Scattering Systems and Characteristic Functions

Jussi Behrndt, Mark M. Malamud and Hagen Neidhardt

***Abstract*— The well-known relation between the scattering matrix of the Lax-Phillips scattering theory and the characteristic function of Foias and Sz.-Nagy found by Adamyan and Arov is extended to a scattering theory of singular perturbations which includes the usual ones as a special case.**

***Keywords*— Lax-Phillips scattering theory, singular perturbation, scattering matrix, characteristic function, dissipative operator**

I. INTRODUCTION

For the needs of acoustic scattering a new type of scattering theory was created by Lax and Phillips in [16] which differed essentially from the classical scattering theory, cf. [5], [6], [19]. Instead of an unperturbed and a perturbed selfadjoint operator a Lax-Phillips scattering system consists only of a "perturbed" operator L acting in a separable Hilbert space $\mathfrak{L}$. It is assumed that the operator L admits an incoming subspace $\mathcal{D}_+ \subseteq \mathfrak{L}$ and an outgoing subspace $\mathcal{D}_- \subseteq \mathfrak{L}$ satisfying the conditions

$$\text{(i)}\ \ e^{-itL}\mathcal{D}_\pm \subseteq \mathcal{D}_\pm,\ \pm t \geq 0, \qquad \text{(ii)}\ \bigcap_{t\in\mathbb{R}} e^{-itL}\mathcal{D}_\pm = \{0\},$$

$$\text{(iii)}\ \bigcup_{t\in\mathbb{R}} e^{-itL}\mathcal{D}_\pm = \mathfrak{L} \quad \text{and} \quad \text{(iv)}\ \mathcal{D}_+ \perp \mathcal{D}_-,$$

cf. [6], [16]. An "unperturbed" operator is not explicitly given. It was shown in [17] (see also [6, Theorem 12.3]) that under the above assumptions (i)-(iv) the isometric semigroups

$$U_\pm(t) := e^{-itL} \upharpoonright \mathcal{D}_\pm, \quad \pm t \geq 0,$$

admit a minimal unitary coupling, that is, there exists a unitary group e^{-itT_0}, $t \in \mathbb{R}$, in the Hilbert space

$$\mathfrak{K} := \mathcal{D}_- \oplus \mathcal{D}_+ \subseteq \mathfrak{L}$$

such that

$$U_\pm(t) = e^{-itT_0} \upharpoonright \mathcal{D}_\pm,\ \ \pm t \geq 0 \qquad \text{and} \qquad \bigcup_{t\in\mathbb{R}} e^{-itT_0}\mathcal{D}_\pm = \mathfrak{K}$$

holds. It turns out that the self-adjoint operator T_0 is unitarily equivalent to the differentiation operator $-i\frac{d}{dx}$ in $L^2(\mathbb{R}, \mathfrak{k})$, where $\mathfrak{k}$ is some auxiliary Hilbert space.

The Lax-Phillips wave operators are defined by

$$\Omega_\pm := \text{s-}\lim_{t\to\pm\infty} e^{itL} J_\pm e^{-itT_0} : \mathfrak{K} \longrightarrow \mathfrak{L},$$

where $J_\pm$ is the embedding of the subspaces $\mathcal{D}_\pm$ into $\mathfrak{L}$, and the Lax-Phillips scattering operator S_{LP} is given by

$$S_{LP} := \Omega_+^* \Omega_- : \mathfrak{K} \longrightarrow \mathfrak{K}.$$

Since S_{LP} is unitary and commutes with T_0 one obtains that S_{LP} is unitarily equivalent to a multiplication operator in $L^2(\mathbb{R}, \mathfrak{k})$ induced by a measurable family $\{S_{LP}(\lambda)\}$ of unitary operators which is called the Lax-Phillips scattering matrix.

