, the following cases can occur on the contact boundary Γ_C.

1. The elastic body is not in contact with rigid foundation, i.e. $u_n < g$ on Γ_C, then
$$\sigma_n(u) = 0 \quad \text{and} \quad \sigma_T(u) = 0 \qquad \text{on } \Gamma_C. \tag{5.65}$$
2. The elastic body is in touch with the rigid foundation, i.e. $u_n = g$ on Γ_C, then
$$\sigma_n(u) \leq 0 \qquad \text{on } \Gamma_C, \tag{5.66}$$
as we have derived in (4.11). If the magnitude of the tangential stress is below the critical value $\nu_F|\sigma_n(u)|$, then there is not enough force to overcome the static friction and no sliding happens, which means
$$u_T = 0 \quad \text{if } |\sigma_T(u)| < \nu_F|\sigma_n(u)|, \tag{5.67}$$
where $u_T = u - u_n n$ is the tangential component of the displacement on Γ_C. If the tangential force overcomes the static friction, meaning that the critical value is reached, then sliding is developed from the kinetic friction in the opposite direction of σ_T, i.e. there exits $\lambda \geq 0$, such that
$$u_T = -\lambda\sigma_T(u) \quad \text{if } |\sigma_T(u)| = \nu_F|\sigma_n(u)|. \tag{5.68}$$
It is possible to combine the contact conditions (5.67) and (5.68) to receive
$$|\sigma_T(u)| < \nu_F|\sigma_n(u)|, \quad (\nu_F|\sigma_n(u)| - |\sigma_T(u)|)\, u_T = 0. \tag{5.69}$$

We recall the equilibrium equation (3.32), the constitutive law (3.35) for linearized strain and the linearized kinematical contact condition (4.24) considering the frictional force to complete the classical form of the Signorini problem with Coulomb friction. Find the displacement u, such that

$$-\frac{\sigma_{ij}(u)}{\partial x_j} = f_i \qquad \text{in } \Omega, \tag{5.70a}$$
$$\sigma_{ij}(u) = \sum_{k,l=1}^{3} C_{ijkl}\epsilon_{ij}(u) \qquad \text{in } \Omega, \tag{5.70b}$$
$$u_i = 0 \qquad \text{on } \Gamma_D, \tag{5.70c}$$
$$\sigma_{ij}(u)n_j = t_i \qquad \text{on } \Gamma_F, \tag{5.70d}$$
$$\left.\begin{array}{ll} \sigma_n(u) = 0 \text{ and } \sigma_T(u) = 0 & \text{if } u_n < g \\ \sigma_n(u) \leq 0 & \text{if } u_n = g \\ u_T = 0 & \text{if } u_n = g \text{ and } |\sigma_T(u)| < \nu_F|\sigma_n(u)| \\ u_T = -\lambda\sigma_T(u) & \text{if } u_n = g \text{ and } |\sigma_T(u)| = \nu_F|\sigma_n(u)| \end{array}\right\} \text{ on } \Gamma_C, \tag{5.70e}$$

where

$$\begin{aligned} u_n &= u \cdot n, \\ u_T &= u - u_n n, \\ \sigma_n(u) &= \sigma_{ij}(u)n_i n_j, \\ \sigma_T(u) &= \sigma(u) \cdot n - \sigma_n(u)n, \\ \nu_F &\in L_\infty(\Gamma_C), \\ \lambda &\geq 0, \\ C_{ijkl} &\in L_\infty(\Omega), \\ C_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} &\geq \alpha\epsilon_{ij}\epsilon_{ij} \quad \text{for an } \alpha > 0. \end{aligned}$$

The classical form of the Signorini problem with Coulomb friction can be characterized by a hemi-variational inequality. We consider the following setting.

$$\begin{aligned}
&\Omega \in \mathbb{R}^3 \text{ and } \partial\Omega \text{ is sufficiently smooth,} \\
&V = \{v \in [H^1(\Omega)]^3 \mid v = 0 \text{ on } \Gamma_D\}, \\
&\mathbb{K} = \{v \in V \mid v_n - g \le 0 \text{ on } \Gamma_C,\ v_n - g \in H^{1/2}(\Gamma_C)\}, \\
&\quad f_i \in L_2(\Omega), \quad t_i \in L_2(\Gamma_F), \qquad \text{for } i = 1, 2, 3.
\end{aligned} \tag{5.71}$$

Indeed the set of the admissible displacement $\mathbb{K}$ is a nonempty, closed and convex subset of V. Moreover, we define the bilinear form $a : V \times V \to \mathbb{R}$, the functional $l \in V^*$ and the nonlinear functional $j : V \times V \to \mathbb{R}$ as follows.

$$a(u, v) = \int_\Omega \sigma(u) : \epsilon(v)\ dx, \tag{5.72}$$

$$l(v) = \int_\Omega f \cdot v\ dx + \int_{\Gamma_F} \vec{t} \cdot v\ ds, \tag{5.73}$$

$$j(u, v) = \int_{\Gamma_C} \nu_F |\sigma_n(u)||v_T|\ ds, \tag{5.74}$$

$$\tag{5.75}$$

for all $u, v \in \mathbb{K}$.

The variational form of the Signorini problem with Coulomb friction reads as: Find the displacement $u \in \mathbb{K}$, such that

$$a(u, v - u) + j(u, v) - j(u, u) \ge l(v - u) \quad \text{for all } v \in \mathbb{K}. \tag{5.76}$$

Theorem 5.61. *Let $u \in \mathbb{K}$ be a sufficiently smooth solution. If u is a solution of (5.76), then it also solves (5.70) and conversely.*

Proof. See [18, Chapter 10]. □

Unfortunately, the hemi-variational inequality (5.76) involves a lot of difficulties when it comes to examine a unique solution. For example, the functional $j(u, u)$ is non-convex and non-differentiable which makes Theorem 5.57, hence Corollary 5.59, not applicable for the Signorini problem with Coulomb friction, see [18, Chapter 10]. Furthermore, if $u \in V = [H^1(\Omega)]^3$ satisfies (5.70a) for $f \in [L_2(\Omega)]^3$, then $\sigma_n(u)$ is only a distribution and $|\sigma_n(u)|$ has no mathematical meaning on the boundary Γ_C. In fact, it is known that $\sigma_n(u) \in H^{-1/2}(\Gamma_C)$, hence $|\sigma_n(u)|$ has no mathematical meaning on Γ_C. However, there are some existence results for very specific problems given in [46]. Another approach for determining a unique solution of (5.76) is to simplify or rather reduce the hemi-variational inequality. This attempt is presented in the works of [1, Chapter 11], [6, Chapter 4], [13, Chapter 2] and [18, Chapter 10].

Remark 5.62. *A possibility to overcome the difficulties of the functional j, i.e. the difficulties on the boundary Γ_C, is presented in [6, Chapter 8]. For this purpose, we suppose that $u \in H^2(\Omega)$ and that there exists a regularization $\sigma_n^*(u)$ for $\sigma_n(u)$, such that*

$$\sigma_n^*(u) \in L_2(\Gamma_C), \tag{5.77}$$

and

$$\|\sigma_n^*(w) - \sigma_n^*(v)\|_{L_2(\Gamma_C)} \le c\|w - v\| \quad \text{for all } w, v \in V, \tag{5.78}$$

where $c > 0$. The functional defined by the regularization $\sigma_n^*(u)$ is denoted by

$$j^*(u,v) = \int\limits_{\Gamma_C} \nu_F |\sigma_n^*(u)||v_T| \; ds. \tag{5.79}$$

In addition, for every $u \in V$, the functional $j^*(u,\cdot) : V \to (-\infty,\infty]$ is proper, convex and lower semi-continuous. Then with the help of the regularization (5.77), we are able to apply Theorem 5.57 on the following regularized Signorini problem with friction.

Proposition 5.63. *Let the assumptions in (5.71) hold and let $C_{ijkl} \in L_\infty(\Omega)$ in (5.70b), such that $C_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq \alpha\epsilon_{ij}\epsilon_{ij}$ for an $\alpha > 0$. In addition, let $\nu_F \in L_\infty(\Gamma_C)$, $\nu_F \geq 0$ on Γ_C and (5.77) hold. Then there exists $\nu_1 > 0$, such that for any $\nu_F \in L_\infty(\Gamma_C)$ with $\nu_F \geq 0$ on Γ_C and $\|\nu_F\|_{L_\infty(\Gamma_C)} \leq \nu_1$, the problem: Find $u \in \mathbb{K}$, such that*

$$a(u, v-u) + j^*(u,v) - j^*(u,u) \geq l(v-u) \quad \textit{for all } v \in \mathbb{K}, \tag{5.80}$$

has a unique solution.

Proof. We apply Corollary 5.59 in order to obtain the existence of a unique solution. Indeed, the subset $\mathbb{K}$ is convex, closed and nonempty. Also, the bilinear form a is bounded and V-elliptic, c.f. [4, Chapter 3] or [50], and $l \in V^*$. The functional j^* satisfies for every $u \in V$, that $j^*(u,\cdot) : V \to (-\infty,\infty]$ is proper, convex and lower semi-continuous by our assumption. It only remains to show property (5.49) for j^*. For this reason, let $u_1, u_2, v_1, v_2 \in V$ and consider

$$\begin{aligned} &|j^*(u_1,v_1) + j^*(u_2,v_2) - j^*(u_1,v_2) - j^*(u_2,v_1)| \\ =& \left| \int\limits_{\Gamma_C} \nu_F \left[(|\sigma_n^*(u_1)| - |\sigma_n^*(u_2)|)(|v_{1T}| - |v_{2T}|) \right] \; ds \right| \\ \leq& \int\limits_{\Gamma_C} \nu_F |\sigma_n^*(u_1) - \sigma_n^*(u_2)||v_{1T} - v_{2T}| \; ds, \end{aligned} \tag{5.81}$$

where we used the triangle inequality in the last step. Since (5.77) holds, we can use the Cauchy-Schwarz inequality in $L_2(\Gamma_C)$ to obtain from (5.81) that

$$\begin{aligned} &\left| \int\limits_{\Gamma_C} \nu_F |\sigma_n^*(u_1) - \sigma_n^*(u_2)||v_{1T} - v_{2T}| \; ds \right| \\ &\leq \|\nu_F\|_{L_\infty(\Gamma_C)} \; \|\sigma_n^*(u_1) - \sigma_n^*(u_2)\|_{L_2(\Gamma_C)} \; \|v_{1T} - v_{2T}\|_{[L_2(\Gamma_C)]^3}. \end{aligned} \tag{5.82}$$

Using now (5.78) and the fact that the trace operator from $V = [H^1(\Omega)]^3$ into $[L_2(\Gamma_C)]^3$ is continuous, we deduce from (5.82) that there exists a constant $C_2 = C_2(\Omega, \Gamma_C)$, such that

$$\begin{aligned} &\|\nu_F\|_{L_\infty(\Gamma_C)} \; \|\sigma_n^*(u_1) - \sigma_n^*(u_2)\|_{L_2(\Gamma_C)} \; \|v_{1T} - v_{2T}\|_{[L_2(\Gamma_C)]^3} \\ &\leq C_2 \|\nu_F\|_{L_\infty(\Gamma_C)} \; \|u_1 - u_2\| \; \|v_1 - v_2\|, \end{aligned} \tag{5.83}$$

for all $u_1, u_2, v_1, v_2 \in V$. Hence, (5.49) is satisfied for $k = C_2\|\nu_F\|_{L_\infty(\Gamma_C)}$. Therefore, if we choose ν_1, such that

$$0 < \nu_1 < \frac{\alpha}{C_2}, \tag{5.84}$$

we obtain that for any $\nu_F \in L_\infty(\Gamma_C)$ with $\nu_F \geq 0$ on Γ_C and $\|\nu_F\|_{L_\infty(\Gamma_C)} \leq \nu_1$, we have $k < \alpha$. Hence, we can use Corollary 5.59 and it follows that (5.80) has a unique solution. □

Remark 5.64. *In [6, Subsection 4.2.3] a similar contact problem to the Signorini problem with Coulomb friction (5.70) is given. It is investigated by hemi-variational inequalities with potential operators, which is an other observation of elliptic hemi-variational inequalities involving the definition of a generalized solution. For the readers interest, the description of variational inequalities with differential operators is given in [6, Chapter 4].*

Chapter 6

Finite Element Discretization

In this chapter, we study the numerical approximations of elliptic variational inequalities (5.38) and elliptic hemi-variational inequalities (5.47) using the Finite Element discretization. In the past, many different Finite Element techniques have bean developed for solving variational inequalities in terms of formulating the problem as saddle point problems or penalization formulations, which are described in [18, Chapter 4] precisely. Other methods, such as regularization techniques or optimization methods, are introduced in [1] or [24], respectively. We start this chapter with the discretization of the variational inequality (5.38) and give some convergence results following the ideas of [1, Chapter 11] and [18, Chapter 4]. The second part of this chapter describes the discretization and the convergence analysis for the elliptic hemi-variational inequality (5.47) as given in [6, Chapter 7].

6.1 Discretization of elliptic variational inequalities

In this section, we consider the discretization of the variational inequality (5.38) and we want to give some convergence results following the ideas of [1, Chapter 11], [6, Chapter 7] and [18, Chapter 4]. Recalling (5.38), we have: Find $u \in \mathbb{K}$, such that

$$(A(u), v - u) + j(v) - j(u) \geq l(v - u) \quad \text{for all } v \in \mathbb{K}. \tag{6.1}$$

We assume throughout this section that

- V is a real Hilbert space with norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$, e.g. $V = H^1(\Omega)$, and $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the inner product in V,
- $\mathbb{K} \subset V$ is a nonempty, closed and convex subset of V,
- $A : V \to V$ is strongly monotone and Lipschitz continuous,
- $j : \mathbb{K} \to \mathbb{R}$ is a convex and lower semi-continuous functional,
- $l \in V^*$.

With this assumptions the elliptic variational inequality (6.1) has a unique solution $u \in \mathbb{K}$ provided by Theorem 5.38.

Before we examine the discretization of (6.1), we want to mention at this point, that the functional $j : \mathbb{K} \to \mathbb{R}$ can be seen as a restriction of another functional $j_0 : V \to (-\infty, +\infty]$ on $\mathbb{K}$, where j_0 is also a convex and l.s.c. functional, i.e.

$$j_0(v) = \begin{cases} j(v) & v \in \mathbb{K}, \\ +\infty & v \notin \mathbb{K}. \end{cases} \tag{6.2}$$

This restriction is usually naturally satisfied. We will use this assumption on the functional j throughout this section. In order to formulate the discrete problem of (6.1), we need to choose a proper finite dimensional subspace $V_h \subset V$. The subset $\mathbb{K}_h \subset V_h$ is a nonempty, closed and convex subset of V_h. The finite element approximation of the problem is: Find $u_h \in \mathbb{K}_h$, such that

$$(A(u_h), v_h - u_h) + j_0(v_h) - j_0(u_h) \geq l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.3}$$

Applying Theorem 5.38 to (6.3) under the assumptions on the discrete data, we deduce that the discrete elliptic variational inequality (6.3) has a unique solution.

