On the nonreal eigenvalues of elliptic differential operators with indefinite weights on Lipschitz domains

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The nonreal spectrum of a second order elliptic differential operator with an indefinite weight function on a Lipschitz domain is investigated with the help of Krein space techniques and perturbation methods.

1 Introduction

In this note the spectral properties of formally symmetric uniformly elliptic second order differential operators with indefinite weights on unbounded Lipschitz domains are studied. The domain Ω is decomposed into two subdomains $\Omega_\pm$ with Lipschitz boundaries and it is assumed that the weight function is the difference of the characteristic functions of the subdomains. Equipped with Dirichlet boundary conditions on $\partial\Omega$ the associated differential operator is selfadjoint with respect to an indefinite inner product on $L^2(\Omega)$. By applying perturbation and coupling techniques the qualitative spectral properties of the differential operator can be described with the help of the orthogonal sum of selfadjoint differential operators with Dirichlet boundary conditions on the boundaries $\partial\Omega_\pm$ of the subdomains $\Omega_\pm$. In particular, it will be shown that the nonreal spectrum of the indefinite elliptic differential operator consists only of normal eigenvalues which may accumulate to certain subsets of $\overline{\mathbb{R}}$.

2 Spectra of elliptic operators with indefinite weights

Let $\Omega \subseteq \mathbb{R}^n$ be an in general unbounded domain which is decomposed in two subdomains Ω_+ and Ω_- such that $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$ is bounded. We assume that the boundaries $\partial\Omega$, $\partial\Omega_+$ and $\partial\Omega_-$ can be parametrized by finitely many Lipschitz functions. The function $\mathrm{sgn}_\Omega(x) := \pm 1$, $x \in \Omega_\pm$, will play the role of an indefinite weight for the elliptic operator defined below. Furthermore, it is assumed that there exists a function $\beta \in C^\infty(\Omega)$, $0 \le \beta \le 1$, β and all its derivatives are bounded, such that β vanishes in an open neighborhood $\widetilde{\omega} \subseteq \Omega$ of Γ and $\beta = 1$ in $\Omega\backslash\omega$, where $\omega \subseteq \Omega$ has a Lipschitz boundary and is a bounded open neighborhood of the closure of $\widetilde{\omega}$ in Ω.

We consider the differential expression Λ and the corresponding Dirichlet form $\mathfrak{a}$ given by

$$\Lambda u = -\sum_{i,j=1}^{n} D^{e_i}(a_{ij} D^{e_j} u) + au \quad \text{and} \quad \mathfrak{a}[u,v] = \int_\Omega \sum_{i,j=1}^{n} a_{ij} D^{e_j} u \overline{D^{e_i} v} + au\overline{v}\, d^n x, \tag{1}$$

where $a \in L^\infty(\Omega)$ is real valued, the functions $a_{ij} = \overline{a_{ji}}$ are bounded and Lipschitz continuous for all $1 \le i, j \le n$ and D^{e_i} denotes the derivative with respect to x_i. Furthermore, the ellipticity condition $\sum_{i,j=1}^{n} a_{ij}\overline{\xi_i}\xi_j \ge E\,|\,\xi\,|^2$ is assumed to hold for some $E > 0$ and all $\xi \in \mathbb{C}^n$. The Dirichlet form $\mathfrak{a}$ defined on the Sobolev space $W_0^{1,2}(\Omega)$ (the closure of the test funtions in the $W^{1,2}$-norm) is a densely defined closed symmetric sesquilinear form in $L^2(\Omega)$ semibounded from below, cf. [7, 13]. Then the corresponding selfadjoint operator $Au = \Lambda u$, $\mathrm{dom}\, A = \{u \in W_0^{1,2}(\Omega) : \Lambda u \in L^2(\Omega)\}$ in $L^2(\Omega)$ is semibounded from below and we will assume in the following that the essential spectrum $\sigma_{\mathrm{ess}}(A)$ of A is nonempty. Our main purpose is to study the spectral properties of the indefinite elliptic operator

$$Tu = \mathrm{sgn}_\Omega A u = \mathrm{sgn}_\Omega \left(-\sum_{i,j=1}^{n} D^{e_i}(a_{ij} D^{e_j} u) + au \right), \qquad u \in \mathrm{dom}\, T = \mathrm{dom}\, A. \tag{2}$$

We mention that T is selfadjoint with respect to the Krein space inner product $[\cdot,\cdot] := (\mathrm{sgn}_\Omega \cdot, \cdot)$ in $L^2(\Omega)$. Ordinary and partial differential operators with indefinite weights were studied with the help of Krein space techniques in, e.g., [3, 5, 6, 8–10, 14].

The following theorem is the main result in this note. Here $\rho(T)$ denotes the resolvent set of T. The case that the essential spectrum of A is empty or contained in $(0,\infty)$ can be treated with the same methods as in [6] and [12].

