J. London Math. Soc. (2) 88 (2013) 319–337 © 2013 London Mathematical Society
doi:10.1112/jlms/jdt012

Trace formulae and singular values of resolvent power differences of self-adjoint elliptic operators

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Abstract

In this note, self-adjoint realizations of second-order elliptic differential expressions with non-local Robin boundary conditions on a domain $\Omega \subset \mathbb{R}^n$ with smooth compact boundary are studied. A Schatten–von Neumann-type estimate for the singular values of the difference of the mth powers of the resolvents of two Robin realizations is obtained, and, for $m > n/2 - 1$, it is shown that the resolvent power difference is a trace class operator. The estimates are slightly stronger than the classical singular value estimates by Birman where one of the Robin realizations is replaced by the Dirichlet operator. In both cases, trace formulae are proved, in which the trace of the resolvent power differences in $L^2(\Omega)$ is written in terms of the trace of derivatives of Neumann-to-Dirichlet and Robin-to-Neumann maps on the boundary space $L^2(\partial\Omega)$.

1. *Introduction*

Let $\Omega \subset \mathbb{R}^n$ be a bounded or unbounded domain with smooth compact boundary and let $\mathcal{L}$ be a formally symmetric, second-order elliptic differential expression with variable coefficients defined on Ω. As a simple example, one may consider $\mathcal{L} = -\Delta$ or $\mathcal{L} = -\Delta + V$ with some real function V. Denote by A_{D} the self-adjoint Dirichlet operator associated with $\mathcal{L}$ in $L^2(\Omega)$ and let $A_{[\beta]}$ be a self-adjoint realization of $\mathcal{L}$ in $L^2(\Omega)$ with Robin boundary conditions of the form $\beta f|_{\partial\Omega} = \partial f/\partial\nu|_{\partial\Omega}$ for functions $f \in \operatorname{dom} A_{[\beta]}$. Here β is a real-valued bounded function on $\partial\Omega$; in the special case $\beta = 0$, one obtains the Neumann operator A_{N} associated with $\mathcal{L}$.

Half a century ago, it was observed by Birman [**9**] in his fundamental paper that the difference of the resolvents of A_{D} and $A_{[\beta]}$ is a compact operator whose singular values s_k satisfy $s_k = \mathrm{O}(k^{-2/(n-1)})$, $k \to \infty$; that is,

$$(A_{[\beta]} - \lambda)^{-1} - (A_{\mathrm{D}} - \lambda)^{-1} \in \mathfrak{S}_{(n-1)/2,\infty}, \quad \lambda \in \rho(A_{[\beta]}) \cap \rho(A_{\mathrm{D}}), \tag{1.1}$$

where $\mathfrak{S}_{p,\infty}$ denotes the weak Schatten–von Neumann ideal of order p; for the latter, see (2.1). The difference of higher powers of the resolvents of A_{D} and $A_{[\beta]}$ leads to stronger decay conditions of the form

$$(A_{[\beta]} - \lambda)^{-m} - (A_{\mathrm{D}} - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/2m,\infty}, \quad \lambda \in \rho(A_{[\beta]}) \cap \rho(A_{\mathrm{D}}); \tag{1.2}$$

see, for example, [**9**, **25**–**27**, **32**]. The estimate (1.1) for the decay of the singular values is known to be sharp if β satisfies some smoothness assumption (see [**10**, **25**–**28**]); the estimate (1.2) is sharp for smooth β by Grubb [**26**, **27**]. Observe that, for $m > (n-1)/2$, the operator in (1.2) belongs to the trace class ideal, and hence the wave operators for the scattering pair $\{A_{\mathrm{D}}, A_{[\beta]}\}$ exist and are complete, and the absolutely continuous parts of A_{D} and $A_{[\beta]}$ are unitarily equivalent. A simple consequence of one of our main results in the present paper is

Received 22 June 2012; published online 21 April 2013.

2010 *Mathematics Subject Classification* 35P05, 35P20 (primary), 47F05, 47L20, 81Q10, 81Q15 (secondary).