The next theorem is a general convergence statement about the discrete solution. However, the convergence rate is not provided by this theorem.

Theorem 6.1. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Moreover, let $A : V \to V$ be strongly monotone and Lipschitz continuous and let $j : \mathbb{K} \to \mathbb{R}$ be a convex and lower semi-continuous restriction of $j_0 : V \to (-\infty, +\infty]$, where j_0 is a continuous, convex and lower semi-continuous functional. Further, we say that $\{\mathbb{K}_h\}_h$ approximates the set $\mathbb{K}$ in the following sense:*

i) For any $v \in \mathbb{K}$ and for every h, there exists a $v_h \in \mathbb{K}_h$, such that $\|v_h - v\|_V \to 0$ as $h \to 0$, i.e. strong convergence in V.

ii) For every sequence $\{v_h\}_h \subset \mathbb{K}_h$ weakly convergent to $v \in V$, i.e. $v_h \rightharpoonup v$ in V as $h \to 0$, we have $v \in \mathbb{K}$.

Then we have the convergence $\|u - u_h\|_V \to 0$ as $h \to 0$, where u and u_h are the solutions of (6.1) and (6.3), respectively.

Proof. The proof is divided into three steps. First, we want to show the boundedness of the set $\{u_h\}_h$ in V. We fix a $v_0 \in \mathbb{K}$ and choose $v_{0,h} \in \mathbb{K}_h$, such that $v_{0,h} \to v_0$ in V as $h \to 0$ as in assumption i). Let $v_h = v_{0,h}$ in (6.3), then

$$\begin{aligned}(A(u_h) - A(v_{0,h}), u_h - v_{0,h}) \leq{} & (A(v_{0,h}) - A(v_0), v_{0,h} - u_h) \\ & + (A(v_0), v_{0,h} - u_h) + j_0(v_{0,h}) - j_0(u_h) - l(v_{0,h} - u_h).\end{aligned} \tag{6.4}$$

We use the strong monotonicity of A in the left hand side of the inequality (6.4) to obtain

$$c_0 \|u_h - v_{0,h}\|^2 \leq (A(u_h) - A(v_{0,h}), u_h - v_{0,h}). \tag{6.5}$$

If we use the Cauchy-Schwarz inequality and apply Lemma 5.45 on $-j_0(u_h)$, i.e. $-j_0(u_h) \leq \|l_{j_0}\|_* \|u_h\| + |c_{j_0}|$, in the right hand side of (6.4), we get that

$$\begin{aligned} & (A(v_{0,h}) - A(v_0), v_{0,h} - u_h) + (A(v_0), v_{0,h} - u_h) \\ & \qquad + j_0(v_{0,h}) - j_0(u_h) - l(v_{0,h} - u_h) + [j_0(v_0) - j_0(v_0)] \\ & \leq (\|A(v_{0,h}) - A(v_0)\| + \|A(v_0)\| + \|l\|_*) \|u_h - v_{0,h}\| + |j_0(v_{0,h}) - j_0(v_0)| + j_0(v_0) \\ & \qquad + |c_{j_0}| + \|l_{j_0}\|_* \|u_h - v_{0,h} + v_{0,h} - v_0 + v_0\| \\ & \leq (\|A(v_{0,h}) - A(v_0)\| + \|A(v_0)\| + \|l\|_* + \|l_{j_0}\|_*) \|u_h - v_{0,h}\| \\ & \qquad + |j_0(v_{0,h}) - j_0(v_0)| + j_0(v_0) + |c_{j_0}| + \|l_{j_0}\|_* (\|v_{0,h} - v_0\| + \|v_0\|), \end{aligned} \tag{6.6}$$

where the last estimate follows from the triangle inequality. Property (6.5) and (6.6) together give

$$\begin{aligned} c_0 \|u_h - v_{0,h}\|^2 \leq{} & (\|A(v_{0,h}) - A(v_0)\| + \|A(v_0)\| + \|l\|_* + \|l_{j_0}\|_*) \|u_h - v_{0,h}\| \\ & + |j_0(v_{0,h}) - j_0(v_0)| + j_0(v_0) + |c_{j_0}| + \|l_{j_0}\|_* (\|v_{0,h} - v_0\| + \|v_0\|). \end{aligned} \tag{6.7}$$

As $h \to 0$, since $\|v_{0,h} - v_0\| \to 0$, we know that $\|A(v_{0,h}) - A(v_0)\| \to 0$ and $|j_0(v_{0,h}) - j_0(v_0)| \to 0$ due to the Lipschitz continuity of A and the continuity of j_0, respectively. So, additionally with the help of Lemma 5.45, $\{\|u_h - v_{0,h}\|\}_h$ is bounded and, therefore, with the choice $v_0 = 0$, $\{\|u_h\|\}_h$ is bounded. Thus, for a subsequence of a bounded sequence $\{u_h\}_h$, still denoted by $\{u_h\}_h$, and for some $w \in V$, we have the weak convergence

$$u_h \rightharpoonup w \quad \text{in } V,$$

since bounded sequences in Hilbert spaces have weakly convergent subsequences, c.f. [17, Chapter 21]. By assumption $ii)$ we get that $w \in \mathbb{K}$.

Secondly, we want to prove that the weak limit w is the solution of the problem (6.1). From the Minty Lemma 5.48, which is also valid for the discrete formulation (6.3), we deduce that the discrete problem (6.3) is equivalent to: Find $u_h \in \mathbb{K}_h$, such that

$$(A(v_h), v_h - u_h) + j_0(v_h) - j_0(u_h) \geq l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.8}$$

By assumption $i)$ we can choose for any fixed $v \in \mathbb{K}$, a $v_h \in \mathbb{K}_h$ such that $v_h \to v$ in V as $h \to 0$. Then

$$\begin{aligned} A(v_h) &\to A(v), \\ (A(v_h), v_h - u_h) &\to (A(v), v - w), \\ j_0(v_h) &\to j_0(v), \\ l(v_h - u_h) &\to l(v - w), \end{aligned} \tag{6.9}$$

as $h \to 0$, which follows from the continuity in every line. From the lower semi-continuity of j_0, we know that

$$j_0(w) \leq \liminf_{h \to 0} j_0(u_h).$$

Thus, taking the limit $h \to 0$ in (6.8) and using the fact that $j_0 = j$ on $\mathbb{K}$, we get the problem: Find $w \in \mathbb{K}$, such that

$$(A(v), v - w) + j(v) - j(w) \geq l(v - w) \quad \text{for all } v \in \mathbb{K}.$$

Applying the Minty Lemma again, we see that $w \in \mathbb{K}$ is the solution of the elliptic variational inequality (6.1). Since (6.1) has a unique solution u, we conclude that $w = u$.

Lastly, we want to prove that u_h converges strongly to u. Therefore we choose $\tilde{u}_h \in \mathbb{K}_h$, such that $\tilde{u}_h \to u$ in V as $h \to 0$. By the strong monotonicity of A, we get

$$\begin{aligned} c_0\|u - u_h\|^2 &\leq (A(u) - A(u_h), u - u_h) \\ &= (A(u), u - u_h) - (A(u_h), \tilde{u}_h - u_h) - (A(u_h), u - \tilde{u}_h). \end{aligned} \tag{6.10}$$

It follows by (6.3) that

$$-(A(u_h), \tilde{u}_h - u_h) \leq j_0(\tilde{u}_h) - j_0(u_h) - l(\tilde{u}_h - u_h),$$

which can be used in (6.10) to obtain

$$c_0\|u - u_h\|^2 \leq (A(u), u - u_h) + j_0(\tilde{u}_h) - j_0(u_h) - l(\tilde{u}_h - u_h) - (A(u_h), u - \tilde{u}_h). \tag{6.11}$$

Now as $h \to 0$, we have that

$$(A(u), u - u_h) \leq \|A(u)\|\|u - u_h\| \to 0 \tag{6.12}$$

by assumption i),

$$l(\tilde{u}_h - u_h) = l(\tilde{u}_h - u + u - u_h) \leq \|l\|_* \|\tilde{u}_h - u\| \|u - u_h\| \to 0 \tag{6.13}$$

by assumption i) and $\tilde{u}_h \to u$, and

$$(A(u_h), u - \tilde{u}_h) \leq \|A(u_h)\| \|u - \tilde{u}_h\| \to 0 \tag{6.14}$$

because $\tilde{u}_h \to u$. Moreover, we have

$$\begin{aligned} \lim_{h \to 0} j_0(\tilde{u}_h) &= j_0(u), \\ \limsup_{h \to 0} (-j_0(u_h)) &\leq -j_0(u). \end{aligned} \tag{6.15}$$

Thus, we get from (6.11) with the help of (6.12) - (6.15) that

$$\limsup_{h \to 0} c_0 \|u - u_h\|^2 \leq 0,$$

and therefore u_h converges strongly to u as $h \to 0$. □

Remark 6.2. *Note that the latter theorem is a general convergence result based on strong assumptions. There are no statements about any convergence rates yet.*

Generally, a standard error estimation can be derived by Céa's Lemma. Our next goal is to find a generalized and adapted Theorem of Céa's Lemma for elliptic variational inequalities. For this purpose, we will introduce the concept of a certain error bound.

Definition 6.3. *The error bound $R : V \times V \to \mathbb{R}$ of the variational inequality (6.1) with solution $u \in \mathbb{K}$ is defined as*

$$R(v, w) = (A(u), v - w) + j_0(v) - j_0(w) - l(v - w). \tag{6.16}$$

We are now in the position to give an error estimate for the solutions of the continuous problem (6.1) and the discrete problem (6.3).

Theorem 6.4. *The error can be estimated as*

$$\frac{c_0}{2} \|u - u_h\|^2 \leq \inf_{v \in \mathbb{K}} R(v, u_h) + \inf_{v_h \in \mathbb{K}_h} \left[R(v_h, u) + \frac{L^2}{2c_0} \|u - v_h\|^2 \right], \tag{6.17}$$

where u and u_h are the solutions of (6.1) and (6.3), and c_0 and L are the constants of the strong monotonicity and Lipschitz continuity of A, respectively.

Proof. We consider our variational inequality

$$0 \leq (A(u), v - u) + j_0(v) - j_0(u) - l(v - u) \quad \text{for all } v \in \mathbb{K},$$

and the discrete variational inequality

$$0 \leq (A(u_h), v_h - u_h) + j_0(v_h) - j_0(u_h) - l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h.$$

Furthermore, we use the strong monotonicity of A to obtain

$$c_0 \|u - u_h\|^2 \leq (A(u) - A(u_h), u - u_h). \tag{6.18}$$

Adding now the general and discrete variational inequality on the right hand side of (6.18) and using the linearity of the inner product and the functional l, we get

$$
\begin{aligned}
c_0\|u-u_h\|^2 &\le (A(u)-A(u_h),u-u_h)+(A(u),v-u)+j_0(v)-j_0(u)-l(v-u)\\
&\quad +(A(u_h),v_h-u_h)+j_0(v_h)-j_0(u_h)-l(v_h-u_h)\\
&=(A(u),u-u_h)-(A(u_h),u-u_h)+(A(u),v-u)+(A(u_h),v_h-u_h)\\
&\quad +j_0(v)-j_0(u_h)-l(v)+l(u_h)+j_0(v_h)-j_0(u)-l(v_h)+l(u)\\
&=(A(u),v-u_h)+(A(u_h),v_h-u)+j_0(v)-j_0(u_h)-l(v-u_h)\\
&\quad +j_0(v_h)-j_0(u)-l(v_h-u)\\
&=R(v,u_h)+(A(u_h)-A(u)+A(u),v_h-u)+j_0(v_h)-j_0(u)-l(v_h-u)\\
&=R(v,u_h)+R(v_h,u)+(A(u_h)-A(u),v_h-u).
\end{aligned}
\tag{6.19}
$$

The last term of (6.19) can be bounded by the Cauchy-Schwarz inequality and Lipschitz continuity of the operator A, which gives

$$(A(u_h)-A(u),v_h-u)\le L\|u_h-u\|\|u-v_h\|.$$

Using Young's inequality $ab\le\frac{a^2}{2}+\frac{b^2}{2}$, yields

$$L\frac{\sqrt{c_0}}{\sqrt{c_0}}\|u_h-u\|\|u-v_h\|\le\frac{c_0}{2}\|u_h-u\|^2+\frac{L^2}{2c_0}\|u-v_h\|^2,$$

which gives in (6.19) the upper estimate

$$\frac{c_0}{2}\|u_h-u\|^2\le R(v,u_h)+R(v_h,u)+\frac{L^2}{2c_0}\|u-v_h\|^2.$$

Since this estimate is valid for every $v\in\mathbb{K}$ and for every $v_h\in\mathbb{K}_h$, it is also valid for their infima, which finally yields (6.17). □

Remark 6.5. *Note that the error estimate of Theorem 6.4 can be rewritten as*

$$\|u-u_h\|^2\le c\left\{\inf_{v\in\mathbb{K}}R(v,u_h)+\inf_{v_h\in\mathbb{K}_h}\left[R(v_h,u)+\|u-v_h\|^2\right]\right\},\tag{6.20}$$

where $c=\max\{2/c_0,\max\{1,L^2/c_0\}\}$.

Remark 6.6. *In many applications,* $\mathbb{K}_h\not\subset\mathbb{K}$*. Hence, it is essential to consider the infima of the error bound (6.17) on* $\mathbb{K}$ *and* $\mathbb{K}_h$ *explicitly. In the case that* $\mathbb{K}_h\subset\mathbb{K}$*, the first term* $\inf_{v\in\mathbb{K}}R(v,u_h)$ *of (6.17) vanishes and the general Lemma of Céa for variational inequalities reduces to*

$$\|u-u_h\|\le c\inf_{v_h\in\mathbb{K}_h}\left[\|u-v_h\|+|R(v_h,u)|^{1/2}\right],\tag{6.21}$$

which is also known as the internal approximation of the elliptic variational inequality.

Remark 6.7. *Note that for variational equalities, we have*

$$R(v,w)=0\quad\text{for all } v,w\in V.$$

Hence, the Lemma of Céa for elliptic variational equalities

$$\|u-u_h\|\le c\inf_{v_h\in V_h}\|u-v_h\|,$$

can be obtained.

Remark 6.8. *In many problems the operator A can be associated with a bilinear form $a(\cdot,\cdot)$ on V, i.e. $(A(u), v) = a(u, v)$. The variational inequality* (6.1) *becomes to: Find $u \in \mathbb{K}$, such that*

$$a(u, v - u) + j(v) - j(u) \geq l(v - u) \qquad \text{for all } v \in \mathbb{K}, \tag{6.22}$$

and the discrete variational inequality (6.3) *becomes to: Find $u_h \in \mathbb{K}_h$, such that*

$$a(u_h, v_h - u_h) + j_0(v_h) - j_0(u_h) \geq l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.23}$$

The error bound (6.16) changes to

$$R(v, w) = a(u, v - w) + j_0(v) - j_0(w) - l(v - w). \tag{6.24}$$

We want to apply now Theorem 6.4 to the obstacle problem (4.60) and to the simplified Signorini problem (4.44). Furthermore, we will observe for these problems how the errors in the L_2-norm and H^1-norm depend on the mesh size h. We start with the obstacle problem.