Theorem 2.1 *Suppose $\rho(T) \neq \varnothing$ and let $\mu := \min \sigma_{\mathrm{ess}}(A) \le 0$. Then $\sigma_{\mathrm{ess}}(T) \subseteq \mathbb{R}$ and $\sigma(T) \cap (\mathbb{C}\backslash\mathbb{R})$ consists of normal eigenvalues with only possible accumulation points in $[\mu, -\mu] \cup \{\infty\}$.*

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P r o o f. Let $\mathfrak{a}_+, \mathfrak{a}_-$ and A_+, A_- be the Dirichlet forms and corresponding selfadjoint operators, respectively, in $L^2(\Omega_+)$ and $L^2(\Omega_-)$ associated to the restriction of the differential expression Λ onto Ω_+ and Ω_-. Then the orthogonal sum $A_+ \oplus A_-$ is a selfadjoint operator in $L^2(\Omega)$ and the assertions follow from general perturbation results for selfadjoint operators in Krein spaces if we show that

$$(A - \lambda)^{-1} - (A_+ \oplus A_- - \lambda)^{-1} \tag{3}$$

is a compact operator for one (and hence for all) $\lambda \in \rho(A) \cap \rho(A_+ \oplus A_-)$. In fact, the compactness of (3) implies $\sigma_{\mathrm{ess}}(A_+) \subseteq [\mu, \infty)$ and $\sigma_{\mathrm{ess}}(-A_-) \subseteq (-\infty, -\mu]$ and according to [2, Proposition 2.3] the operator $(T - \nu)^{-1} - (A_+ \oplus (-A_-) - \nu)^{-1}$ is compact for all $\nu \in \rho(T) \cap \rho(A_+ \oplus (-A_-))$. Therefore, $\sigma_{\mathrm{ess}}(T) \subseteq \mathbb{R}$ and by [2, Theorem 2.4] the nonreal eigenvalues of T can not accumulate to the intervals $(-\infty, \mu)$ and $(-\mu, \infty)$. We refer the reader to [11] for a detailed study of the general class of operators involved here.

In the sequel we show that (3) is a compact operator. Observe first that $A_+ \oplus A_-$ is the selfadjoint operator associated to the form $\mathfrak{a}_+ \oplus \mathfrak{a}_-$ defined on $W_0^{1,2}(\Omega_+) \oplus W_0^{1,2}(\Omega_-)$ and that $\mathfrak{a}_+ \oplus \mathfrak{a}_- \subseteq \mathfrak{a}$ holds. The ellipticity condition and the assumption $a \in L^\infty(\Omega)$ implies that the forms $\mathfrak{a}$, $\mathfrak{a}_+$, $\mathfrak{a}_-$ and the associated operators A, A_+ and A_- are bounded from below by some constant $\eta \in \mathbb{R}$. Let us fix some $\nu \in \mathbb{R}$, $\nu < \eta$. Then $\mathrm{dom}(\mathfrak{a})$ equipped with the scalar product $\mathfrak{a} - \nu$ is a Hilbert space. Let $U_\nu = (\mathrm{dom}(\mathfrak{a}_1 \oplus \mathfrak{a}_2))^{\perp_{\mathfrak{a}-\nu}} \subseteq \mathrm{dom}(\mathfrak{a})$ and denote by $\mathfrak{b}_\nu$ the restriction of $\mathfrak{a}$ to U_ν. The closure of U_ν with respect to $\|\cdot\|_{L^2(\Omega)}$ will be denoted by $\overline{U}_\nu$. If B_ν denotes the selfadjoint operator associated to $\mathfrak{b}_\nu$ in the Hilbert space $(\overline{U}_\nu, (\cdot,\cdot)_{L^2(\Omega)})$, then a similar reasoning as in [1] implies the representation

$$(A - \nu)^{-1} - (A_1 \oplus A_2 - \nu)^{-1} = \begin{bmatrix} 0 & & \\ & (B_\nu - \nu)^{-1} & \\ & & 0 \end{bmatrix} \tag{4}$$

with respect to the decomposition

$$L^2(\Omega) = \mathrm{ran}(S - \nu) \oplus \overline{U}_\nu \oplus \Big(\mathrm{dom}(B_\nu)^{\perp_{L^2(\Omega)}} \cap \ker(S^* - \nu) \Big).$$

Here S denotes the orthogonal sum of the closed minimal operators associated to Λ in $L^2(\Omega_+)$ and $L^2(\Omega_-)$, respectively. Hence it remains to show that $(B_\nu - \nu)^{-1}$ is a compact operator in the Hilbert space $\overline{U}_\nu$. Let $(v_k)_k \subseteq \overline{U}_\nu$ be a bounded sequence and set $u_k = (B_\nu - \nu)^{-1} v_k$. Since $B_\nu \geq \eta > \nu$ we have $\nu \in \rho(B_\nu)$ and hence $(u_k)_k$ is bounded with respect to $\|\cdot\|_{L^2(\Omega)}$. As a consequence of the ellipticity condition and the first representation theorem in [13] one obtains

$$\| u_k \|^2_{W^{1,2}(\Omega)} \leq c_1 (\mathfrak{b}_\nu - \nu)[u_k, u_k] = c_1 ((B_\nu - \nu) u_k, u_k)_{L^2(\Omega)} \leq c_1 \left\| (B_\nu - \nu)^{-1} \right\| \| v_k \|^2_{L^2(\Omega)}$$

for some constant $c_1 > 0$ independent of u_k, i.e., $(u_k)_k$ is also bounded in $W^{1,2}(\Omega)$. By $\| u_k|_\omega \|_{W^{1,2}(\omega)} \leq \| u_k \|_{W^{1,2}(\Omega)}$ also the restrictions $u_k|_\omega$ of u_k to ω are bounded. Since ω is bounded and has a Lipschitz boundary, the embedding $\mathrm{id} : W^{1,2}(\omega) \to L^2(\omega)$ is compact, which yields a convergent subsequence $(u_{k_j}|_\omega)_j$ with respect to $\|\cdot\|_{L^2(\omega)}$. As in [4] one verifies that the estimate $\int_\Omega |u|^2 \, d^n x \leq c \int_\omega |u|^2 \, d^n x$ holds for some $c > 0$ and all $u \in U_\nu$. Thus $(u_{k_j})_j$ is a convergent subsequence of $(u_k)_k$ in $L^2(\Omega)$ and it follows that (4), and hence also (3), is compact. □

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