Denoting by $P^{\mathfrak{L}}_{\mathfrak{H}}$ the orthogonal projection from $\mathfrak{L}$ onto $\mathfrak{H} := \mathfrak{L} \ominus \mathfrak{K}$ one defines a contraction C_0-semigroup by

$$Z(t) := P^{\mathfrak{L}}_{\mathfrak{H}} e^{-itL} \upharpoonright \mathfrak{H}, \qquad t \geq 0,$$

and it follows that there is a completely non-selfadjoint maximal dissipative operator H in $\mathfrak{H}$ such that the representation

$$Z(t) = e^{-itH}, \quad t \geq 0,$$

J. Behrndt is with Institute of Mathematics MA 6-4, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany behrndt@math.tu-berlin.de

M.M. Malamud is with the Department of Mathematics, Donetsk National University, Universitetskaya 24, 83055 Donetsk, Ukraine mmm@telenet.dn.ua

H. Neidhardt is with the Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, D-10117 Berlin, Germany neidhard@wias-berlin.de

holds. We note that L and H are related via

$$P^{\mathfrak{L}}_{\mathfrak{H}}(L-z)^{-1}\upharpoonright\mathfrak{H}=(H-z)^{-1},\quad z\in\mathbb{C}_+. \tag{I.1}$$

By [13] the operator H is determined up to unitary equivalence by its characteristic function $W_H(\cdot):\mathbb{C}_-\to\mathcal{L}(\mathfrak{k})$, where $\mathfrak{k}$ is an auxiliary Hilbert space, see above, and $\mathcal{L}(\mathfrak{k})$ denotes the space of bounded linear operators defined on $\mathfrak{k}$. Recall that $W_H(\cdot)$ is holomorphic and contraction-valued. In [1], [2], [3], [4] it was shown by Adamyan and Arov that the scattering matrix $\{S_{LP}(\lambda)\}$ of Lax and Phillips and the characteristic function $W_H(\lambda-i0)$ of Foias and Sz.-Nagy are related by

$$S_{LP}(\lambda)=W_H(\lambda-i0)^*\quad\text{for a.e. }\lambda\in\mathbb{R}. \tag{I.2}$$

This unexpected connection offers a nice possibility to calculate the Lax-Phillips scattering matrix via the characteristic function of a maximal dissipative operator.

In this note we consider scattering systems $\{L,L_0\}$, where it is assumed that L_0 coincides with the orthogonal sum of two self-adjoint operators A_0 and T_0 in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and that L is special self-adjoint singular perturbation of L_0 in $\mathfrak{L}=\mathfrak{H}\oplus\mathfrak{K}$. Moreover we suppose that L and L_0 are self-adjoint extension of some symmetric operator $A\oplus T$ in $\mathfrak{L}$, where both A and T are one-dimensional restrictions of A_0 and T_0 in $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then we obtain a family $\{H(z)\}_{z\in\mathbb{C}_+}$ of maximal dissipative operators in $\mathfrak{H}$ such that the relation

$$P^{\mathfrak{L}}_{\mathfrak{H}}(L-z)^{-1}\upharpoonright\mathfrak{H}=(H(z)-z)^{-1},\quad z\in\mathbb{C}_+, \tag{I.3}$$

holds, cf. Theorem 3.1. Note that (I.3) can be regarded as an extension of the relation (I.1). The family $\{H(z)\}_{z\in\mathbb{C}_+}$ of extensions of A is called *Strauss family*. Naturally, the question arises whether the family of characteristic functions of the Strauss family is related to the usual scattering matrix $\{S(\lambda)\}$ of the complete scattering system $\{L,L_0\}$ (see [6], [15], [19] and Section III) like the characteristic function of H is related to the Lax-Phillips scattering matrix $\{S_{LP}(\lambda)\}$ by (I.2). In fact, under the additional assumption that the spectrum of A_0 is purely singular, here we are able to prove that the scattering matrix $\{S(\lambda)\}$ admits the representation

$$S(\lambda)=W_{H(\lambda+i0)}(\lambda-i0)^*\quad\text{for a.e. }\lambda\in\mathbb{R},$$

where $H(\lambda+i0)$ is a suitable defined continuation of the Strauss family to $\mathbb{R}$ and $W_{H(\lambda+i0)}(\cdot)$ denotes the characteristic function of $H(\lambda+i0)$.