Example 6.9. We assume that Ω is a polygonial domain and $u, \psi \in H^2(\Omega)$, where ψ denotes the obstacle function. Further, we assume that Ω is divided in a shape regular triangulation $\mathcal{T}_h$, c.f. [4, Chapter 2], and we use linear elements on the mesh of triangles, i.e. P_1-elements, where $P_1(T) = \{a_0 + a_1 x \mid x \in T\}$ for the triangle $T \in \mathcal{T}_h$. The discrete admissible set is

$$\mathbb{K}_h = \{v_h \in H_0^1(\Omega) \mid v_h \text{ is piecewise linear, } v_h(x_{node}) \geq \psi(x_{node}) \text{ for any node } x_{node} \text{ of } \mathcal{T}_h\}.$$

As we have remarked before, $\mathbb{K}_h$ need not to be a subset of $\mathbb{K}$. For any $u \in H^2(\Omega)$ and for any $v, w \in H_0^1(\Omega)$, the error bound of the obstacle problem is

$$R(v, w) = \int_\Omega [\nabla u \cdot \nabla(v - w) - f(v - w)] \; dx. \tag{6.25}$$

Using integration by parts for the first expression in (6.25) we receive

$$R(v, w) = \int_\Omega - \operatorname{div}(\nabla u)(v - w) \; dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(v - w) \; ds - \int_\Omega f(v - w) \; dx.$$

Since $\operatorname{div} \nabla = \Delta$ and $v, w = 0$ on the boundary $\partial\Omega$, therefore $v - w = 0$ on $\partial\Omega$, we get

$$R(v, w) = \int_\Omega -\Delta u(v - w) - f(v - w) \; dx = \int_\Omega (-\Delta u - f)(v - w) \; dx. \tag{6.26}$$

By Theorem 6.4 we obtain with (6.26) the following error bound,

$$\begin{aligned} \|u - u_h\|_1 \leq c\Bigg\{ & \| - \Delta u - f\|_0^{1/2} \inf_{v \in \mathbb{K}} \|v - u_h\|_0^{1/2} \\ & + \inf_{v_h \in \mathbb{K}_h} \left[\| - \Delta u - f\|_0^{1/2} \|u - v_h\|_0^{1/2} + \|u - v_h\|_1 \right] \Bigg\}, \end{aligned} \tag{6.27}$$

where the Cauchy-Schwarz inequality and $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$, for $a, b \geq 0$, were used in the last step.

We want to make further estimations on the error and examine the dependency on the mesh size h. For this purpose, we assumed $u \in H^2(\Omega)$ in order to use helpful estimates for the interpolant of u, which we denote by $\Pi_h u \in \mathbb{K}_h$. Choosing $v_h = \Pi_h u \in \mathbb{K}_h$, the second term of the error estimation (6.27) can be estimated as follows:

$$\inf_{v_h \in \mathbb{K}_h} \left[\| -\Delta u - f\|_0^{1/2} \|u - v_h\|_0^{1/2} + \|u - v_h\|_1 \right] \leq \| -\Delta u - f\|_0^{1/2} \|u - \Pi_h u\|_0^{1/2} + \|u - \Pi_h u\|_1.$$

Following the interpolation error estimates by [5, Chapter 4], we get

$$\| -\Delta u - f\|_0^{1/2} \underbrace{\|u - \Pi_h u\|_0^{1/2}}_{\leq (ch^2|u|_2)^{1/2}} + \underbrace{\|u - \Pi_h u\|_1}_{\leq ch|u|_2} \leq ch \left[\| -\Delta u - f\|_0^{1/2} |u|_2^{1/2} + |u|_2 \right]. \tag{6.28}$$

It remains to identify the dependency of h for the first expression in (6.27), which reads as

$$\| -\Delta u - f\|_0^{1/2} \inf_{v \in \mathbb{K}} \|v - u_h\|_0^{1/2}.$$

For this reason we define

$$u^{h,*} = \max\{u_h, \psi\}.$$

First, we must show that $u^{h,*}$ is always in our region $\mathbb{K}$. Since $u_h, \psi \in H^1(\Omega)$, it follows that $u^{h,*} \in H^1(\Omega)$, [28, Chapter 4]. By the definition of $u^{h,*}$, we always have $u^{h,*} \geq \psi$. Now, since $\psi \leq 0$ and $u_h = 0$ on the boundary $\partial\Omega$, we have $u^{h,*} = u_h = 0$ on $\partial\Omega$. So we can deduce that $u^{h,*} \in \mathbb{K}$.
Let Ω^* be the set of points, which are not in $\mathbb{K}$, i.e.

$$\Omega^* = \{x \in \Omega \mid u_h(x) < \psi(x)\}.$$

Then over $\Omega \setminus \Omega^*$ we have $u^{h,*} = u_h$ and therefore

$$\inf_{v \in \mathbb{K}} \|v - u_h\|_0^2 \leq \|u^{h,*} - u_h\|_0^2 = \int_{\Omega^*} |u_h - \psi|^2 \, dx. \tag{6.29}$$

We use again the interpolant to obtain interpolation error estimates. This time we use the continuous and piece-wise linear interpolant of ψ, which is denoted as $\Pi_h \psi$. We know that at any node x_{node}, $u_h \geq \psi = \Pi_h \psi$, so $u_h \geq \Pi_h \psi$ in Ω. Over Ω^* we have

$$0 < |u_h - \psi| = \psi - u_h \leq \psi - \Pi_h \psi = |\psi - \Pi_h \psi|.$$

Thus, it follows from (6.29) that

$$\inf_{v \in \mathbb{K}} \|v - u_h\|_0^2 \leq \int_{\Omega^*} |u_h - \psi|^2 \, dx \leq \int_{\Omega^*} |\psi - \Pi_h \psi|^2 \, dx \leq \int_{\Omega} |\psi - \Pi_h \psi|^2 \, dx = \|\psi - \Pi_h \psi\|_0^2.$$

With the interpolation error estimate from [5, Chapter 4], we get

$$\inf_{v \in \mathbb{K}} \|v - u_h\|_0^{1/2} \leq \|\psi - \Pi_h \psi\|_0^{1/2} \leq ch|\psi|_2^{1/2}. \tag{6.30}$$

Finally, we obtain with (6.28) and (6.30) an optimal error estimate

$$\|u - u_h\|_1 \leq c \left\{ \| -\Delta u - f\|_0^{1/2} |\psi|_2^{1/2} + \| -\Delta u - f\|_0^{1/2} |u|_2^{1/2} + |u|_2 \right\} h = c(u)h. \tag{6.31}$$

The next example shows the application of Theorem 6.4 for the simplified Signorini problem (4.44) and gives an error bound dependent on the mesh size h.

Example 6.10. We assume that Ω is a polygonial domain and $u \in H^2(\Omega) \cap \mathbb{K}$. Furthermore, we assume that the boundary $\partial\Omega = \Gamma$ is divided into line segments Γ_i, i.e. $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, where Γ_i for $i = 1, ..., m$ denote the line segments only on the boundary Γ. Also, we assume for every Γ_i, that $u|_{\Gamma_i} \in H^2(\Gamma_i)$ and $\partial u/\partial n \in L_\infty(\Gamma_i)$. We can interpret the boundary assumptions as the boundedness of the trace of u on Γ_i and of its first derivative in the direction of the side Γ_i. Further, we consider Ω to be divided into a shape regular triangulation $\mathcal{T}_h$ and define the FE space V_h and the discrete admissible set $\mathbb{K}_h$ as

$$V_h = \{v \in C(\overline{\Omega}) \mid v|_T \in P_1(T) \text{ for all } T \in \mathcal{T}_h\} \subset V = H^1(\Omega)$$
$$\mathbb{K}_h = \{v_h \in V_h \mid v_h(x_{node}) \geq 0 \text{ for any node } x_{node} \text{ of } \partial\mathcal{T}_h\}.$$

In this particular case, $\mathbb{K}_h \subset \mathbb{K}$ and we can consider the reduced error estimate (6.21). For any $u \in H^2(\Omega) \cap \mathbb{K}$ and for any $v, w \in H^1(\Omega)$, the error bound for the simplified Signorini problem (4.44) is

$$R(v, w) = \int_\Omega \left[\nabla u \cdot \nabla(v - w) + u(v - w) - f(v - w)\right] \, dx - \int_\Gamma g(v - w) \, ds. \tag{6.32}$$

Using integration by parts for the first expression in (6.32) we deduce

$$R(v, w) = \int_\Omega (-\Delta u + u - f)(v - w) \, dx + \int_\Gamma \left(\frac{\partial u}{\partial n} - g\right)(v - w) \, ds. \tag{6.33}$$

Considering now the reduced error bound (6.21) in Theorem 6.4 we obtain with (6.33), the Cauchy-Schwarz inequality and $(a + b)^{1/2} \leq a^{1/2} + b^{1/2}$ the following error bound,

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} \leq c\Big\{ \inf_{v_h \in \mathbb{K}_h} \Big[&\| - \Delta u + u - f\|_{L_2(\Omega)}^{1/2} \|u - v_h\|_{L_2(\Omega)}^{1/2} \\ &+ \left\|\frac{\partial u}{\partial n} - g\right\|_{L_2(\Gamma)}^{1/2} \|u - v_h\|_{L_2(\Gamma)}^{1/2} + \|u - v_h\|_{H^1(\Omega)} \Big]\Big\}, \end{aligned} \tag{6.34}$$

or summarized

$$\|u - u_h\|_{H^1(\Omega)} \leq c(u) \inf_{v_h \in \mathbb{K}_h} \left[\|u - v_h\|_{H^1(\Omega)} + \|u - v_h\|_{L_2(\Omega)}^{1/2} + \|u - v_h\|_{L_2(\Gamma)}^{1/2}\right] \tag{6.35}$$

In order to obtain the dependency of the error on the mesh size h we choose for v_h the Lagrange interpolant denoted by $\Pi_h u \in \mathbb{K}_h$, which gives in (6.35) that

$$\begin{aligned} &\inf_{v_h \in \mathbb{K}_h} \left[\|u - v_h\|_{H^1(\Omega)} + \|u - v_h\|_{L_2(\Omega)}^{1/2} + \|u - v_h\|_{L_2(\Gamma)}^{1/2}\right] \\ &\qquad \leq \|u - \Pi_h u\|_{H^1(\Omega)} + \|u - \Pi_h u\|_{L_2(\Omega)}^{1/2} + \|u - \Pi_h u\|_{L_2(\Gamma)}^{1/2} \end{aligned}$$

Following the interpolation error estimates by [5, Chapter 4], we get

$$\underbrace{\|u - \Pi_h u\|_{H^1(\Omega)}}_{\leq ch|u|_2} + \underbrace{\|u - \Pi_h u\|_{L_2(\Omega)}^{1/2}}_{\leq (ch^2|u|_2)^{1/2}} + \underbrace{\|u - \Pi_h u\|_{L_2(\Gamma)}^{1/2}}_{\leq (ch^2|u|_2)^{1/2}} \leq c(u)h, \tag{6.36}$$

due to the assumptions for u on Ω and on the boundary Γ. Finally, we can estimate the error in (6.34) with the help of (6.36) by

$$\|u - u_h\|_{H^1(\Omega)} \leq c(u)h. \tag{6.37}$$

Remark 6.11. *Another error estimate for the simplified Signorini problem is given in [16, Chapter 1]. It shows the same result as in (6.37), but under stronger assumptions imposed on the solution u.*

6.2 Discretization of elliptic hemi-variational inequalities

Similar to the section before, we want to give an overview about the discretization of elliptic hemi-variational inequalities in this section. The contribution of this section is based on the work of [6, Chapter 7]. For more detailed description we refer to [30] and [31]. Recalling the hemi-variational inequality (5.47), we consider the problem: Find $u \in \mathbb{K}$, such that

$$\langle A(u), v-u \rangle + j(u,v) - j(u,u) \geq l(v-u) \quad \text{for all } v \in \mathbb{K}, \tag{6.38}$$

where $A : V \to V^*$ is an operator, $j : V \times V \to (-\infty, +\infty]$ is a functional and $l \in V^*$. The goal is to find an error estimate for hemi-variational inequalities (6.38), as in Theorem 6.4 for variational inequalities of the form (6.1). We assume the following throughout this section.

- V and V^* denote a real Hilbert space and its dual with respective norms $\|\cdot\|$ and $\|\cdot\|_*$, e.g. $V = H^1(\Omega)$ and $V^* = H^{-1}(\Omega)$, and $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the duality product.
- $\mathbb{K} \subset V$ is a nonempty, closed and convex subset.
- $A : V \to V^*$ is strongly monotone and Lipschitz continuous.
- $j : V \times V \to (-\infty, +\infty]$ satisfies for every $u \in V$ that $j(u, \cdot) : V \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Moreover, we assume that there exists $k < c_0$, such that

  $$|j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1)| \leq k\|u_1 - u_2\|\|v_1 - v_2\|, \tag{6.39}$$

  for all $u_1, u_2, v_1, v_2 \in \mathbb{K}$, where c_0 is the constant of the strong monotonicity of A.

Considering these assumptions for the operator A and the functional j, we conclude that the elliptic hemi-variational inequality has a unique solution by Theorem 5.57.

Remark 6.12. *The functional j in the previous section is defined on the subset $\mathbb{K}$ and we considered it to be the restriction of another functional j_0, naturally existing, defined on V. In this section, the functional j is already defined on V. In order to keep the notation up, we write $j_0 = j$.*

We are now in the position to formulate the discrete analogue of problem (6.38). For this purpose, we choose an appropriate finite dimensional subspace $V_h \subset V$ and a nonempty, closed and convex subset $\mathbb{K}_h \subset V_h$. The discrete variational inequality of (6.38) reads as: Find $u_h \in \mathbb{K}_h$, such that

$$\langle A(u_h), v_h - u_h \rangle + j_0(u_h, v_h) - j_0(u_h, u_h) \geq l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.40}$$

Applying Theorem 5.57 on (6.40) under these assumptions on the discrete data, we deduce that the discrete elliptic hemi-variational inequality (6.38) has a unique solution.

As a next step, we want to give a general convergence statement of the discrete solution comparable to Theorem 6.1 for elliptic variational inequalities (6.1). However, the proof requires more technical treatment. Therefore, we will give a part of the proof as a Lemma and refer to [6, Chapter 7] for the complete proof.

Lemma 6.13. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Moreover, let $A : V \to V^*$ be strongly monotone and Lipschitz continuous and let $j_0 : V \times V \to \mathbb{R}$ be a proper, convex and lower semi-continuous functional in the second argument satisfying (6.39). Further, we consider a family $\{V_h\}_h$ of finite dimensional closed subspaces of V and a family $\{\mathbb{K}_h\}_h$ of closed and convex subsets of V_h, which approximates $\mathbb{K}$ in the following sense:*

i) For every $v \in \mathbb{K}$, there exists a $v_h \in \mathbb{K}_h$, such that $\|v_h - v\| \to 0$ as $h \to 0$, i.e. strong convergence in V.

ii) For every $v_h \in \mathbb{K}_h$ with $v_h \rightharpoonup v$ in V as $h \to 0$, then $v \in \mathbb{K}$.

