Boundary triples and quasi boundary triples for elliptic operators

Jussi Behrndt[1]* and **Till Micheler**[2]

[1] Institut für Numerische Mathematik, Technische Universität Graz, Steyergasse 30, 8010 Graz, Austria
[2] Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

The abstract concepts of boundary triples and their recent generalizations are useful tools to parametrize the self-adjoint and maximal dissipative/maximal accumulative extensions of formally symmetric elliptic differential expressions with the help of explicit boundary conditions. In the present note the parametrizations induced by the "natural" quasi boundary triple with the Dirichlet and Neumann trace as boundary maps are compared with the parametrizations induced by a "classical" ordinary boundary triple, where a regularized Neumann trace is used for one of the boundary maps.

1 Boundary triples, quasi boundary triples and Weyl functions

The notion of quasi boundary triples and their Weyl functions was introduced in [2] in connection with boundary value problems for elliptic differential equations. This concept is a generalization of the notions of ordinary and generalized boundary triples which appear frequently in extension and spectral theory of symmetric and selfadjoint operators. In the following we recall the relevant definitions and refer the reader to [2, 5, 6] for further details and references.

Definition 1.1 Let A be a densely defined closed symmetric operator in a Hilbert space $\mathcal{H}$ and let $T \subset A^*$ be a linear operator such that $\overline{T} = A^*$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called *quasi boundary triple* for A^* if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings such that

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

holds for all $f, g \in \operatorname{dom} T$, the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range, and $A_0 := T \upharpoonright \ker \Gamma_0$ is self-adjoint in $\mathcal{H}$. In the case $T = A^*$ a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called *ordinary boundary triple*.

Note that a quasi or ordinary boundary triple for A^* exists if and only if A has equal defect numbers, and that for the case of finite defect numbers both notions coincide; cf. [2]. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for A^* and let A_0 be the corresponding selfadjoint extension of A defined on $\ker \Gamma_0$. Since for $\lambda \in \rho(A_0)$ each element $f \in \operatorname{dom} T$ can be decomposed uniquely in $f = f_0 + f_\lambda$, where $f_0 \in \operatorname{dom} A_0$ and $f_\lambda \in \mathcal{N}_\lambda(T) = \ker(T - \lambda)$, the map $\Gamma_0 \upharpoonright \mathcal{N}_\lambda(T)$ is invertible. The *Weyl function* M corresponding to the quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is defined by $M(\lambda) := \Gamma_1(\Gamma_0 \upharpoonright \mathcal{N}_\lambda(T))^{-1}$, $\lambda \in \rho(A_0)$. Observe that the values $M(\lambda)$, $\lambda \in \rho(A_0)$, of the Weyl function are linear operators in $\mathcal{G}$ with $\operatorname{dom} M(\lambda) = \operatorname{ran} \Gamma_0$.

2 Quasi boundary triples for second order elliptic differential operators in $L^2(\Omega)$

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $\mathcal{C}^\infty$-domain with boundary $\partial\Omega$. We denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces of order $s \in \mathbb{R}$ on Ω and $\partial\Omega$, respectively, and with $H^s_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. Recall that there exist isometric isomorphisms $\iota^\pm : H^{\pm 1/2}(\partial\Omega) \to L^2(\partial\Omega)$ such that $(\iota^- x', \iota^+ x)_{L^2(\partial\Omega)} = \langle x', x \rangle_{-1/2 \times 1/2}$ holds for all $x' \in H^{-1/2}(\partial\Omega)$, $x \in H^{1/2}(\partial\Omega)$; here $\langle \cdot, \cdot \rangle_{-1/2 \times 1/2}$ denotes the extension of the $L^2(\partial\Omega)$ inner product to $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$. For $s \in \mathbb{R}$ the restriction or extension of ι^+ (ι^-) from $H^s(\partial\Omega)$ onto $H^{s-1/2}(\partial\Omega)$ (from $H^{s-1/2}(\partial\Omega)$ onto $H^s(\partial\Omega)$, respectively) is an isometric isomorphism which is denoted by ι^+_s (ι^-_s, respectively). Note that $\iota^-_s \iota^+_s = \mathrm{Id}_{H^s(\partial\Omega)}$ and $\iota^+_s \iota^-_s = \mathrm{Id}_{H^{s-1/2}(\partial\Omega)}$ hold.