II. EXTENSION THEORY OF SYMMETRIC OPERATORS

A. Boundary triples and closed extensions

Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A)=\dim\ker(A^*\mp i)\le\infty$. Recall that a triple $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is said to be a *boundary triple* for the adjoint operator A^* if $(\mathcal{H},(\cdot,\cdot))$ is a Hilbert space and $\Gamma_0,\Gamma_1:\ \mathrm{dom}\,(A^*)\to\mathcal{H}$ are linear mappings such that

$$(A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)-(\Gamma_0 f,\Gamma_1 g)$$

for all $f,g\in\mathrm{dom}\,(A^*)$, and the mapping

$$\Gamma:=\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}:\ \mathrm{dom}\,(A^*)\longrightarrow\mathcal{H}\times\mathcal{H}$$

is surjective, see [14] and e.g. [9], [10], [12].

We refer to [9], [10] and [12] for a detailed study of boundary triples and recall only some important facts. If $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a boundary triple for A^*, then the mapping

$$\Theta\mapsto A_\Theta:=\Gamma^{-1}\Theta=\left\{f\in\mathrm{dom}\,(A^*):\begin{pmatrix}\Gamma_0 f\\ \Gamma_1 f\end{pmatrix}\in\Theta\right\}$$

establishes a bijective correspondence between the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ and the set of closed extensions $A_\Theta\subset A^*$ of A. Moreover the extension A_Θ is symmetric (self-adjoint, dissipative, maximal dissipative) if and only if Θ is symmetric (resp. self-adjoint, dissipative, maximal dissipative). Note that in particular the operator $A_0:=A^*\upharpoonright\ker(\Gamma_0)$ is a self-adjoint extension of A. Here a linear relation Θ is called *dissipative* if $\Im\mathrm{m}\,(h',h)\le 0$ for all $(h,h')^\top\in\Theta$ and it is called *maximal dissipative* if it is dissipative and has no dissipative extensions.

Let $\mathcal{N}_\lambda=\ker(A^*-\lambda)$ be the defect subspace of A at the point λ. The operator valued functions

$$\gamma(\cdot):\ \rho(A_0)\to\mathcal{L}(\mathcal{H},\mathfrak{H})\qquad\text{and}\qquad M(\cdot):\ \rho(A_0)\to\mathcal{L}(\mathcal{H})$$

defined by

$$\gamma(\lambda) := \bigl(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\bigr)^{-1}, \quad \lambda \in \rho(A_0), \qquad \text{and} \qquad M(\lambda) := \Gamma_1\gamma(\lambda), \quad \lambda \in \rho(A_0),$$

are called the *γ-field* and the *Weyl function*, respectively, corresponding to the boundary triple Π, cf. [9], [10], [12]. We note that $M(\cdot)$ is a so-called Nevanlinna function with the additional property $0 \in \rho(\Im\mathrm{m}\,(M(\lambda)))$ for $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

The spectrum and the resolvent set of a proper (not necessarily self-adjoint) extension $A_\Theta \subseteq A^*$ of A can be described with the help of the Weyl function. Namely a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ ($\sigma_i(A_\Theta)$, $i = p, c, r$) if and only if $0 \in \rho(\Theta - M(\lambda))$ (resp. $0 \in \sigma_i(\Theta - M(\lambda))$, $i = p, c, r$). Moreover, for $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\bigl(\Theta - M(\lambda)\bigr)^{-1}\gamma(\overline{\lambda})^*$$

holds, see [9], [10], [12]. If $\Theta \in \mathcal{L}(\mathcal{H})$ is a dissipative operator, then the closed extension

$$A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$$

of A is maximal dissipative and $\mathbb{C}_+$ belongs to $\rho(A_\Theta)$. It follows from [11] that the *characteristic function* of A_Θ is given by