Then the sequence $\{u_h^n\}_n$, defined by

$$u_h^n = T_h u_h^{n-1}, \quad n \geq 1, \tag{6.41}$$

which approximates the solution u_h of (6.40) with the mapping $T_h : \mathbb{K}_h \to \mathbb{K}_h$, is bounded.

Proof. The proof is divided into three steps. First, we show the mapping T_h is a contraction. For this purpose, we consider for every $w_h \in \mathbb{K}_h$ the auxiliary problem of (6.40): Find $u_{w_h} \in \mathbb{K}_h$, such that

$$\langle A(u_{w_h}), v_h - u_{w_h}\rangle + j_0(w_h, v_h) - j_0(w_h, u_{w_h}) \geq l(v_h - u_{w_h}) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.42}$$

We define the mapping $T_h : \mathbb{K}_h \to \mathbb{K}_h$, which associates every $w_h \in \mathbb{K}_h$ with the unique solution of

$$T_h w_h = u_{w_h}. \tag{6.43}$$

The set of all fixed points coincides with the set of all solutions of (6.40), therefore the existence of a unique solution of (6.40) is equivalent to the existence of a unique fixed point of (6.43). We show that $T_h : \mathbb{K}_h \to \mathbb{K}_h$ is a contraction and deduce from Banach's fixed point theorem 2.14, that T_h admits a unique fixed point. For this purpose, let $w_{1,h}, w_{2,h} \in \mathbb{K}_h$ and let $u_{w_{1,h}}, u_{w_{2,h}}$ be the corresponding solutions obtained by (6.42), i.e.

$$\begin{aligned}
&\langle A(u_{w_{1,h}}), v_h - u_{w_{1,h}}\rangle + j_0(w_{1,h}, v_h) - j_0(w_{1,h}, u_{w_{1,h}}) \geq l(v_h - u_{w_{1,h}}) \quad \text{for all } v_h \in \mathbb{K}_h,\\
&\langle A(u_{w_{2,h}}), v_h - u_{w_{2,h}}) + j_0(w_{2,h}, v_h\rangle - j_0(w_{2,h}, u_{w_{2,h}}) \geq l(v_h - u_{w_{2,h}}) \quad \text{for all } v_h \in \mathbb{K}_h.
\end{aligned}$$

Choosing $v_h = u_{w_{2,h}}$ in the first inequality and $v_h = u_{w_{1,h}}$ in the second inequality and adding the resulting inequalities we deduce

$$\begin{aligned}
\langle A(u_{w_{1,h}}) - A(u_{w_{2,h}}), u_{w_{1,h}} - u_{w_{2,h}}\rangle \leq\\
j_0(w_{1,h}, u_{w_{2,h}}) - j_0(w_{1,h}, u_{w_{1,h}}) + j_0(w_{2,h}, u_{w_{1,h}}) - j_0(w_{2,h}, u_{w_{2,h}}).
\end{aligned}$$

Using the strong monotonicity of A and property (6.39) of j_0, we get that

$$c_0\|u_{w_{1,h}} - u_{w_{2,h}}\|^2 \leq k\|w_{1,h} - w_{2,h}\|\|u_{w_{1,h}} - u_{w_{2,h}}\|,$$

which gives

$$\|u_{w_{1,h}} - u_{w_{2,h}}\| \leq \frac{k}{c_0}\|w_{1,h} - w_{2,h}\|,$$

with $\frac{k}{c_0} < 1$. Recalling the definition of T_h in (6.43), we deduce that T_h is a contraction. Since a solution $u_h \in \mathbb{K}_h$ of (6.40) is also a solution of (6.42) with $w_h = u_h$, it follows that $T_h u_h = u_h$. So u_h is a fixed point and therefore (6.40) has a unique solution by Banach's fixed point theorem.

Secondly, we prove that the sequence of solutions $\{u_h\}_h$ of (6.40) is bounded. For this reason, let $v_0 \in \mathbb{K}$ and $v_{0,h} \in \mathbb{K}_h$, such that $v_{0,h} \to v_0$ strongly in V as $h \to 0$ following from assumption *i*). Taking $v_h = v_{0,h}$ in (6.40), we obtain

$$\langle A(u_h), v_{0,h} - u_h\rangle + j_0(u_h, v_{0,h}) - j_0(u_h, u_h) \geq l(v_{0,h} - u_h),$$

which is equivalent to

$$-\langle A(u_h), v_{0,h} - u_h\rangle \le j_0(u_h, v_{0,h}) - j_0(u_h, u_h) - l(v_{0,h} - u_h). \tag{6.44}$$

Using now the strong monotonicity of A and (6.44) we obtain

$$\begin{aligned} c_0\|u_h - v_{0,h}\|^2 &\le \langle A(u_h) - A(v_{0,h}), u_h - v_{0,h}\rangle = \langle A(v_{0,h}), v_{0,h} - u_h\rangle - \langle A(u_h), v_{0,h} - u_h\rangle \\ &\le \langle A(v_{0,h}), v_{0,h} - u_h\rangle + j_0(u_h, v_{0,h}) - j_0(u_h, u_h) - l(v_{0,h} - u_h) \\ &= \langle A(v_{0,h}), v_{0,h} - u_h\rangle + j_0(u_h, v_{0,h}) - j_0(u_h, u_h) + j_0(u, u_h) - j_0(u, v_{0,h}) \\ &\quad - j_0(u, u_h) + j_0(u, v_{0,h}) - l(v_{0,h} - u_h). \end{aligned} \tag{6.45}$$

Due to the lower semi-continuity of j_0, i.e. $\lim_{h\to 0} j_0(u, v_{0,h}) = j_0(u, v_0)$ for all $v_0 \in \mathbb{K}$, we have that $|j_0(u, v_{0,h})| < C_1(u)$, where $C_1(u)$ is independent of h. Moreover, the sequence $\{v_{0,h}\}_h$ is bounded and from the Lipschitz continuity of A we have that $\|A(v_{0,h})\|_* \le C_2(v_0)$, where $C_2(v_0)$ is independent of h. Using now the last bounds, property (6.39) and Lemma 5.56 in the inequality (6.45), we get that

$$\begin{aligned} c_0\|u_h - v_{0,h}\|^2 - \|l_{j_0}\|_*\|u_h\| \ - |c_{j_0}| &\le \langle A(u_h) - A(v_{0,h}), u_h - v_{0,h}\rangle + j_0(u, u_h) \\ &\le C_2(v_0)\|v_{0,h} - u_h\| + k\|u_h - u\|\|v_{0,h} - u_h\| + C_1(u) + \|l\|_*\|v_{0,h} - u_h\|. \end{aligned} \tag{6.46}$$

Applying the triangle inequalities

$$\begin{aligned} \|l_{j_0}\|_*\|u_h\| &\le \|l_{j_0}\|_*\|u_h - v_{0,h}\| + \|l_{j_0}\|_*\|v_{0,h}\|, \\ k\|u_h - u\| &\le k\|u_h - v_{0,h}\| + k\|u - v_{0,h}\|, \end{aligned}$$

and then Young's inequality $2\epsilon a \frac{1}{\epsilon} b \le \epsilon^2 a^2 + \frac{1}{\epsilon^2} b^2$ on (6.46), we obtain

$$\begin{aligned} &\left(c_0 - k - \frac{k\epsilon_1 + \epsilon_2 + \epsilon_3}{2}\right)\|u_h - v_{0,h}\|^2 \\ &\quad \le \frac{\|l_{j_0}\|_*}{2\epsilon_2} + \|v_{0,h}\|\|l_{j_0}\|_* + \frac{k}{2\epsilon_1}\|v_{0,h} - u\|^2 + \frac{(C_2(v_0) + \|l\|_*)^2}{2\epsilon_2} + C_1(u) + |c_{j_0}| \le \ C^*, \end{aligned} \tag{6.47}$$

with C^* independent of h and $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are chosen, such that $c_0 - k - \frac{k\epsilon_1+\epsilon_2+\epsilon_3}{2} > 0$. Therefore, according to the choice of $\{v_{0,h}\}_h$ we conclude that $\{u_h - v_{0,h}\}_h$ is bounded and thus the sequence $\{u_h\}_h$ is bounded as well.

Lastly, we consider the mapping $T_h : \mathbb{K}_h \to \mathbb{K}_h$ which is defined through an iterative procedure, i.e. $T_h u_h^{n-1} = u_h^n$. Thus, we can rewrite (6.42) as: Find $u_h^n \in \mathbb{K}_h$, such that

$$\langle A(u_h^n), v_h - u_h^n\rangle + j_0(u_h^{n-1}, v_h) - j_0(u_h^{n-1}, u_h^n) \ge l(v_h - u_h^n) \quad \text{for all } v_h \in \mathbb{K}_h. \tag{6.48}$$

The problem (6.48) is an iterative approximation of the discrete elliptic hemi-variational inequality (6.40). This means, that the solution u_h in (6.40) is approximated by the sequence of solutions $\{u_h^n\}_n$ computed by (6.48). Since u_h is a fixed point of T_h, i.e. $T_h u_h = u_h$, and using the contraction property, we have

$$\|u_h^n - u_h\| = \|T_h u_h^{n-1} - T_h u_h\| \le \frac{k}{c_0}\|u_h^{n-1} - u_h\| = \frac{k}{c_0}\|T_h u_h^{n-2} - T_h u_h\| \le \left(\frac{k}{c_0}\right)^2 \|u_h^{n-2} - u_h\|,$$

which gives after n steps that

$$\|u_h^n - u_h\| \le \left(\frac{k}{c_0}\right)^n \|u_h^0 - u_h\|, \qquad \text{with } \left(\frac{k}{c_0}\right)^n \to 0 \ \text{ for } \ n \to \infty. \tag{6.49}$$

Thus, we can deduce from the boundedness of $\{u_h\}_h$ and (6.49), that

$$\|u_h^n - u_h^0\| \leq Cq^n,$$

with $q = \frac{k}{c_0} < 1$ and $C > 0$ independent of h and n. Hence, for every $\epsilon > 0$ there exists $n_0(\epsilon)$, such that

$$\|u_h^n - u_h\| \leq \epsilon,$$

for all $n > n_0(\epsilon)$ and $h > 0$, i.e. $\{u_h^n\}_n$ is bounded. □

With the help of Lemma 6.13 it is possible to prove the next important result.

Theorem 6.14. *Let $\mathbb{K} \subset V$ be a nonempty, closed and convex subset. Moreover, let $A : V \to V^*$ be strongly monotone and Lipschitz continuous and let $j_0 : V \times V \to \mathbb{R}$ be a proper, convex and lower semi-continuous functional in the second argument satisfying (6.39). Further, we consider a family $\{V_h\}_h$ of finite dimensional closed subspaces of V and a family $\{\mathbb{K}_h\}_h$ of closed and convex subsets of V_h, which approximates $\mathbb{K}$ in the following sense:*

i) For every $v \in \mathbb{K}$, there exists a $v_h \in \mathbb{K}_h$, such that $\|v_h - v\| \to 0$ as $h \to 0$, i.e. strong convergence in V.

ii) For every $v_h \in \mathbb{K}_h$ with $v_h \rightharpoonup v$ in V as $h \to 0$, then $v \in \mathbb{K}$.

Then we have the convergence $\|u - u_h\|_V \to 0$ as $h \to 0$, where u and u_h are the solutions of (6.38) and (6.40), respectively.

Proof. See [6, Chapter 7] and Lemma 6.13. □

Remark 6.15. *Note that the latter theorem is a general convergence result based on strong assumptions. There are no statements about any convergence rates yet.*

We close this chapter with an error approximation estimate for elliptic hemi-variational inequalities (6.38), which is comparable to Céa's Lemma for variational equalities. For this purpose, we introduce two more Hilbert spaces H and U in addition to V, where we can think of $V = H^1(\Omega)$, $H = L_2(\Omega)$ and $U = H^1(\Omega)$ in mechanical applications.

Theorem 6.16. *Let V be a real Hilbert space and let the assumptions from Theorem 6.14 hold, where u and u_h are the solutions of (6.38) and (6.40), respectively. Moreover, we suppose that there exist two Hilbert spaces $(H, \|\cdot\|_H)$ and $(U, \|\cdot\|_U)$, such that V is densely embedded in H, i.e. $V \hookrightarrow H$ dense, $V \subset U$ and*

$$A(u) - l \in H, \tag{6.50}$$

$$|j_0(u, v_h) - j_0(u, v)| \leq c_1 \|v_h - v\|_V \quad \text{for all } v_h \in \mathbb{K}_h, v \in \mathbb{K}, \tag{6.51}$$

where c_1 is a positive constant independent of h. Then there exists a positive constant c independent of h, such that the estimate

$$\begin{aligned} \|u - u_h\| \leq c \Big\{ &\inf_{v_h \in \mathbb{K}_h} \left(\|u - v_h\|^2 + \|A(u) - l\|_H \|u - v_h\|_H + c_1 \|u - v_h\|_U\right) \\ &+ \inf_{v \in \mathbb{K}} \left(\|A(u) - l\|_H \|u_h - v\|_H + c_1 \|u_h - v\|_U\right) \Big\}^{\frac{1}{2}} \end{aligned} \tag{6.52}$$

holds.