In the following we consider differential operators in $L^2(\Omega)$ which are realizations of the differential expression

$$\mathscr{L} := -\sum_{j,\,k=1}^{n} \partial_j a_{jk} \partial_k + a,$$

where $a_{jk} \in \mathcal{C}^\infty(\overline{\Omega})$ are real valued and satisfy $a_{jk} = a_{kj}$, $1 \leq j, k \leq n$, $a \in L^\infty(\Omega)$ is real valued, and the ellipticity condition $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c \sum_{k=1}^n \xi_k^2$ holds for some $c > 0$ and all $\xi \in \mathbb{R}^n$, $x \in \Omega$. Set $H^s_{\mathscr{L}}(\Omega) := \{f \in H^s(\Omega) \mid \mathscr{L} f \in L^2(\Omega)\}$, $s \in [0, 2]$. The *minimal operator* corresponding to $\mathscr{L}$ is defined as $A := \mathscr{L} \upharpoonright H^2_0(\Omega)$. It is well known that A is a densely defined closed symmetric operator with infinite defect in $L^2(\Omega)$ and that the adjoint operator is the *maximal operator* $A^* = \mathscr{L} \upharpoonright H^0_{\mathscr{L}}(\Omega)$. For $s \in [0, 2]$ let $T_s := \mathscr{L} \upharpoonright H^s_{\mathscr{L}}(\Omega)$, so that $\overline{T_s} = A^*$ holds; cf. [9]. Note that the Dirichlet operator $A_D := \mathscr{L} \upharpoonright H^2(\Omega) \cap H^1_0(\Omega)$ is a self-adjoint restriction of T_s, $s \in [0, 2]$.

* Corresponding author: Email behrndt@math.tugraz.at, phone +43 316 873 8127, fax +43 316 873 8621

In the next proposition we define a quasi boundary triple for T_s, for $s = 0$ see also [8] and [1, 3, 4, 7]. For this we fix $\eta \in \rho(A_D) \cap \mathbb{R}$ and decompose $f \in \operatorname{dom} T_s$ in the form $f = f_D + f_\eta$, where $f_D \in \operatorname{dom} A_D$ and $f_\eta \in \mathcal{N}_\eta(T_s) = \ker(T_s - \eta)$. We denote by $f\restriction_{\partial\Omega}$ and $\partial_N f\restriction_{\partial\Omega} := \sum_{j,k=1}^n a_{jk}\mathfrak{n}_j \partial_k f\restriction_{\partial\Omega}$ the trace and (oblique) Neumann trace of a function f on Ω. Here $\mathfrak{n}_j$, $j = 1, \ldots, n$, are the components of the unit vector $\mathfrak{n}$ pointing out of Ω.

Proposition 2.1 *Let $s \in [0, 2]$ and T_s be defined as above. Then $\{L^2(\partial\Omega), \Gamma_0^s, \Gamma_1^s\}$, where $\Gamma_0^s, \Gamma_1^s : H^s_{\mathscr{L}}(\Omega) \to L^2(\partial\Omega)$ are defined by $\Gamma_0^s f := \iota_s^- f\restriction_{\partial\Omega}$ and $\Gamma_1^s f := -\iota^+ \partial_N f_D\restriction_{\partial\Omega}$ is a quasi boundary triple for A^* such that*

$$\ker \Gamma^s = \operatorname{dom} A, \quad \ker \Gamma_0^s = \operatorname{dom} A_D, \quad \ker \Gamma_1^s = \operatorname{dom} A \dot{+} \mathcal{N}_\eta(T_s), \quad \operatorname{ran} \Gamma^s = H^s(\partial\Omega) \times L^2(\partial\Omega)$$

hold. The quasi boundary triple $\{L^2(\partial\Omega), \Gamma_0^s, \Gamma_1^s\}$ is an ordinary boundary triple if and only if $s = 0$.