The second author (M. L.) thanks the Graz University of Technology for the hospitality.

the following representation for the trace of the operator in (1.2) (see Theorem 3.10):

$$\begin{aligned}&\operatorname{tr}\big((A_{[\beta]}-\lambda)^{-m}-(A_{\mathrm{D}}-\lambda)^{-m}\big)\\&\quad=\frac{1}{(m-1)!}\operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\Big(\big(I-M(\lambda)\beta\big)^{-1}M(\lambda)^{-1}M'(\lambda)\Big)\right),\end{aligned}\tag{1.3}$$

where $M(\lambda)$ is the Neumann-to-Dirichlet map (that is, the inverse of the Dirichlet-to-Neumann map) associated with $\mathcal{L}$; see also [**7**, Corollary 4.12] for $m=1$. In the special case where $A_{[\beta]}$ is the Neumann operator A_{N}, that is, $\beta=0$, the above formula simplifies to

$$\operatorname{tr}\big((A_{\mathrm{N}}-\lambda)^{-m}-(A_{\mathrm{D}}-\lambda)^{-m}\big)=\frac{1}{(m-1)!}\operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\big(M(\lambda)^{-1}M'(\lambda)\big)\right),\tag{1.4}$$

which is an analogue of [**14**, Théorème 2.2] and reduces to [**2**, Corollary 3.7] in the case $m=1$. We point out that the right-hand sides in (1.3) and (1.4) consist of traces of operators in the boundary space $L^2(\partial\Omega)$, whereas the left-hand sides are traces of operators in $L^2(\Omega)$. Some related reductions for ratios of Fredholm perturbation determinants can be found in [**20**]. We also refer to [**17**] for other types of trace formulae for Schrödinger operators.

Recently, it was shown in [**6**] that if one considers two self-adjoint Robin realizations $A_{[\beta_1]}$ and $A_{[\beta_2]}$ of $\mathcal{L}$, then the estimate (1.1) can be improved to

$$(A_{[\beta_1]}-\lambda)^{-1}-(A_{[\beta_2]}-\lambda)^{-1}\in\mathfrak{S}_{(n-1)/3,\infty},\tag{1.5}$$

so that, roughly speaking, any two Robin realizations with bounded coefficients β_j are closer to each other than to the Dirichlet operator A_{D}; see also [**7**] and the paper [**28**] by Grubb where the estimate (1.5) was shown to be sharp under some smoothness conditions on the functions β_1 and β_2. One of the main objectives of this note is to prove a counterpart of (1.2) for higher powers of resolvents of $A_{[\beta_1]}$ and $A_{[\beta_2]}$. For that we apply abstract boundary triple techniques from the extension theory of symmetric operators and a variant of Krein's formula which provides a convenient factorization of the resolvent difference of two self-adjoint realizations of $\mathcal{L}$; see [**4**, **5**, **7**]; for related approaches see [**12**, **15**, **18**, **19**, **24**, **29**, **32**, **34**, **35**] for related approaches. Our tools allow us to consider general non-local Robin-type realizations of $\mathcal{L}$ of the form

$$\begin{aligned}A_{[B]}f&=\mathcal{L}f,\\ \operatorname{dom}A_{[B]}&=\left\{f\in H^{3/2}(\Omega):\mathcal{L}f\in L^2(\Omega),\ Bf|_{\partial\Omega}=\frac{\partial f}{\partial\nu}\bigg|_{\partial\Omega}\right\},\end{aligned}\tag{1.6}$$

where B is an arbitrary bounded self-adjoint operator in $L^2(\partial\Omega)$ and $H^{3/2}(\Omega)$ denotes the L^2-based Sobolev space of order $\frac{3}{2}$. In the special case where B is the multiplication operator with a bounded real-valued function β on $\partial\Omega$, the differential operator in (1.6) coincides with the usual corresponding Robin realization $A_{[\beta]}$ of $\mathcal{L}$ in $L^2(\Omega)$. It is proved in Theorem 3.7 that, for two self-adjoint realizations $A_{[B_1]}$ and $A_{[B_2]}$ as in (1.6), the difference of the mth powers of the resolvents satisfies

$$(A_{[B_1]}-\lambda)^{-m}-(A_{[B_2]}-\lambda)^{-m}\in\mathfrak{S}_{(n-1)/(2m+1),\infty},\quad \lambda\in\rho(A_{[B_1]})\cap\rho(A_{[B_2]}),$$

and if, in addition, B_1-B_2 belongs to some weak Schatten–von Neumann ideal, then the estimate improves accordingly. Moreover, for $m>n/2-1$ the resolvent difference is a trace class operator and for the trace we obtain

$$\begin{aligned}&\operatorname{tr}\big((A_{[B_1]}-\lambda)^{-m}-(A_{[B_2]}-\lambda)^{-m}\big)\\&\quad=\frac{1}{(m-1)!}\operatorname{tr}\left[\frac{d^{m-1}}{d\lambda^{m-1}}\Big(\big(I-B_1M(\lambda)\big)^{-1}(B_1-B_2)\big(I-M(\lambda)B_2\big)^{-1}M'(\lambda)\Big)\right].\end{aligned}\tag{1.7}$$

As in (1.3) and (1.4), the right-hand side in (1.7) consists of the trace of derivatives of Robin-to-Neumann and Neumann-to-Dirichlet maps on the boundary $\partial\Omega$, so that (1.7) can be viewed as a reduction of the trace in $L^2(\Omega)$ to the boundary space $L^2(\partial\Omega)$.