$$W_{A_\Theta} : \mathbb{C}_- \to \mathcal{L}(\mathcal{H}_\Theta), \quad \mu \mapsto I_{\mathcal{H}_\Theta} - 2i\sqrt{-\Im\mathrm{m}\,\Theta}\bigl(\Theta^* - M(\mu)\bigr)^{-1}\sqrt{-\Im\mathrm{m}\,\Theta}, \tag{II.1}$$

where $\mathcal{H}_\Theta = \mathrm{clo}\{\mathrm{ran}\,(\Im\mathrm{m}\,(\Theta))\}$.

B. The Strauss family and its characteristic function

Let now A be a densely defined closed symmetric operator with deficiency indices $n_\pm(A) = 1$ and let $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* with corresponding (scalar) Weyl function $M(\cdot)$. Further, let $\tau(\cdot)$ be a scalar Nevanlinna function. The family $\{H(\lambda)\}_{\lambda\in\mathbb{C}_+}$ of maximal dissipative extensions of A defined by

$$H(\lambda) := A^* \upharpoonright \bigl\{f \in \mathrm{dom}\,(A^*) : \Gamma_1 f = -\tau(\lambda)\Gamma_0 f\bigr\},$$

$\lambda \in \mathbb{C}_+$, is called the *Strauss family of* A (associated with the function τ), cf. [18]. It follows from (II.1) that for any $\lambda \in \mathbb{C}_+$ with $\Im\mathrm{m}\,(\tau(\lambda)) \neq 0$ the characteristic function of $H(\lambda)$ is given by

$$W_{H(\lambda)}(\mu) = \frac{\tau(\lambda) + M(\mu)}{\overline{\tau(\lambda)} + M(\mu)}, \quad \mu \in \mathbb{C}_-. \tag{II.2}$$

In the following we make the convention that $W_{H(\lambda)}(\mu) \equiv 1$ if $\Im\mathrm{m}\,(\tau(\lambda)) = 0$.

Since τ is a Nevanlinna function the limit $\tau(\lambda + i0) = \lim_{\epsilon\to+0}\tau(\lambda + i\epsilon)$ from the upper half-plane exists for a.e. $\lambda \in \mathbb{R}$. We set

$$\Sigma^\tau := \bigl\{\lambda \in \mathbb{R} :\ \tau(\lambda) = \lim_{\epsilon\to+0}\tau(\lambda + i\epsilon) \text{ exists}\bigr\}.$$

Then the Strauss family $\{H(\lambda)\}_{\lambda\in\mathbb{C}_+}$ admits a continuation to $\mathbb{C}_+ \cup \Sigma^\tau$ which is also denoted by $H(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \Sigma^\tau$. If $\Im\mathrm{m}\,(\tau(\lambda + i0)) \neq 0$, then the characteristic function $\mu \mapsto W_{H(\lambda)}(\mu)$, $\mu \in \mathbb{C}_-$, will be defined as in (II.2). The next proposition shows that $W_{H(\lambda)}(\lambda - i0)$ exists for a.e. $\lambda \in \Sigma^\tau$ with $\Im\mathrm{m}\,(\tau(\lambda + i0)) \neq 0$.

Proposition 2.1: Let A be a densely defined closed symmetric operator with deficiency indices $n_\pm(A) = 1$, let $\Pi = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* and let $M(\cdot)$ be the corresponding Weyl function. Further, let $\tau(\cdot)$ be a Nevanlinna function and let $\mu \mapsto W_{H(\lambda)}(\mu)$, $\mu \in \mathbb{C}_-$, be the family of characteristic functions (II.2) of the Strauss family $\{H(\lambda)\}$. Then for a.e. $\lambda \in \Sigma^\tau$ the limit $\tau(\lambda) + M(\lambda)$ exists, is invertible and

$$\bigl(\tau(\lambda) + M(\lambda)\bigr)^{-1} = \lim_{\epsilon\to0}\bigl(\tau(\lambda + i\epsilon) + M(\lambda + i\epsilon)\bigr)^{-1}$$