Proof. We use the strong monotonicity of A to obtain

$$
\begin{aligned}
c_0\|u_h - u\|^2 &\leq \langle A(u_h) - A(u), u_h - u\rangle = \langle A(u_h), u_h - u\rangle - \langle A(u), u_h - u\rangle \\
&= \langle A(u_h), u_h - u + (v_h - v_h)\rangle - \langle A(u), u_h - u + (v - v)\rangle \\
&= -\langle A(u_h), v_h - u_h\rangle + \langle A(u_h), v_h - u\rangle + \langle A(u), v - u_h\rangle - \langle A(u), v - u\rangle
\end{aligned}
\tag{6.53}
$$

Using the inequalities (6.38) and (6.40) and rewriting them as

$$
\begin{aligned}
-\langle A(u), v - u\rangle &\leq j_0(u, v) - j_0(u, u) - l(v - u) \quad \text{for all } v \in \mathbb{K}, \\
-\langle A(u_h), v_h - u_h\rangle &\leq j_0(u_h, v_h) - j_0(u_h, u_h) - l(v_h - u_h) \quad \text{for all } v_h \in \mathbb{K}_h,
\end{aligned}
$$

we get by inserting them into (6.53) that

$$
\begin{aligned}
c_0\|u_h - u\|^2 &\leq \langle A(u_h), v_h - u\rangle - \langle A(u), v_h - u\rangle + \langle A(u), v_h - u\rangle + \langle A(u), v - u_h\rangle \\
&\quad + j_0(u_h, v_h) - j_0(u_h, u_h) + j_0(u, v) - j_0(u, u) - l(v - u + v_h - u_h) \\
&= \underbrace{\langle A(u) - l, v - u_h + v_h - u\rangle}_{T_1} + \underbrace{\langle A(u_h) - A(u), v_h - u\rangle}_{T_2} \\
&\quad \underbrace{+ j_0(u_h, v_h) - j_0(u_h, u_h) + j_0(u, v) - j_0(u, u)}_{T_3}.
\end{aligned}
\tag{6.54}
$$

Considering now the Cauchy-Schwarz and triangle inequality for T_1 and the Cauchy-Schwarz inequality and the Lipschitz continuity for T_2, we obtain

$$
\begin{aligned}
&\langle A(u) - l, v - u_h + v_h - u\rangle + \langle A(u_h) - A(u), v_h - u\rangle \\
&\quad \leq \|A(u) - l\|_H(\|v - u_h\|_H + \|v_h - u\|_H) + L\|u - u_h\|\|v_h - u\|.
\end{aligned}
\tag{6.55}
$$

Using (6.39) and (6.51) for T_3 yields

$$
\begin{aligned}
&j_0(u_h, v_h) - j_0(u_h, u_h) + j_0(u, v) - j_0(u, u) \\
&\leq |j_0(u_h, v_h) - j_0(u_h, u_h) + j_0(u, u_h) - j_0(u, v_h)| + |j_0(u, v_h) - j_0(u, u)| + |j_0(u, v) - j_0(u, u_h)| \\
&\leq k\|u_h - u\|\|v_h - u_h\| + c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right).
\end{aligned}
\tag{6.56}
$$

The last line in (6.56) can be further estimated by

$$
\begin{aligned}
&k\|u_h - u\|\|v_h - u_h\| + c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right) \\
&= k\|u_h - u\|\|v_h - u + u - u_h\| + c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right) \\
&\leq k\|u_h - u\|^2 + k\|u_h - u\|\|v_h - u\| + c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right).
\end{aligned}
\tag{6.57}
$$

Hence, it follows from (6.54) with the help of (6.55) and (6.57) that

$$
\begin{aligned}
(c_0 - k)\|u_h - u\|^2 \leq& (L + k)\|u_h - u\|\|v_h - u\| + \|A(u) - l\|_H(\|v - u_h\|_H + \|v_h - u\|_H) \\
&+ c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right).
\end{aligned}
\tag{6.58}
$$

If we use now Young's inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$, for $\epsilon = \frac{c_0 - k}{L + k}$, where $c_0 > k$, we obtain from (6.58) that

$$
\begin{aligned}
\frac{c_0 - k}{2}\|u_h - u\|^2 \leq& \frac{L + k}{2(c_0 - k)}\|v_h - u\|^2 + \|A(u) - l\|_H(\|v - u_h\|_H + \|v_h - u\|_H) \\
&+ c_1\left(\|v_h - u\|_U + \|v - u_h\|_U\right),
\end{aligned}
\tag{6.59}
$$

for all $v \in \mathbb{K}$ and $v_h \in \mathbb{K}_h$. Thus, the infima over $\mathbb{K}$ and $\mathbb{K}_h$ in (6.59) gives

$$\|u - u_h\| \leq c\Big\{ \inf_{v_h \in \mathbb{K}_h} \big(\|u - v_h\|^2 + \|A(u) - l\|_H \|u - v_h\|_H + c_1 \|u - v_h\|_U \big) + \inf_{v \in \mathbb{K}} \left(\|A(u) - l\|_H \|u_h - v\|_H + c_1 \|u_h - v\|_U \right) \Big\}^{\frac{1}{2}},$$

with $c = \max\left\{ \frac{L+k}{(c_0-k)^2}, \frac{2}{c_0-k}, \frac{2c_1}{c_0-k} \right\}$, which completes the proof. □

Remark 6.17. *If we take a closer look to inequality (6.56) in the latest proof, we may estimate the last expression in another way instead of (6.57). The inequality in (6.56) reads as*

$$\begin{aligned} &j_0(u_h, v_h) - j_0(u_h, u_h) + j_0(u, v) - j_0(u, u) \\ &\quad \leq \underbrace{k\|u_h - u\|\|v_h - u_h\|}_{T} + c_1 \left(\|v_h - u\|_U + \|v - u_h\|_U \right). \end{aligned}$$

We can use for example Young's inequality for the term T, which gives

$$k\|u_h - u\|\|v_h - u_h\| \leq \frac{\epsilon\|u_h - u\|^2}{2} + \frac{k^2\|v_h - u_h\|^2}{2\epsilon}, \tag{6.60}$$

for $\epsilon > 0$. The expression $\|v_h - u_h\|$ in (6.60) can be further estimated by special techniques. We only remark this possible approach at this point and leave it as an outlook for further research after this work.

Remark 6.18. *As mentioned in the previous section, $\mathbb{K}_h \not\subset \mathbb{K}$ in general. In the case that $\mathbb{K}_h \subset \mathbb{K}$, the expression T_1 in (6.54) vanishes and the estimate (6.52) reduces to*

$$\|u - u_h\| \leq c \left\{ \inf_{v_h \in \mathbb{K}_h} \big(\|u - v_h\|^2 + \|A(u) - l\|_H \|u - v_h\|_H + c_1 \|u - v_h\|_U \big) \right\}^{\frac{1}{2}}. \tag{6.61}$$

We can see that the error estimate (6.61) depends on the approximation properties of V_h in V, i.e. the distance between u and the elements in $v_h \in V_h$. Thus, if we consider high regularity assumptions for the solution u, then we can achieve optimal convergence rates under a suitable choice of V_h.

Remark 6.19. *If $j_0 = 0$, hence $c_1 = 0$ in (6.51), we deduce the following estimate*

$$\|u - u_h\| \leq c \left\{ \inf_{v_h \in \mathbb{K}_h} \big(\|u - v_h\|^2 + \|A(u) - l\|_H \|u - v_h\|_H \big) + \|A(u) - l\|_H \inf_{v \in K} \|u_h - v\|_H \right\}^{\frac{1}{2}},$$

which is comparable to the estimate in Remark 6.5. Furthermore, if $\mathbb{K} = V$, and thus $\mathbb{K}_h = V_h$, then we can obtain the classical Finite Element estimate by Céa's Lemma

$$\|u - u_h\| \leq c \inf_{v_h \in V_h} \|u - v_h\|.$$

Chapter 7

Numerical results

The aim of this chapter is to perform numerical tests for the obstacle problem (4.60) the simplified Signorini problem (4.44) and validate the estimates, which have been presented in Example 6.9 and Example 6.10, respectively. Many effective different methods have been developed to compute Finite Element solutions for variational inequalities. The first section is dedicated to the obstacle problem and we solve its discrete problem in terms of dual problems following the ideas of [37] and [38] and penalization methods referring to [13, Chapter 1], [18, Chapter 4] and [19, Chapter 4]. We compare the different methods and analyze the errors of the discrete solutions and the convergence rates. For the simplified Signorini problem, we only use the penalization technique to compute numerical solutions and compare the results with Example 6.10. The last section of this chapter is dedicated to a more realistic friction problem, analyzing the displacements of an elastic body in frictional contact with a rigid plate for different friction coefficients. The numerical examples have been performed using the free Finite Element software *FreeFem++*, see [36].

7.1 Numerical results for the obstacle problem

Recalling the variational form of the obstacle problem (4.59), we have:
Find $u \in \mathbb{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ in } \Omega\}$, such that

$$\int_\Omega \nabla u \cdot \nabla (v-u)\, dx \geq \int_\Omega f(v-u)\, dx \quad \text{for all } v \in \mathbb{K}, \tag{7.1}$$

where f and ψ are sufficiently smooth and $\psi \leq 0$ on $\partial\Omega$. The discrete formulation is:
Find $u_h \in \mathbb{K}_h = \{v_h \in V_h \mid v_h(x_{node}) \geq \psi(x_{node}) \text{ in every node } x_{node}\}$, such that

$$\int_\Omega \nabla u_h \cdot \nabla (v_h-u_h)\, dx \geq \int_\Omega f(v_h-u_h)\, dx \quad \text{for all } v_h \in \mathbb{K}_h. \tag{7.2}$$

We always consider the unit square $\overline{\Omega} = [0,1]\times[0,1]$ as the computational domain and use a triangulation $\mathcal{T}_h = \{T_1, T_2, \ldots, T_M\}$ of Ω, where $T_i = \overline{T}_i$ are triangular elements and $\bigcup_i^M T_i = \overline{\Omega}$, c.f. [4, Chapter 2]. Further, we choose the finite dimensional space $V_h \subset V$ as

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_T \in P_1 \text{ for all } T \in \mathcal{T}_h \text{ and } v_h = 0 \text{ on } \Gamma = \partial\Omega\}, \tag{7.3}$$

where

$$P_1 = \{v(x,y) = \sum_{\substack{i+j\leq 1\\ i,j\geq 0}} a_{ij}x^i y^j,\ a_{ij} \in \mathbb{R}\}.$$

For the fundamental Finite Element theory we highly recommend [4, Chapter 2] and do not go further into detail. Since our problem is a variational inequality, we cannot convert the variational formulation into a linear system directly. Instead, we introduce two different techniques, the saddle point problem (dual method) and the penalty method, and formulate the variational inequality as an equality, where additional terms must be added in order to satisfy the inequality.

7.1.1 Dual methods

We start with a dual method and consider the saddle point problem for the obstacle problem as described in [36]. We do not go into the theory of dual problems, but we just use the algorithm in [36] in order to compare it with the penalty method in the upcoming subsection. For the mathematical description and analysis of saddle point problems we refer to [37] and [38]. The variational form (7.1) can be rewritten as a saddle point problem: Find $u \in H_0^1(\Omega), \lambda \in L_2(\Omega)$ such that

$$\max_{\substack{\lambda \in L_2(\Omega) \\ \lambda \geq 0}} \min_{u \in H_0^1(\Omega)} \mathcal{L}(u, \lambda) = \frac{1}{2} \int_\Omega \left[\nabla u \cdot \nabla v - fu + \lambda(\psi - u)^+ \right] \, dx, \tag{7.4}$$

where $(\psi - u)^+ = \max(0, \psi - u)$. This saddle point problem is equivalent to:
Find $u \in H_0^1(\Omega), \lambda \in L_2(\Omega)$, such that

$$\begin{aligned} \int_\Omega \left[\nabla u \cdot \nabla v + \lambda v^+ \right] \, dx &= \int_\Omega fv \, dx && \text{for all } v \in H_0^1(\Omega), \\ \int_\Omega \mu(\psi - u)^+ \, dx &= 0 && \text{for all } \mu \in L_2(\Omega), \mu \geq 0, \lambda \geq 0. \end{aligned} \tag{7.5}$$

A possible algorithm to solve (7.5) is the following, c.f. [36], [37] and [38].

Algorithm 1: Solve saddle point problem (7.5)

$k = 0$, $k_{max} = 100$, choose $\lambda_0 \in H^{-1}(\Omega)$, choose penalty parameter c large enough;
for $k = 0, 1, ..., k_{max}$ **do**
  Set $I_k = \{x \in \Omega \mid \lambda_k + c(\psi - u_{k+1}) \leq 0\}$;
  Set $V_{\psi,k+1} = \{v \in H_0^1(\Omega) \mid v = \psi \text{ on } I_k\}$;
  Set $V_{0,k+1} = \{v \in H_0^1(\Omega) \mid v = 0 \text{ on } I_k\}$;
  Find $u_{k+1} \in V_{\psi,k+1}$ and $\lambda_{k+1} \in H^{-1}(\Omega)$, such that

$$\begin{aligned} \int_\Omega \nabla u_{k+1} \cdot \nabla v_{k+1} \, dx &= \int_\Omega f v_{k+1} \, dx && \text{for all } v_{k+1} \in V_{0,k+1}, \\ \langle \lambda_{k+1}, v \rangle &= \int_\Omega \nabla u_{k+1} \cdot \nabla v - fv \, dx && \text{for all } v \in H_0^1(\Omega) \end{aligned}$$

end

We solve the saddle point problem (7.5) with Algorithm 1 for two different values for f and ψ.

Example 7.1. Let $f = -2$ and $\psi = (x - x^2)(y - y^2)$. First, we compute the reference solution, meaning that we compute the solution via Algorithm 1 on a very fine triangular mesh and choose higher order polynomials for the shape functions. In our case, Ω is divided into a mesh

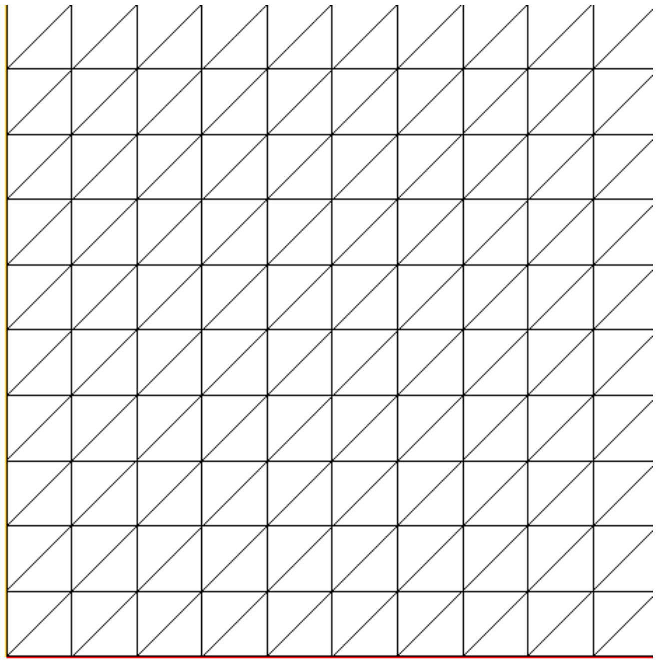


Figure 7.1: Zoomed computational domain $\overline{\Omega} = [0,1] \times [0,1]$, which is divided into a triangular mesh with 200×200 grid points.

with 200×200 grid points, see Figure 7.1, and we use P_2-elements, where

$$P_2 = \{v(x,y) = \sum_{\substack{i+j\leq 2\\ i,j\geq 0}} a_{ij}x^iy^j,\ a_{ij} \in \mathbb{R}\}.$$

We solve now the variational inequality (7.2) with Algorithm 1 for different meshes sizes $h_j = (1/2)^j$, $j = 1, ..., 8$, and with P_1-elements, where these solutions are referred to as discrete solutions u_h. The errors of the discrete solutions u_h with respect to the reference solution u and the convergence rates can be observed with respect to the H^1-norm and L_2-norm, as given in Table 7.1. We compute the numerical convergence rates r_i by using the formula

$$r_i = \frac{\log\left(\frac{e_{j+1}}{e_j}\right)}{\log\left(\frac{h_{j+1}}{h_j}\right)}, \quad j = 1, 2, \ldots, 8, \tag{7.6}$$

where $e_j = \|u - u_h\|_i$ for mesh size h_j and $i = 1, 2$.

dof	h	$\|u - u_h\|_1$	r_1	$\|u - u_h\|_0$	r_0
9	0.5	0.278838		0.0454333	
25	0.25	0.159107	0.809433	0.0143888	1.65881
81	0.125	0.0829626	0.939463	0.00386289	1.89719
289	0.0625	0.0420212	0.981344	0.000986677	1.96903
1089	0.03125	0.0210929	0.994358	0.000248258	1.99074
4225	0.015625	0.0105597	0.998188	6.21819e-05	1.99727
16641	0.0078125	0.0052813	0.999607	1.55539e-05	1.99922
66049	0.00390625	0.00264053	1.00006	3.88889e-06	1.99984

Table 7.1: Errors for different mesh sizes h and their convergence rates r_i for $f = -2$ and $\psi = (x - x^2)(y - y^2)$ with respect to H^1- norm and L_2-norm, respectively.