The paper is organized as follows. We first recall some necessary facts about singular values and (weak) Schatten–von Neumann ideals in Subsection 2.1. In Subsection 2.2, the abstract concept of quasi-boundary-triples, γ-fields and Weyl functions from [**4**] is briefly recalled. Furthermore, we prove some preliminary results on the derivatives of the γ-field and Weyl function, and we provide some Krein-type formulae for the resolvent differences of self-adjoint extensions of a symmetric operator. Section 3 contains our main results on singular value estimates and traces of resolvent power differences of Dirichlet, Neumann and non-local Robin realizations of $\mathcal{L}$. In Subsection 3.1, the elliptic differential expression is defined and a family of self-adjoint Robin realizations is parametrized with the help of a quasi-boundary-triple. A detailed analysis of the smoothing properties of the derivatives of the corresponding γ-field and Weyl function together with Krein-type resolvent formulae and embeddings of Sobolev spaces then leads to the estimates and trace formulae in Theorems 3.6, 3.7 and 3.10.

2. *Schatten–von Neumann ideals and quasi-boundary-triples*

This section starts with preliminary facts on singular values and (weak) Schatten–von Neumann ideals. Furthermore, we review the concepts of quasi-boundary-triples, associated γ-fields and Weyl functions, which are convenient abstract tools for the parametrization and spectral analysis of self-adjoint realizations of elliptic differential expressions.

2.1. *Singular values and Schatten–von Neumann ideals*

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We denote by $\mathcal{B}(\mathcal{H},\mathcal{K})$ the space of bounded operators from $\mathcal{H}$ to $\mathcal{K}$ and by $\mathfrak{S}_\infty(\mathcal{H},\mathcal{K})$ the space of compact operators. Moreover, we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathfrak{S}_\infty(\mathcal{H}) := \mathfrak{S}_\infty(\mathcal{H},\mathcal{H})$.

The *singular values* (or *s-numbers*) $s_k(K)$, $k = 1, 2, \ldots,$ of a compact operator $K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{K})$ are defined as the eigenvalues of the non-negative compact operator $(K^*K)^{1/2} \in \mathfrak{S}_\infty(\mathcal{H})$, which are enumerated in non-increasing order and with multiplicities taken into account. Note that the singular values of K and K^* coincide: $s_k(K) = s_k(K^*)$ for $k = 1, 2, \ldots$; see, for example, [**22**, II.Section 2.2]. Recall that, for $p > 0$, the *Schatten–von Neumann ideals* $\mathfrak{S}_p(\mathcal{H},\mathcal{K})$ and *weak Schatten–von Neumann ideals* $\mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{K})$ are defined by

$$\begin{aligned} \mathfrak{S}_p(\mathcal{H},\mathcal{K}) &:= \left\{ K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{K})\colon \sum_{k=1}^{\infty}(s_k(K))^p < \infty \right\}, \\ \mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{K}) &:= \{K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{K})\colon s_k(K) = \mathrm{O}(k^{-1/p}),\ k \longrightarrow \infty\}. \end{aligned} \tag{2.1}$$

If no confusion can arise, then the spaces $\mathcal{H}$ and $\mathcal{K}$ are suppressed and we write $\mathfrak{S}_p$ and $\mathfrak{S}_{p,\infty}$. For $0 < p' < p$, the inclusions

$$\mathfrak{S}_p \subset \mathfrak{S}_{p,\infty} \quad \text{and} \quad \mathfrak{S}_{p',\infty} \subset \mathfrak{S}_p \tag{2.2}$$

hold; for $s, t > 0$, one has

$$\mathfrak{S}_{1/s} \cdot \mathfrak{S}_{1/t} = \mathfrak{S}_{1/(s+t)} \quad \text{and} \quad \mathfrak{S}_{1/s,\infty} \cdot \mathfrak{S}_{1/t,\infty} = \mathfrak{S}_{1/(s+t),\infty}, \tag{2.3}$$