holds. Moreover, the boundary value

$$W_{H(\lambda)}(\lambda - i0) := \lim_{\epsilon\to+0} W_{H(\lambda)}(\lambda - i\epsilon)$$

of the characteristic function $W_{H(\lambda)}(\mu)$, $\mu \in \mathbb{C}_-$, exists and is given by

$$W_{H(\lambda)}(\lambda - i0) = \frac{\tau(\lambda) + \overline{M(\lambda)}}{\overline{\tau(\lambda)} + \overline{M(\lambda)}}$$

for a.e. $\lambda \in \Sigma^\tau$, where we have used the convention $W_{H(\lambda)}(\lambda - i0) \equiv 1$ if $\Im\mathrm{m}\,(\tau(\lambda)) = 0$.

III. SCATTERING SYSTEMS

A. Coupling of symmetric operators

Let A and T be densely defined closed simple symmetric operators in the separable Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, assume that their deficiency indices are $n_\pm(A) = n_\pm(T) = 1$, and let $\Pi_A = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ and $\Pi_T = \{\mathbb{C}, \Upsilon_0, \Upsilon_1\}$ be boundary triples for A^* and T^* with

$$A_0 := A^* \upharpoonright \ker(\Gamma_0) \quad \text{and} \quad T_0 := T^* \upharpoonright \ker(\Upsilon_0).$$

The next theorem can be found in a slightly different form in [8].

Theorem 3.1: Let A, T, Π_A and Π_T be as above and denote the corresponding γ-fields and Weyl functions by γ, ν and M and τ, respectively. Then the following assertions (i)-(iv) hold.

(i) $\Pi_A \oplus \Pi_T = \{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0 := (\Gamma_0, \Upsilon_0)^\top$ and $\widetilde{\Gamma}_1 := (\Gamma_1, \Upsilon_1)^\top$, is a boundary triple for the operator $A^* \oplus T^*$ with corresponding γ-field $\widetilde{\gamma}$ and Weyl function $\widetilde{M}$ given by

$$\lambda \mapsto \widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \nu(\lambda) \end{pmatrix} \qquad \text{and} \qquad \lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}.$$

(ii) The closed extension $L := A^* \oplus T^* \upharpoonright \widetilde{\Gamma}^{-1}\widetilde{\Theta}$ corresponding to the relation

$$\widetilde{\Theta} := \left\{ \begin{pmatrix} (v,v)^\top \\ (w,-w)^\top \end{pmatrix} : v, w \in \mathbb{C} \right\} \in \widetilde{\mathcal{C}}(\mathbb{C}^2)$$

is self-adjoint in the Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$ and is given by

$$L = A^* \oplus T^* \upharpoonright \Big\{ f_1 \oplus f_2 \in \operatorname{dom}(A^* \oplus T^*) : \Gamma_0 f_1 - \Upsilon_0 f_2 = \Gamma_1 f_1 + \Upsilon_1 f_2 = 0 \Big\}.$$

(iii) For $\lambda \in \mathbb{C} \backslash \mathbb{R}$ we have

$$\big(L - \lambda\big)^{-1} = (L_0 - \lambda)^{-1} + \widetilde{\gamma}(\lambda)\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1}\widetilde{\gamma}(\overline{\lambda})^*, \tag{III.1}$$

where $L_0 := A_0 \oplus T_0 = A^* \oplus T^* \upharpoonright \ker \widetilde{\Gamma}_0$. The compressed resolvent of L onto $\mathfrak{H}$ is given by

$$P_{\mathfrak{H}}\big(L - \lambda\big)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^*.$$

(iv) The Strauss family $H(\lambda) = A^* \upharpoonright \ker(\Gamma_1 + \tau(\lambda)\Gamma_0)$ satisfies

$$(H(\lambda) - \lambda)^{-1} = P_{\mathfrak{H}}\big(L - \lambda\big)^{-1} \upharpoonright \mathfrak{H}$$

for all $\lambda \in \mathbb{C}_+$.