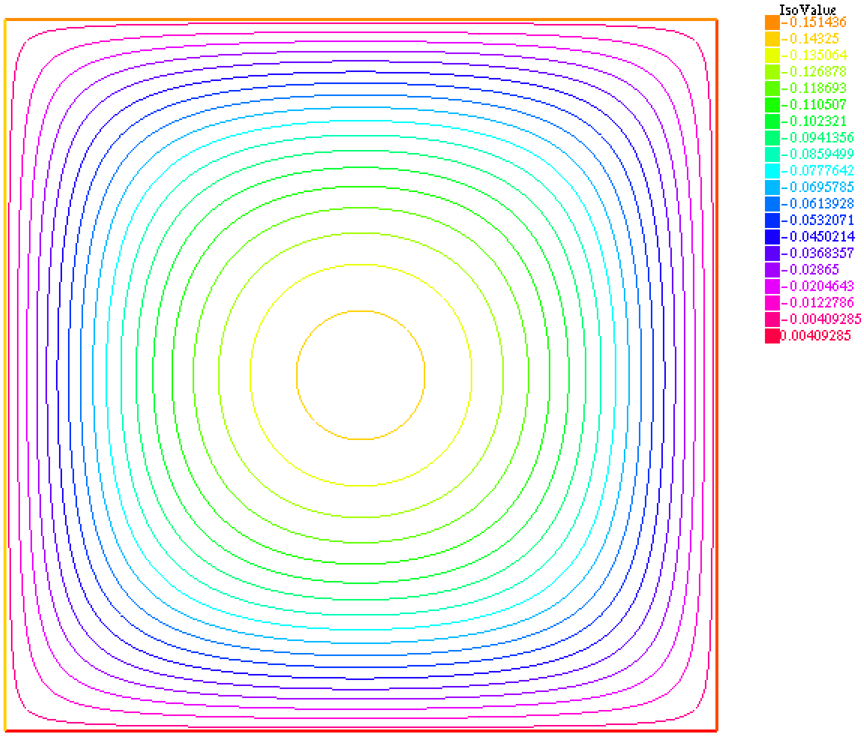


Figure 7.2: Iso-values of the reference solution u for $f = -2$ and $\psi = (x - x^2)(y - y^2)$ computed by means of the saddle point problem.

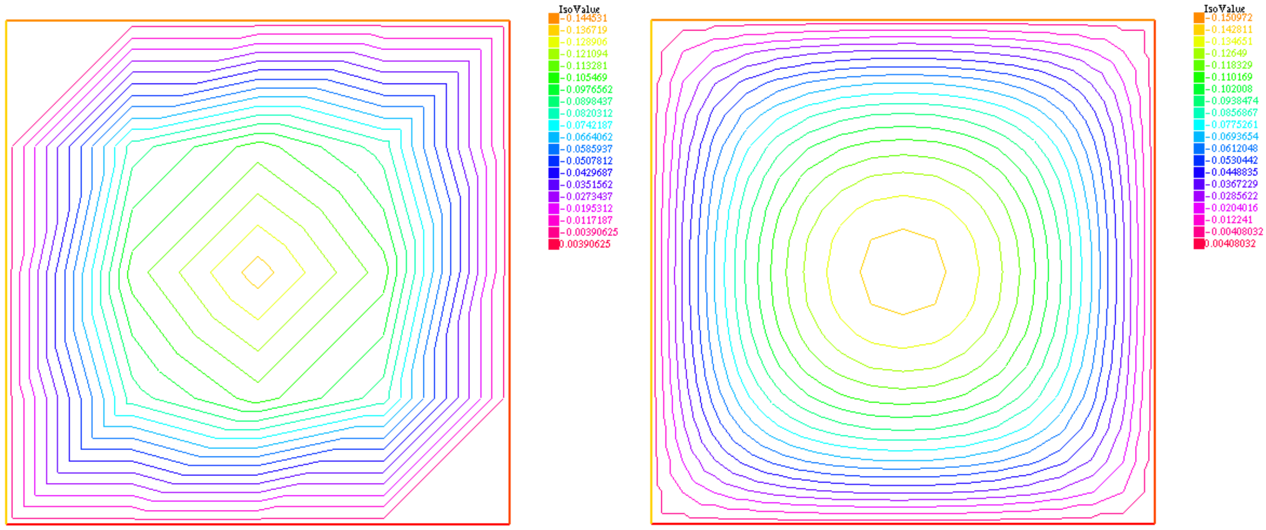
Recalling the error estimate (6.31) of Example 6.9, i.e.

$$\|u - u_h\|_1 \leq ch,$$
$$\|u - u_h\|_0 \leq ch^2,$$

we observe from Table 7.1, that the dependency on the mesh size h of the error agrees with the computed error of the numerical example in the respective norms.

Furthermore, we investigate the iso-values of the reference solution and compare them with the discrete solutions. Figure 7.2 shows the values of the reference solution u. We can see from Figure 7.3, that the discrete solution u_h converges to u as $h \to 0$. In addition, the iso-values of u_h agrees with the ones of the reference solution u as $h \to 0$. We will compare this example with itself but computed by a different method, i.e. the penalty method.

Example 7.2. Let $f = -x$ and $\psi = -0.5$. As in the previous example, we first compute the reference solution u and investigate the dependency on the mesh size h of the error. Table 7.2 shows the error and the convergence rates in the H^1-norm and L_2-norm, respectively. Analyzing the error and convergence rates in Table 7.2, we observe that the dependency of the mesh size h in the error agrees with the a-priori estimates in Example 6.9, i.e. $\|u - u_h\|_1 \leq ch$ and $\|u - u_h\|_0 \leq ch^2$. Figure 7.4 shows the iso-values of the reference solution u.



(a) Iso-values of u_h for $h = 0.25$.

(b) Iso-values of u_h for $h = 0.625$.

Figure 7.3: Iso-values of u_h for $h = 0.25$ and $h = 0.625$, where $f = -2$ and $\psi = (x - x^2)(y - y^2)$ computed by means of the saddle point problem.

dof	h	$\|u - u_h\|_1$	r_1	$\|u - u_h\|_0$	r_0
9	0.5	0.0762753		0.0121097	
25	0.25	0.0446101	0.773847	0.00395721	1.61361
81	0.125	0.023499	0.924771	0.00107725	1.87713
289	0.0625	0.0119476	0.975879	0.000276677	1.96107
1089	0.03125	0.00600501	0.992482	6.97523e-05	1.98789
4225	0.015625	0.00300762	0.997544	1.74825e-05	1.99633
16641	0.0078125	0.00150439	0.999448	4.37389e-06	1.99892
66049	0.00390625	0.000752149	1.00008	1.09364e-06	1.99978

Table 7.2: Errors for different mesh sizes h and their convergence rates r_i for $f = -x$ and $\psi = -0.5$ with respect to H^1- norm and L_2-norm, respectively.

7.1.2 Penalty methods

We introduce another method for solving variational inequalities called the penalty method. The contribution of this section is based on the ideas of [13, Chapter 1], [18, Chapter 4], [19, Chapter 4] and [45]. We emphasize, that only a short introduction of the penalty method is given in this work and refer to the latest references for its detailed description and analysis.

We consider the following general problem. Find $u \in \mathbb{K}$, such that

$$\begin{aligned} F(u) &= \min_{v \in \mathbb{K}} F(v), \\ F(v) &= \frac{1}{2} a(v, v) - l(v), \\ \mathbb{K} &= \{v \in V \mid Bv - g \leq 0 \text{ in } Q\} \subset V, \end{aligned} \tag{7.7}$$

where $(V, \|\cdot\|)$ and $(Q, \|\cdot\|_Q)$ are two Hilbert spaces, $B : V \to Q$ is a linear mapping and $g \in Q$. By Theorem 5.27 we can characterize the minimization problem (7.7) by the variational inequality (5.19a) using the G-derivative of F. In order to convert the inequality into an

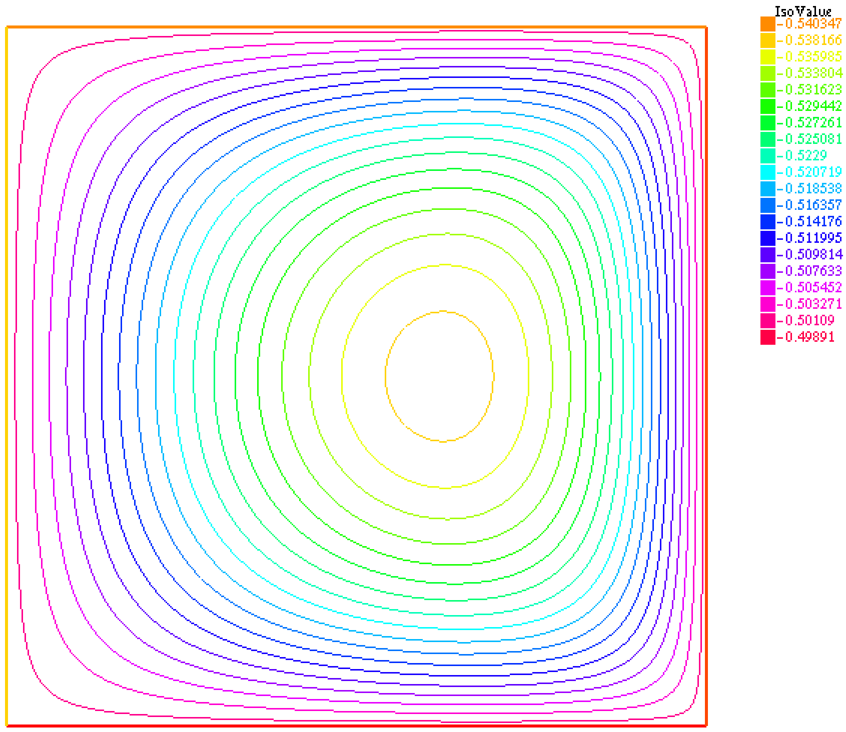


Figure 7.4: Iso-values of the reference solution u for $f = -x$ and $\psi = -0.5$.

equality we must introduce the penalty term $P : V \to \mathbb{R}$, $P(v) = \frac{1}{2}\|(Bv - g)^+\|_Q$, where $(f)^+ = \max(0, f)$. The penalty term is large the constraint of the set $\mathbb{K}$ is not satisfied. Thus, we rewrite the minimization problem (7.7) as: Find $u_\kappa \in V$, such that

$$\begin{aligned} F_\kappa(u_\kappa) &= \min_{v \in V} F_\kappa(v), \\ F_\kappa(v) &= F(v) + \frac{1}{\kappa} P(v), \end{aligned} \tag{7.8}$$

where $\kappa > 0$ and $F(v)$ as in (7.7). In order to apply Theorem 5.27 on (7.8), we suppose that the penalization term P is (weakly) lower semi-continuous and G-differentiable. Furthermore, P has only an impact if the constraint of $\mathbb{K}$ is not satisfied, i.e.

$$P(v) \geq 0, \quad P(v) = 0 \text{ if and only if } v \in \mathbb{K}.$$

With this assumptions on P, we can apply Theorem 5.27, where the G-derivative of the penalization term must be computed. Following the steps of [18, Chapter 3] we obtain, that the minimization problem (7.8) can be characterized by the variational form:
Find $u_\kappa \in V$, such that

$$a(u_\kappa, v) + \frac{1}{\kappa}[j(Bu_\kappa - g)^+, Bv] = l(v) \quad \text{for all } v \in V, \tag{7.9}$$

where $j : Q \to Q^*$ is the Riesz map, Q^* is the dual space of Q and $[\cdot, \cdot] : Q^* \times Q \to \mathbb{R}$ is the duality paring in $Q^* \times Q$. We choose the finite dimensional subspace $V_h \subset V$ as in (7.3). Hence, the discrete formulation of (7.9) reads as: Find u_h in V_h, such that

$$a(u_h, v_h) + \frac{1}{\kappa}[j(Bu_h - g)^+, Bv_h] = l(v_h) \quad \text{for all } v_h \in V_h. \tag{7.10}$$

We will use the penalty method to solve the same problems as in Example 7.1 and Example 7.2.

Example 7.3. As in Example 7.1, we set $f = -2$ and $\psi = (x - x^2)(y - y^2)$. In this case, the Hilbert space $Q = V = H_0^1(\Omega)$, $B = I$ and $g = \psi$. We compute a reference solution u on a fine mesh, again 200×200 grid, using P_2-elements. With the help of the reference solutions we can compute the error and convergence rates with the solutions for different mesh sizes and P_1-elements, see Table 7.3. We discover that the mesh size dependency of the error agrees with the obtained error estimates of Example 6.9, i.e. $\|u - u_h\|_1 \leq ch$ and $\|u - u_h\|_0 \leq ch^2$. Furthermore, it can be seen that the errors obtained using the penalty methods are very similar to the errors computed by means of the saddle point problems. Hence, both methods give quite the same result. This result can be also detected by comparing the iso-values of the solutions in Figure 7.5 and Figure 7.6 with Figure 7.2 and Figure 7.3.

dof	h	$\|u - u_h\|_1$	r_1	$\|u - u_h\|_0$	r_0
9	0.5	0.278836		0.0454333	
25	0.25	0.159111	0.809383	0.0143888	1.65881
81	0.125	0.0829578	0.939585	0.00386289	1.89719
289	0.0625	0.04202	0.9813	0.000986671	1.96904
1089	0.03125	0.0210991	0.993894	0.000248262	1.9907
4225	0.015625	0.0105598	0.9986	6.21993e-05	1.99689
16641	0.0078125	0.00527925	1.00018	1.57638e-05	1.98028
66049	0.00390625	0.0026413	0.999083	5.54499e-06	1.50736

Table 7.3: Errors for different mesh sizes h and their convergence rates r_i with respect to H^1-norm and L_2-norm, respectively.

Remark 7.4. *The convergence rate r_0 in the last line with respect to the L_2-norm in Table 7.3 is 1.50736, which does not agree with the rates above. The reason is, that the error as well as the convergence rates are computed in terms of the reference solution, which is an approximation of the exact solution as well. Thus, the approximate solution u_h may have the same accuracy as the reference solution u, such that the errors and convergence rates give unexpected values.*

Example 7.5. Next, we want to investigate the penalty method with the data as in Example 7.2, $f = -x$ and $\psi = -0.5$. The errors and convergence rates can be observed in Table 7.4 and agree with the theory in Example 6.9. The error values and convergence rates are quite similar to Table 7.2. The iso-values of the reference solution are even equal to Figure 7.4. As explained in Remark 7.4, the convergence rate r_0 in the last line of Table 7.4 does not give an expected value due to the accuracy issues.

dof	h	$\|u - u_h\|_1$	r_1	$\|u - u_h\|_0$	r_0
9	0.5	0.0762746		0.0121097	
25	0.25	0.0446117	0.773778	0.0039572	1.61361
81	0.125	0.0234975	0.924918	0.00107725	1.87713
289	0.0625	0.0119473	0.975818	0.00027669	1.961
1089	0.03125	0.00600719	0.991926	6.97688e-05	1.98762
4225	0.015625	0.00300765	0.998052	1.74899e-05	1.99606
16641	0.0078125	0.00150362	1.0002	4.39397e-06	1.99293
66049	0.00390625	0.000752668	0.998355	1.55686e-06	1.49688

Table 7.4: Errors for different mesh sizes h and their convergence rates r_i with respect to H^1-norm and L_2-norm, respectively.