When dealing with elliptic boundary value problems it is more appropriate to use the Neumann trace for the boundary map Γ_1^s instead of the regularized Neumann trace in the above proposition. In order to generate a quasi boundary triple such that the Weyl function is the usual Dirichlet-to-Neumann map (up to a minus sign) in $L^2(\partial\Omega)$ it is necessary to restrict the boundary mappings to $H^s_{\mathscr{L}}(\Omega)$ with $s \in [\frac{3}{2}, 2]$; cf. [2, 3].

Proposition 2.2 *Let $s \in [\frac{3}{2}, 2]$ and T_s be defined as above. Then $\{L^2(\partial\Omega), \Upsilon_0^s, \Upsilon_1^s\}$, where $\Upsilon_0^s, \Upsilon_1^s : H^s_{\mathscr{L}}(\Omega) \to L^2(\partial\Omega)$ are defined by $\Upsilon_0^s f := f\restriction_{\partial\Omega}$ and $\Upsilon_1^s f := -\partial_N f\restriction_{\partial\Omega}$ is a quasi boundary triple for $T_s \subset A^*$ such that $\ker \Upsilon^s = \operatorname{dom} A$, $\ker \Upsilon_0^s = \operatorname{dom} A_D$ and*

$$\operatorname{ran} \Upsilon^s = \left\{ \begin{pmatrix} x \\ x' \end{pmatrix} \in H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega) \mid x' = M_s(\eta)\, x + y,\ y \in H^{1/2}(\partial\Omega) \right\},$$

where M_s is the Weyl function of $\{L^2(\partial\Omega), \Upsilon_0^s, \Upsilon_1^s\}$. Moreover, $\operatorname{ran} \Upsilon^s = H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega)$ if and only if $s = 2$.

Let $s \in [\frac{3}{2}, 2]$ and let $\{L^2(\partial\Omega), \Gamma_0^s, \Gamma_1^s\}$ and $\{L^2(\partial\Omega), \Upsilon_0^s, \Upsilon_1^s\}$ be the quasi boundary triples from above. Let Θ and ϑ be linear operators or relations in $L^2(\partial\Omega)$ and define the extensions A_Θ and $\mathcal{A}_\vartheta$ by

$$A_\Theta := \mathscr{L} \restriction \ker(\Gamma_1^s - \Theta\Gamma_0^s) \qquad \text{and} \qquad \mathcal{A}_\vartheta := \mathscr{L} \restriction \ker(\Upsilon_1^s - \vartheta\Upsilon_0^s).$$

For the case that Θ or ϑ is multivalued the above definitions are understood in the sense of linear relations. It follows from Propositions 2.1 and 2.2 that A_Θ and $\mathcal{A}_\vartheta$ are the realizations of $\mathscr{L}$ defined on

$$\operatorname{dom} A_\Theta = \left\{ f \in H^s_{\mathscr{L}}(\Omega) \mid \Theta\iota_s^- f\restriction_{\partial\Omega} + \iota^+ \partial_N f_D\restriction_{\partial\Omega} = 0 \right\} \quad \text{and} \quad \operatorname{dom} \mathcal{A}_\vartheta = \left\{ f \in H^s_{\mathscr{L}}(\Omega) \mid \vartheta f\restriction_{\partial\Omega} + \partial_N f\restriction_{\partial\Omega} = 0 \right\},$$

respectively. In order to compare the boundary conditions for A_Θ and $\mathcal{A}_\vartheta$ define the mapping

$$W_s := \begin{pmatrix} \iota_s^+ & 0 \\ M_s(\eta)\iota_s^+ & \iota_{1/2}^- \end{pmatrix} : H^s(\partial\Omega) \times L^2(\partial\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega)$$