B. Coupling and scattering

In this section we consider the scattering system $\{L, L_0\}$ consisting of the self-adjoint operators L and L_0 in the Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$ defined in Theorem 3.1. Since by (III.1) the resolvents of L and L_0 differ by a rank two operator the wave operators

$$W_\pm(L, L_0) = s - \lim_{t \to \pm\infty} e^{itL} e^{-itL_0} P^{ac}(L_0)$$

exist and are complete, where $P^{ac}(L_0)$ denotes the orthogonal projection onto the absolutely continuous subspace $\mathfrak{H}^{ac}(L_0)$ of L_0. Completeness means that the ranges of $W_\pm(L, L_0)$ coincide with the absolutely continuous subspace $\mathfrak{H}^{ac}(L)$ of L, cf. [6], [15], [19]. The *scattering operator* S of the *scattering system* $\{L, L_0\}$ is then defined by

$$S := W_+(L, L_0)^* W_-(L, L_0). \tag{III.2}$$

Since the scattering operator S commutes with L_0 it follows that S is unitarily equivalent to a multiplication operator induced by a family $\{S(\lambda)\}$ of unitary operators in a spectral representation of

$$L_0^{ac} := L_0 \upharpoonright \operatorname{dom}(L_0) \cap \mathfrak{H}^{ac}(L_0).$$

With the help of Theorem 3.1 and [7, Theorem 3.8] we obtain a representation of the scattering matrix $\{S(\lambda)\}$ of the scattering system $\{L, L_0\}$ in the next theorem.

Theorem 3.2: Let A, T, Π_A and Π_T be as in Theorem 3.1 and let γ, ν and M and τ be the corresponding γ-fields and Weyl functions, respectively. Assume that A_0 has no absolutely continuous spectrum, let $L_0 = A_0 \oplus T_0$ and let L

be the coupling of the operators A_0 and T_0 defined in Theorem 3.1 (ii). Then there is a direct integral representation $L^2(\mathbb{R}, d\lambda, \mathcal{H}_\lambda)$ of the absolutely continuous part T_0^{ac} of T_0,

$$\mathcal{H}_\lambda = \begin{cases} \mathbb{C} & \text{if } \Im\mathrm{m}\,(\tau(\lambda)) \neq 0 \\ \{0\} & \text{if } \Im\mathrm{m}\,(\tau(\lambda)) = 0 \end{cases},$$

such that the scattering matrix $\{S(\lambda)\}$ of the scattering system $\{L, L_0\}$ admits the representation

$$S(\lambda) = \frac{\overline{\tau(\lambda)} + M(\lambda)}{\tau(\lambda) + M(\lambda)} \tag{III.3}$$

for a.e. $\lambda \in \mathbb{R}$, where $\tau(\lambda) = \tau(\lambda+i0)$, $M(\lambda) = M(\lambda+i0)$ and we have used the convention $S(\lambda) \equiv 1$ if $\Im\mathrm{m}\,(\tau(\lambda)) = 0$.

Proof: Let $\Pi_A \oplus \Pi_T = \{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_0 = (\Gamma_0, \Upsilon_0)^\top \text{ and } \widetilde{\Gamma}_1 = (\Gamma_1, \Upsilon_1)^\top,$$

be the boundary triple for $A^* \oplus T^*$ from Theorem 3.1. Then the corresponding Weyl function is a 2×2-matrix function given by

$$\lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}, \qquad \lambda \in \rho(L_0),$$

and a simple calculation shows that

$$\big(\widetilde{\Theta} - \widetilde{M}(\lambda)\big)^{-1} = -\begin{pmatrix} (\tau(\lambda) + M(\lambda))^{-1} & (\tau(\lambda) + M(\lambda))^{-1} \\ (\tau(\lambda) + M(\lambda))^{-1} & (\tau(\lambda) + M(\lambda))^{-1} \end{pmatrix} \tag{III.4}$$