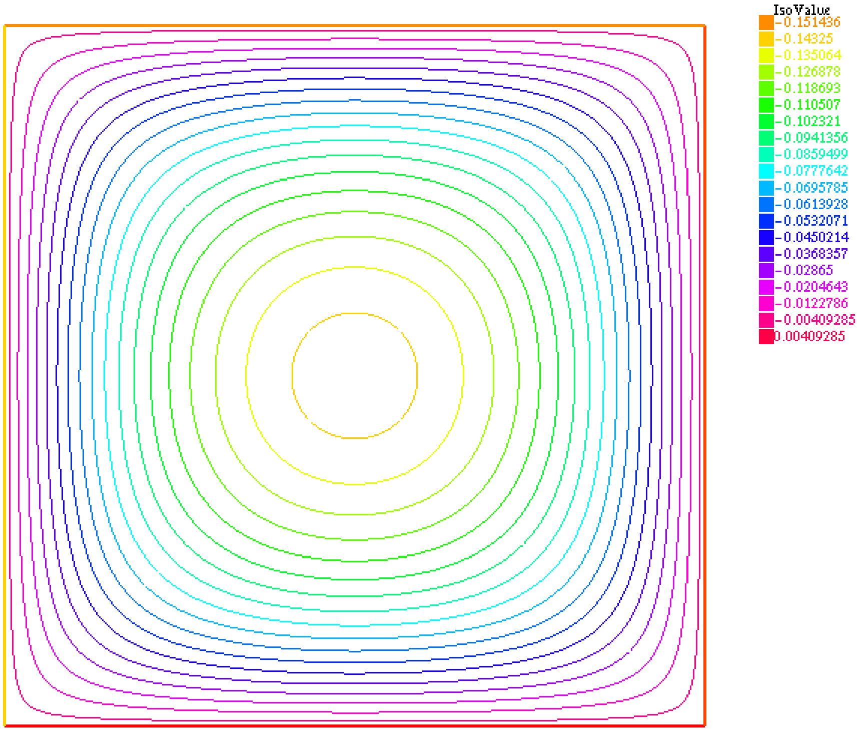
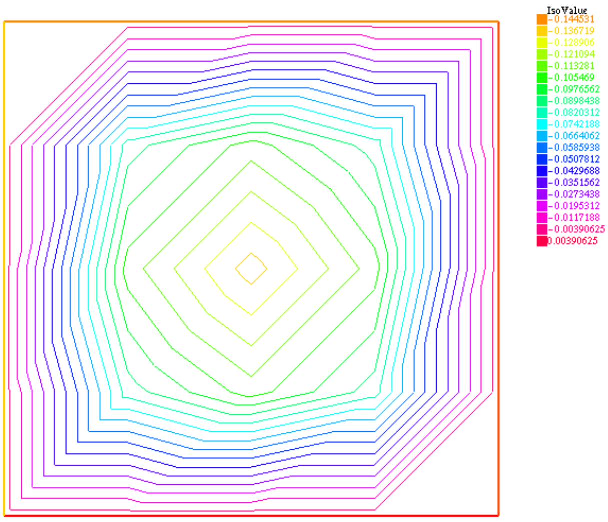
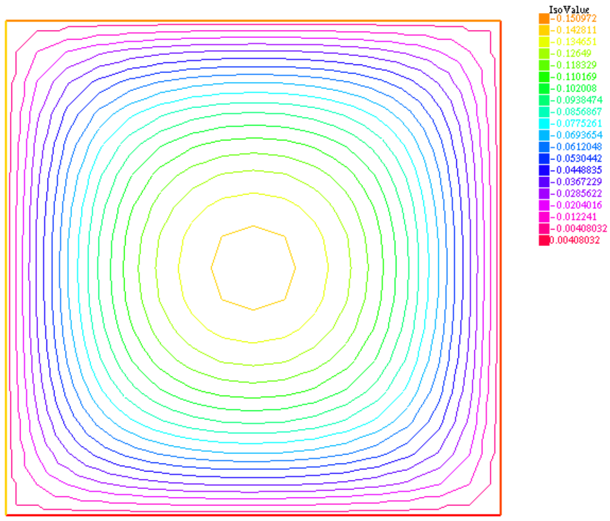


Figure 7.5: Iso-values of the reference solution u for $f = -x$ and $\psi = -0.5$ computed by the penalty method.



(a) Iso-values of u_h for $h = 0.25$.



(b) Iso-values of u_h for $h = 0.625$.

Figure 7.6: Iso-values of u_h for $h = 0.25$ and $h = 0.625$, where $f = -2$ and $\psi = (x - x^2)(y - y^2)$ computed by the penalty method.

7.2 Numerical results for the simplified Signorini problem

In this section, we turn our attention to the numerical results for the simplified Signorini problem (4.25). Recalling the variational form of the simplified Signorini problem (4.27), we have: Find $u \in \mathbb{K} = \{v \in H^1(\Omega) \mid v \geq 0 \text{ on } \Gamma = \partial\Omega\}$, such that

$$\int_\Omega \nabla u \cdot \nabla(v-u) + u(v-u)\, dx \geq \int_\Omega f(v-u)\, dx + \int_\Gamma g(v-u)\, ds \quad \text{for all } v \in \mathbb{K}, \tag{7.11}$$

where f and g are sufficiently smooth. The discrete formulation is:
Find $u_h \in \mathbb{K}_h = \{v_h \in V_h \mid v_h(x_{node}) \geq 0$ for every node x_{node} on the boundary $\Gamma\}$, such that

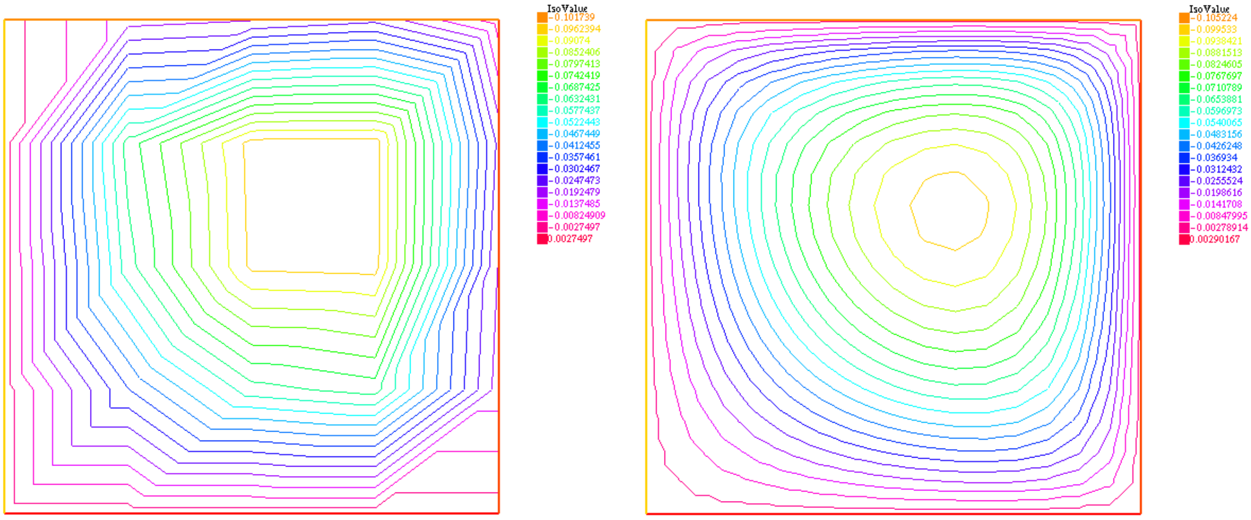
$$\int_\Omega \nabla u_h \cdot \nabla(v_h-u_h) + u_h(v_h-u_h)\, dx \geq \int_\Omega f(v_h-u_h)\, dx + \int_\Gamma g(v_h-u_h)\, ds \quad \text{for all } v_h \in \mathbb{K}_h. \tag{7.12}$$

We solve the simplified Signorini problem only with the help of the penalty method. For our example, we choose $f = -2x^2 - 2y^2 + x^2y^2 - xy$ and $g = 0$. This time, the Hilbert space Q in (7.7) denotes the space $L_2(\Gamma)$. The computational domain is the unit square $\overline{\Omega} = [0,1] \times [0,1]$. As a slight simplification, we fixed two boundary sides of the domain, i.e. $u = 0$, and computed the reference solution on a 250×250 grid using P_2-elements. Solving the discrete problem (7.12) with the penalty method for different mesh sizes h and with P_1-elements, we can observe in Table 7.5 the errors and convergence rates with respect to the reference solution u.

dof	h	$\|u-u_h\|_1$	r_1	$\|u-u_h\|_0$	r_0
9	0.5	0.216582		0.0210473	
25	0.25	0.123927	0.805414	0.00525725	2.00126
81	0.125	0.0663637	0.90103	0.00139896	1.90995
289	0.0625	0.0341887	0.956877	0.000364405	1.94074
1089	0.03125	0.0172744	0.984879	0.000102346	1.83209
4225	0.015625	0.00866511	0.995348	5.91655e-05	0.790628
16641	0.0078125	0.00434309	0.996497	5.76312e-05	0.0379065
66049	0.00390625	0.00218799	0.989117	5.81437e-05	-0.0127735

Table 7.5: Errors for different mesh sizes h and their convergence rates r_i for $f = -2x^2 - 2y^2 + x^2y^2 - xy$ and $g = 0$ with respect to H^1- norm and L_2-norm, respectively.

From Table 7.5 it can be deduced that the dependency on the mesh size h of the error agrees with the a-priori estimates of Example 6.10, i.e. $\|u-u_h\|_1 \leq ch$ and $\|u-u_h\|_0 \leq ch^2$. The error and convergence rates with respect to the L_2-norm in the last three lines of Table 7.5 do not give expected values due to the accuracy issue as explained in Remark 7.4. Lastly, we plot the reference solution u in Figure 7.8 and the discrete solutions u_h for different mesh sizes h in Figure 7.7. It can be observed that the discrete solution converges to the reference solution and that the iso-values are very close to each other.



(a) Iso-values of u_h for $h = 0.25$.

(b) Iso-values of u_h for $h = 0.625$.

Figure 7.7: Iso-values of u_h for $h = 0.25$ and $h = 0.625$, where $f = -2x^2 - 2y^2 + x^2y^2 - xy$ and $g = 0$ computed by the penalty method.

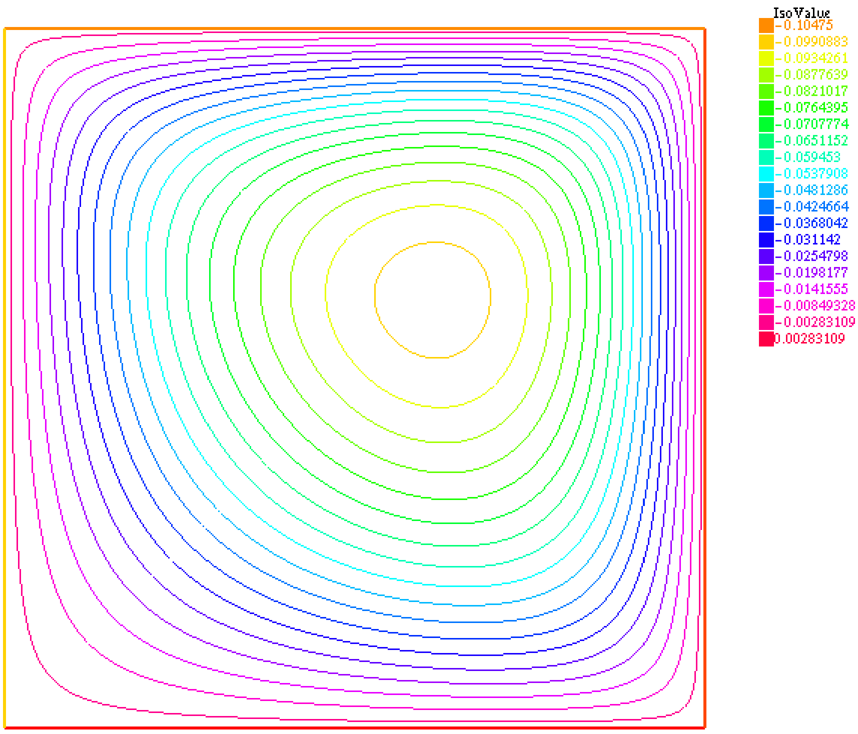


Figure 7.8: Iso-values of the reference solution u for $f = -2x^2 - 2y^2 + x^2y^2 - xy$ and $g = 0$ computed by the penalty method.