which is an isomorphism from $\operatorname{ran} \Gamma^s$ onto $\operatorname{ran} \Upsilon^s$ such that the equation $W_s\Gamma^s f = \Upsilon^s f$ holds for all $f \in H^s_{\mathscr{L}}(\Omega)$. For a linear operator or relation $\Theta \subset \operatorname{ran} \Gamma^s$ we obtain $W_s(\Theta) = M_s(\eta) + \iota_{1/2}^- \Theta \iota_s^-$, so that $W_s(\Theta) \subset \operatorname{ran} \Upsilon^s$. Also this equation is understood in the sense of relations if Θ is multivalued. It can be shown that $\Gamma_1^s f = \Theta\Gamma_0^s f$ if and only if $\Upsilon_1^s f = W_s(\Theta)\Upsilon_0^s f$, $f \in H^s_{\mathscr{L}}(\Omega)$. In particular, this implies $A_\Theta = \mathcal{A}_{W_s(\Theta)}$.

Theorem 2.3 *Let $s \in [\frac{3}{2}, 2]$. Then the mapping $\Theta \mapsto \mathcal{A}_{W_s(\Theta)} = \mathscr{L} \restriction \ker(\Upsilon_1^s - W_s(\Theta)\Upsilon_0^s)$ induces a bijective correspondence between all closed symmetric, (self-adjoint, (maximal) dissipative, (maximal) accumulative) operators and relations Θ in $L^2(\partial\Omega)$ with $\operatorname{dom} \Theta \subset H^s(\partial\Omega)$ and all closed symmetric (self-adjoint, (maximal) dissipative, (maximal) accumulative, respectively) extensions $\mathcal{A}_{W_s(\Theta)}$ of A with $\operatorname{dom} \mathcal{A}_{W_s(\Theta)} \subset H^s_{\mathscr{L}}(\Omega)$.*

Proof. Let $s \in [\frac{3}{2}, 2]$ and let Θ be a closed symmetric (self-adjoint, (maximal) dissipative, (maximal) accumulative) relation in $L^2(\partial\Omega)$ with $\operatorname{dom} \Theta \subset H^s(\partial\Omega)$. As a consequence of Proposition 2.1 we have $\Theta \subset \operatorname{ran} \Gamma^s \subset \operatorname{ran} \Gamma^0$. Since $\{L^2(\partial\Omega), \Gamma_0^0, \Gamma_1^0\}$ is an ordinary boundary triple for A^* it follows that $A_\Theta = \mathscr{L} \restriction \ker(\Gamma_1^0 - \Theta\Gamma_0^0)$ is a closed symmetric (self-adjoint, (maximal) dissipative, (maximal) accumulative, respectively) extension of A. Furthermore, we have $A_\Theta = \mathcal{A}_{W_s(\Theta)}$ and $\operatorname{dom} \mathcal{A}_{W_s(\Theta)} = \operatorname{dom} A_\Theta \subset \operatorname{dom} \Gamma^s = H^s_{\mathscr{L}}(\Omega)$. The converse implication can be shown with similar arguments. $\square$

Acknowledgements Till Micheler is grateful to the "Studienstiftung des Deutschen Volkes" for financial support.

References

[1] J. Behrndt, J. Differential Equations **249**, 2663–2687 (2010).
[2] J. Behrndt and M. Langer, J. Funct. Anal. **243**, 536–565 (2007).
[3] J. Behrndt and M. Langer, submitted
[4] B. M. Brown, G. Grubb and I.G. Wood, Math. Nachr. **282**, 314–347 (2009).
[5] V. Derkach, S. Hassi, M. Malamud, and H. de Snoo, Trans. Amer. Math. Soc. **358**, 5351–5400 (2006).
[6] V. Derkach and M. Malamud, J. Funct. Anal. **95**, 1–95 (1991).
[7] F. Gesztesy and M. Mitrea, J. Anal. Math. **113**, 53–172 (2011).
[8] G. Grubb, Ann. Scuola Norm. Sup. Pisa **22**, 425–513 (1968).
[9] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications I (Springer, Berlin, 1972).