where a product of operator ideals is defined as the set of all products. We refer the reader to [**22**, III.Section 7 and III.Section 14] and [**36**, Chapter 2] for a detailed study of the classes $\mathfrak{S}_p$ and $\mathfrak{S}_{p,\infty}$; see also [**7**, Lemma 2.3]. The ideal of *nuclear* or *trace class operators* $\mathfrak{S}_1$ plays an important role later on. The trace of a compact operator $K \in \mathfrak{S}_1(\mathcal{H})$ is defined as

$$\operatorname{tr} K := \sum_{k=1}^{\infty} \lambda_k(K),$$

where $\lambda_k(K)$ are the eigenvalues of K and the sum converges absolutely. It is well known (see, for example, [**22**, III.Section 8]) that, for $K_1, K_2 \in \mathfrak{S}_1(\mathcal{H})$,

$$\operatorname{tr}(K_1 + K_2) = \operatorname{tr} K_1 + \operatorname{tr} K_2 \tag{2.4}$$

holds. Moreover, if $K_1 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $K_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ are such that $K_1K_2 \in \mathfrak{S}_1(\mathcal{K})$ and $K_2K_1 \in \mathfrak{S}_1(\mathcal{H})$, then

$$\operatorname{tr}(K_1K_2) = \operatorname{tr}(K_2K_1). \tag{2.5}$$

The next useful lemma can be found in, for example, [**6**, **7**] and is based on the asymptotics of the eigenvalues of the Laplace–Beltrami operator. For a smooth compact manifold Σ, we denote the usual L^2-based Sobolev spaces by $H^r(\Sigma)$, $r \geqslant 0$.

LEMMA 2.1. *Let Σ be an $(n-1)$-dimensional compact C^∞ manifold without boundary and let $\mathcal{K}$ be a Hilbert space and let $K \in \mathcal{B}(\mathcal{K}, H^{r_1}(\Sigma))$ with $\operatorname{ran} K \subset H^{r_2}(\Sigma)$, where $r_2 > r_1 \geqslant 0$. Then K is compact and its singular values $s_k(K)$ satisfy*

$$s_k(K) = \mathrm{O}(k^{-(r_2-r_1)/(n-1)}), \quad k \longrightarrow \infty,$$

that is, $K \in \mathfrak{S}_{(n-1)/(r_2-r_1),\infty}(\mathcal{K}, H^{r_1}(\Sigma))$ and hence $K \in \mathfrak{S}_p(\mathcal{K}, H^{r_1}(\Sigma))$ for every $p > (n-1)/(r_2-r_1)$.

2.2. *Quasi-boundary-triples and their Weyl functions*

In this subsection, we recall the definitions and some important properties of quasi-boundary-triples, corresponding γ-fields and associated Weyl functions; see [**4**, **5**, **7**] for more details. Quasi-boundary-triples are particularly useful when dealing with elliptic boundary value problems from an operator and extension-theoretic point of view.

DEFINITION 2.2. Let A be a closed, densely defined, symmetric operator in a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_\mathcal{H})$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *quasi-boundary-triple* for A^* if $(\mathcal{G}, (\cdot,\cdot)_\mathcal{G})$ is a Hilbert space and, for some linear operator $T \subset A^*$ with $\overline{T} = A^*$, the following hold.

- (i) $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings, and the mapping $\Gamma := \binom{\Gamma_0}{\Gamma_1}$ has dense range in $\mathcal{G} \times \mathcal{G}$;
- (ii) $A_0 := T \upharpoonright \ker \Gamma_0$ is a self-adjoint operator in $\mathcal{H}$;
- (iii) for all $f, g \in \operatorname{dom} T$, the *abstract Green identity* holds:

$$(Tf, g)_\mathcal{H} - (f, Tg)_\mathcal{H} = (\Gamma_1 f, \Gamma_0 g)_\mathcal{G} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{G}.$$

We remark that a quasi-boundary-triple for A^* exists if and only if the deficiency indices of A coincide. Moreover, in the case of finite deficiency indices a quasi-boundary-triple is automatically an ordinary boundary triple; cf. [**4**, Proposition 3.3]. For the notion of (ordinary) boundary triples and their properties, we refer to [**13**, **15**, **16**, **23**, **30**]. If $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi-boundary-triple for A^*, then A coincides with $T \upharpoonright \ker \Gamma$ and the operator $A_1 := T \upharpoonright \ker \Gamma_1$ is symmetric in $\mathcal{H}$. We also mention that a quasi-boundary-triple with the additional property $\operatorname{ran} \Gamma_0 = \mathcal{G}$ is a generalized boundary triple in the sense of [**16**]; see [**4**, Corollary 3.7(ii)].