7.3 A more realistic example

The aim of this section is to use the mathematical model (5.70) for frictional contact, or rather, its variational formulation (5.76), to consider the contact of a more realistic three-dimensional problem. We investigate the contact between an elastic body Ω and a rigid medium $\mathcal{F}$ (rigid plate). The elastic body Ω has a cube shape with edge side length A equal to $A = 2 \cdot 10^{-4}$ and is centered at the origin of the axes. The rigid plate $\mathcal{F}$ has the shape of a parallelepiped

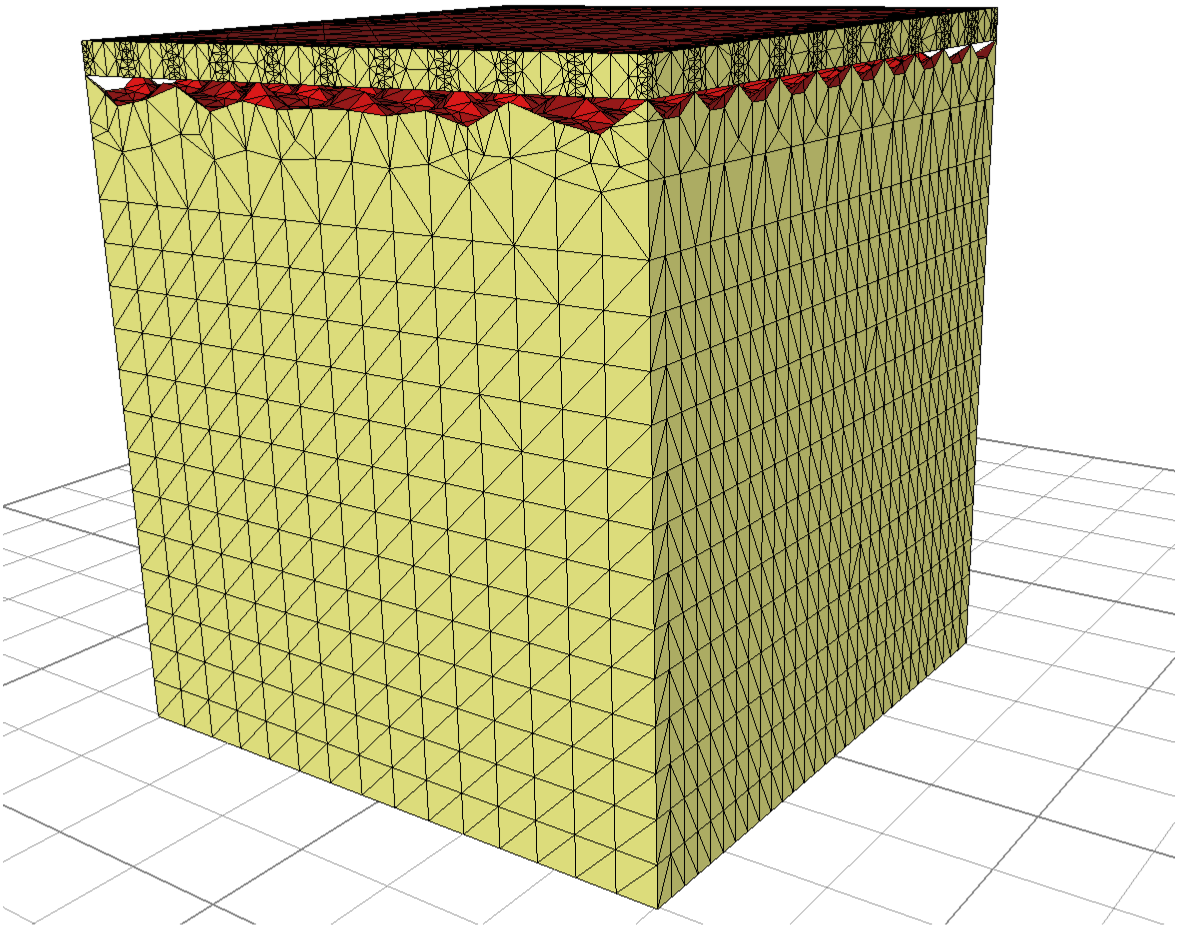


Figure 7.9: The elastic body Ω (lower medium) possesses asperities which comes in contact with the rigid plate $\mathcal{F}$ (upper medium).

with length in x- and y- directions equal to $A_p = 2 \cdot 10^{-3}$ and length in z-direction equal to $A_p^z = 10^{-6}$. The two mediums are parallel with respect to xy plane and the elastic body Ω possesses asperities on the surface boundary, which comes into contact with the rigid plate, as illustrated in Figure 7.9. We consider the simple case, where the contact boundary part of the rigid medium is parallel to the xy plane and the contact boundary part of the elastic medium can be a rough surface, i.e. asperities. For our purpose, the rough surface of the elastic body is described by

$$S(x, y) = 0.1A \cos\left(\frac{5(x+y)\pi}{A}\right) \cos\left(\frac{12y\pi}{A}\right), \tag{7.13}$$

as visualized in Figure 7.10. The two contact surfaces are parallel to each other with maximum distance defined by the gap function $g = 0.1A$. The contact area between these two surfaces are determined by the contact points which have zero distance, i.e. the peaks of the asperities are in touch with the rigid plate.

In our problem, the rigid plate above the elastic material is moving in x-direction with velocity $v_1 = 0.01A$. Thus, the contact surfaces move relative to each other. Due to this motion, friction forces (shear stresses) are developed across the contact surfaces, which, in turn, cause a deformation of the lower elastic body and a deformation of the asperities in a boundary layer in the contact region. This deformation is described by the mathematical model (5.70). We solve this problem numerically, considering its variational formulation (5.76), and compute the maximal displacement u for the points on the surface S, for different friction

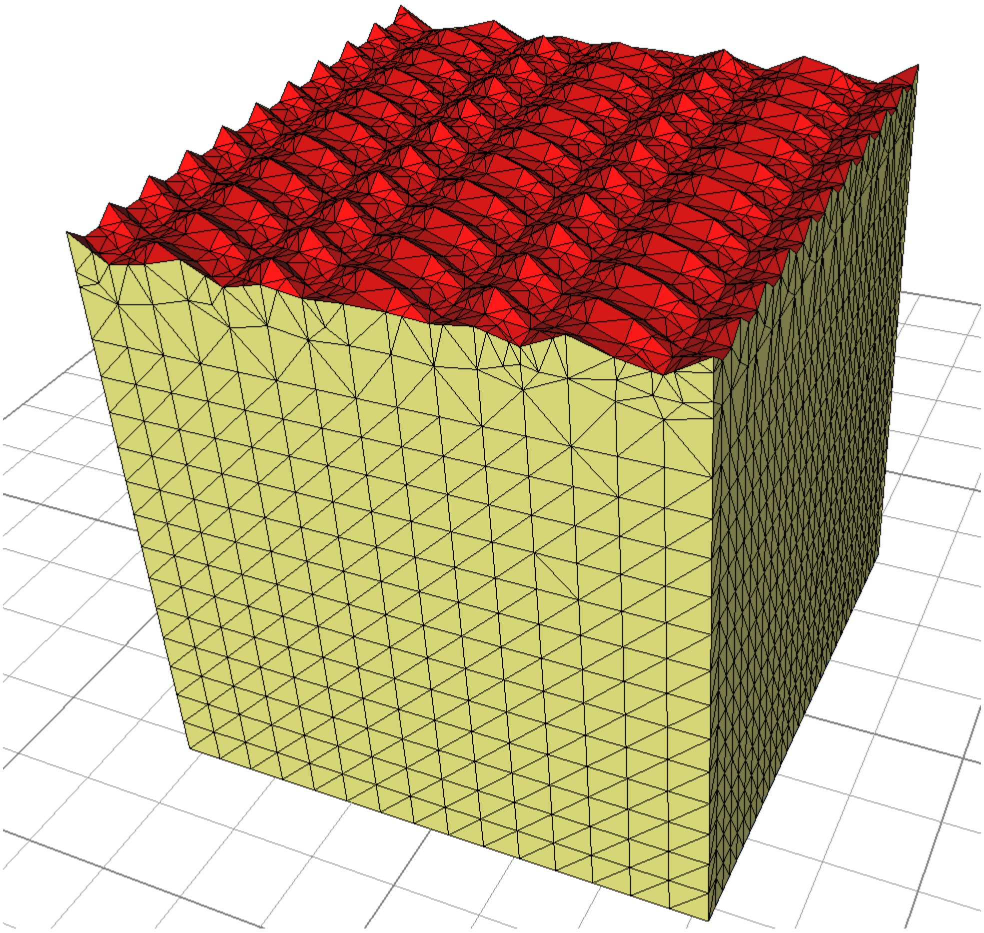


Figure 7.10: Rough surface, i.e. asperities, on the contact boundary part of the elastic body Ω.

coefficients ν_F. Furthermore, we always consider Young's Modulus E and Poisson's ration ν to be equal to $E = 69.5 \cdot 10^4$ and $\nu = 0.29$, and compute the Lamé coefficients as in (3.38). In addition, we consider $\sigma_n(u) = -1$, so $\sigma_n(u) \leq 0$ holds, for our computations. We have computed the maximal displacements u for different values of the friction coefficients ν_F. The results are displayed in Table 7.6.

Input data	Output data
Friction coefficient ν_F	Maximal displacement u
0.3	1.56028e-05
0.4	1.87371e-05
0.47	1.97386e-05
0.53	4.15374e-05

Table 7.6: Maximal displacements u of the asperity points for different friction coefficients ν_F.

We can observe from Table 7.6 that higher friction coefficients result in higher maximal displacements. This is physically reasonable, since materials with lower friction coefficients have slippery surfaces, therefore the rigid plate slides over the asperities without pulling them with it strongly. On the other hand, if the friction coefficient is high, then the elastic material is more afflicted with the rigid plate, which causes higher displacements. Additionally, we observe that the variations of the displacement do not follow a linear relation with respect to the variations of the friction coefficients. In Figure 7.11 we can see the initial configuration of the elastic body Ω and its deformed configuration after the movement of the rigid plate.

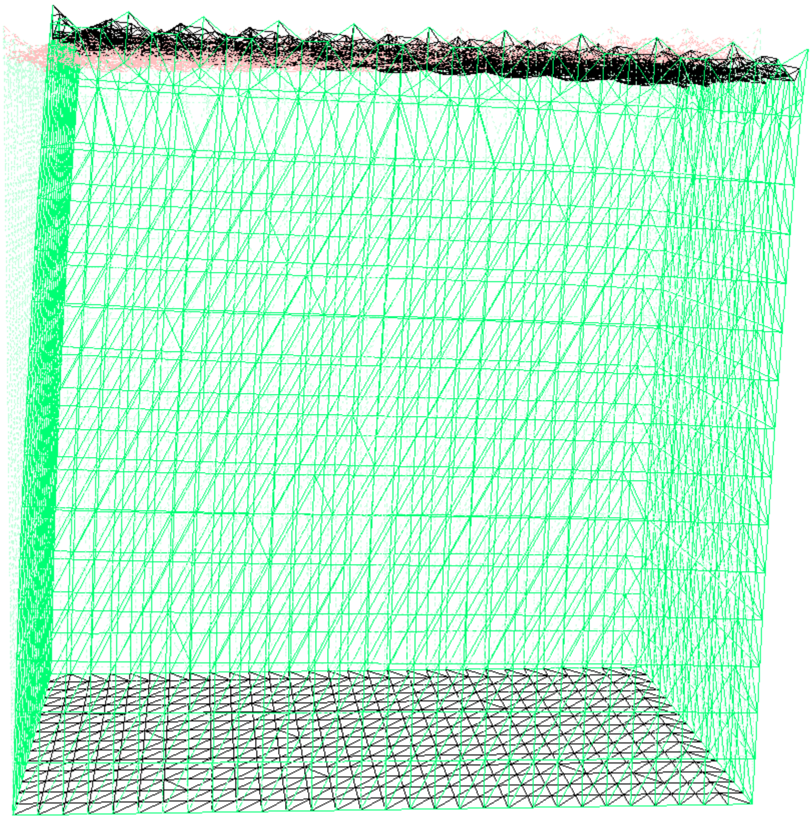


Figure 7.11: The elastic body Ω deforms relative to the movement of the rigid plate for $\nu_F = 0.3$. Both, the original and deformed shape of Ω are illustrated for comparison.

Chapter 8

Conclusion and Outlook

8.1 Conclusion

In this thesis, we focused on the analysis of abstract elliptic variational and hemi-variational inequalities. The motivation was the classical Signorini problem (4.25), where we derived its variational formulation (4.27). We saw that the contact conditions led to restrictions in the mathematical model, which turned out to be in the form of inequalities and defining the solution on an admissible set. Furthermore, we obtained a generalized form of variational inequalities (5.35). We successfully answered the question about the existence of a unique solution under strong assumptions as strong monotonicity for the operator A and convexity and lower semi-continuity for the functional j, see Theorem 5.38. Additionally, we provided an analysis of variational inequalities derived from minimization problems, c.f. Theorem 5.31. The variational formulation (5.35) can be associated with frictionless or simplified frictional contact problems. However, in order to provide frictional contact problems, hemi-variational inequalities are essential. They are characterized by the dependence of the functional j on the solution. We introduced elliptic hemi-variational inequalities (5.47) and answered the question about a unique solution as well, see Theorem 5.57. For this purpose, we considered a strong assumption imposed on the functional j, which was (5.49). We examined an application of hemi-variational inequalities (5.70), where we discovered that frictional problems are by far no simply solvable problems and that under very strong assumptions on the data a solution can found. In many works, c.f. [6], [16] and [18], this problem has been investigated and they faced a lot of difficulties.

In addition, we considered the Finite Element discretization of elliptic variational and hemi-variational inequalities. For both cases, we obtained the convergence of the discrete solution to the exact solution, c.f. Theorem 6.1 and Theorem 6.14, and derived discretization error estimates in Theorem 6.4 and Theorem 6.16. Unfortunately, solving the variational inequality numerically is associated with lots of computational difficulties. Hence, the elaborated convergence theory is hard to verify. Recent works of [34], [35], [41] and [45] address this problem and introduce different methods for solving variational inequalities efficiently.

Lastly, we considered simplified contact problems, i.e. the obstacle problem (7.1) and the simplified Signorini problem (7.11). We verified the error dependency on the mesh size h derived in Example 6.9 and Example 6.10 for each problem. We introduced two different numerical methods, i.e. the dual method by means of saddle point problems and the penalty method. We observed similar results for the error behavior solving the obstacle problem with these methods. The last numerical example was dedicated to a more realistic frictional contact problem, where we investigated the connection between the displacement and friction coefficient using the mathematical model for the Signorini problem with Coulomb friction (5.70).

8.2 Outlook

Considering this thesis, there are many possibilities for future work and research. Firstly, we always considered the contact between an elastic body and a rigid medium. In the literature, this type of contact is usually called unilateral contact. If both bodies are elastic, then the deformation of both bodies must be considered and the contact condition becomes more complicated. However, this type of contact problem can be also characterized by variational inequalities.

The analysis of variational inequalities presented in this work has been done for real-valued Hilbert spaces. It is possible to extend this analysis to vector-valued spaces, in order to describe realistic three-dimensional contact problems. However, as we have already remarked, the numerical treatment of more-dimensional contact problems involves a lot of mathematical difficulties for complicated contact conditions such as friction. In general, frictional contact problems are of special interest and many researchers focus on the numerical simulation of these type of contact problems. Numerical methods for solving hemi-variational inequalities require deep mathematical investigation. The computation of an appropriate solution for non-simplified contact problems is still an open research field.

Another possible outlook is to extend the analysis of this work for another types of contact. In terms of tribology, not only firctional contact is of special importance, but also lubrication or wear are essential in engineering applications. Similar mathematical models can be introduced and it is possible to find analogies to this thesis.

In this work, we did not go into detail about Finite Element solvers for variational inequalities. We used well known methods as the penalty method or the dual method by means of the saddle point problem, for solving the variational inequalities. A direct continuation to this work would be the investigation of different Finite Element solvers and an approach to combine parallelization or adaptive methods with the Finite Element method.

Furthermore, the examination of variational inequalities can be enlarged by considering time dependent contact problems, i.e. hyperbolic variational inequalities. Time-dependent problems do not only complicate the mathematical model, but also involve more numerical difficulties compared to elliptic variational inequalities.

Finally, due to the actual research about Artificial Intelligence (AI), there are several approaches to combine the Finite Element methods with AI systems. A possible experiment would be to go into this direction and solve variational inequalities by Finite Element methods in terms of AI systems. However, this requires the correct connection between the mathematical foundation and the AI systems.

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Ich erkläre an Eides statt, dass ich die vorliegende Masterarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.
Die vorliegende Masterarbeit ist mit dem elektronisch übermittelten Textdokument identisch.

Linz, August 2020

Mario Gobrial