holds for all $\lambda \in \rho(L_0) \cap \rho(L)$. By [7, Theorem 3.8] there is a direct integral representation $L^2(\mathbb{R}, d\lambda, \widetilde{\mathcal{H}}_\lambda)$, where

$$\widetilde{\mathcal{H}}_\lambda := \mathrm{ran}\,\big(\Im\mathrm{m}\,(\widetilde{M}(\lambda + i0))\big),$$

of the absolutely continuous part L_0^{ac} of L_0 such that the scattering matrix $\{S(\lambda)\}$ admits the representation

$$S(\lambda) = I_{\widetilde{\mathcal{H}}_\lambda} + 2i\big(\Im\mathrm{m}\,\big(\widetilde{M}(\lambda + i0)\big)\big)^{1/2}\big(\widetilde{\Theta} - \widetilde{M}(\lambda + i0)\big)^{-1}\big(\Im\mathrm{m}\,\big(\widetilde{M}(\lambda + i0)\big)\big)^{1/2}$$

for a.e. $\lambda \in \mathbb{R}$. Since $\sigma(A_0)$ is purely singular we get $L_0^{ac} = 0 \oplus T_0^{ac}$ and therefore we have

$$\Im\mathrm{m}\,\big(\widetilde{M}(\lambda + i0)\big) = \begin{pmatrix} 0 & 0 \\ 0 & \Im\mathrm{m}\,(\tau(\lambda + i0)) \end{pmatrix} \tag{III.5}$$

for a.e. $\lambda \in \mathbb{R}$ and hence by inserting (III.4) and (III.5) we conclude that the scattering matrix admits the representation (III.3). ■

In the following theorem we establish a connection between the scattering matrix $\{S(\lambda)\}$ of the scattering system $\{L, L_0\}$ and the characteristic functions of the Strauss family $\{H(\lambda)\}$. In the framework of Lax-Phillips scattering theory relation (III.6) below can be regarded as a generalization of the Adamyan-Arov result discussed in the introduction. A more detailed exposition with illustrating examples will be published elsewhere.

Theorem 3.3: Let A, T, Π_A and Π_T be as in Theorem 3.1 and Theorem 3.2 and let γ, ν and M and τ be the corresponding γ-fields and Weyl functions, respectively. Assume that $A_0 = A^* \upharpoonright \ker(\Gamma_0)$ has no absolutely continuous spectrum, let $L_0 = A_0 \oplus T_0$ and let L be the coupling of the operators A_0 and T_0 defined in Theorem 3.1 (ii). Then the scattering matrix $\{S(\lambda)\}$ of the scattering system $\{L, L_0\}$ is connected with the characteristic functions $W_{H(\lambda)}(\cdot)$ of the the Strauss family $\{H(\lambda)\}_{\lambda\in\Sigma^\tau}$, cf. (II.2), by

$$\overline{S(\lambda)} = W_{H(\lambda)}(\lambda - i0) \tag{III.6}$$

for a.e. $\lambda \in \mathbb{R}$ where we have used the convention $S(\lambda) \equiv W_{H(\lambda)}(\lambda - i0) \equiv 1$ if $\Im\mathrm{m}\,(\tau(\lambda)) = 0$.

Proof: By Proposition 2.1 the characteristic functions $W_{H(\lambda)}(\cdot)$ of the Strauss family $\{H(\lambda)\}_{\lambda\in\Sigma^\tau}$ satisfy

$$W_{H(\lambda)}(\lambda - i0) = \frac{\tau(\lambda) + \overline{M(\lambda)}}{\overline{\tau(\lambda)} + \overline{M(\lambda)}}$$

for a.e. $\lambda \in \mathbb{R}$. Since A_0 has no absolutely continuous spectrum one has $\overline{M(\lambda)} = M(\lambda)$ for a.e. $\lambda \in \mathbb{R}$. Comparing this with relation (III.3) we obtain (III.6). ■

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