Next we recall the definition of the γ-field and the Weyl function associated with the quasi-boundary-triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^*. Note that the decomposition

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda)$$

holds for all $\lambda \in \rho(A_0)$, so that $\Gamma_0 \upharpoonright \ker(T-\lambda)$ is invertible for all $\lambda \in \rho(A_0)$. The (operator-valued) functions γ and M defined by

$$\gamma(\lambda) := (\Gamma_0 \upharpoonright \ker(T-\lambda))^{-1} \quad \text{and} \quad M(\lambda) := \Gamma_1\gamma(\lambda), \quad \lambda \in \rho(A_0),$$

are called the *γ-field* and the *Weyl function* corresponding to the quasi-boundary-triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, respectively. These definitions coincide with the definitions of the γ-field and the Weyl function in the case where $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple; see [**15**]. Note that, for each $\lambda \in \rho(A_0)$, the operator $\gamma(\lambda)$ maps $\operatorname{ran}\Gamma_0 \subset \mathcal{G}$ into $\operatorname{dom} T \subset \mathcal{H}$ and $M(\lambda)$ maps $\operatorname{ran}\Gamma_0$ into $\operatorname{ran}\Gamma_1$. Furthermore, as an immediate consequence of the definition of $M(\lambda)$, we obtain

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \quad f_\lambda \in \ker(T-\lambda), \quad \lambda \in \rho(A_0).$$

In the next proposition, we collect some properties of the γ-field and the Weyl function associated with the quasi-boundary-triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^*; most statements were proved in [**4**].

PROPOSITION 2.3. *For all $\lambda, \mu \in \rho(A_0)$, the following assertions hold.*

(i) *The mapping $\gamma(\lambda)$ is a bounded, densely defined operator from $\mathcal{G}$ into $\mathcal{H}$. The adjoint of $\gamma(\bar\lambda)$ has the representation*

$$\gamma(\bar\lambda)^* = \Gamma_1(A_0-\lambda)^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{G}).$$

(ii) *The mapping $M(\lambda)$ is a densely defined (and in general unbounded) operator in $\mathcal{G}$ that satisfies $M(\lambda) \subset M(\bar\lambda)^*$ and*

$$M(\lambda)h - M(\bar\mu)h = (\lambda - \bar\mu)\gamma(\mu)^*\gamma(\lambda)h$$

for all $h \in \mathcal{G}_0$. If $\operatorname{ran}\Gamma_0 = \mathcal{G}$, then $M(\lambda) \in \mathcal{B}(\mathcal{G})$ and $M(\lambda) = M(\bar\lambda)^*$.

(iii) *If $A_1 = T \upharpoonright \ker\Gamma_1$ is a self-adjoint operator in $\mathcal{H}$ and $\lambda \in \rho(A_0) \cap \rho(A_1)$, then $M(\lambda)$ maps $\operatorname{ran}\Gamma_0$ bijectively onto $\operatorname{ran}\Gamma_1$ and*

$$M(\lambda)^{-1}\gamma(\bar\lambda)^* \in \mathcal{B}(\mathcal{H}, \mathcal{G}).$$

Proof. Items (i), (ii) and the first part of (iii) follow from [**4**, Proposition 2.6(i)–(iii), (v) and Corollary 3.7(ii)]. For the second part of (iii), note that $\{\mathcal{G}, \Gamma_1, -\Gamma_0\}$ is also a quasi-boundary-triple if A_1 is self-adjoint. It is easy to see that in this case the corresponding γ-field is $\tilde\gamma(\lambda) = \gamma(\lambda)M(\lambda)^{-1}$. Since $\operatorname{ran}(\gamma(\bar\lambda)^*) \subset \operatorname{ran}\Gamma_1$ by item (ii), the operator $M(\lambda)^{-1}\gamma(\bar\lambda)^*$ is defined on $\mathcal{H}$. Now the boundedness of $\tilde\gamma(\lambda)$, which follows from (i), and the relation $M(\lambda) \subset M(\bar\lambda)^*$ imply that $M(\lambda)^{-1}\gamma(\bar\lambda)^*$ is bounded. □

In the following, we shall often use product rules for holomorphic operator-valued functions. Let $\mathcal{H}_i$, $i = 1, \ldots, 4$, be Hilbert spaces, let U be a domain in $\mathbb{C}$ and let $A\colon U \to \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$, $B\colon U \to \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$ and $C\colon U \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be holomorphic operator-valued functions. Then

$$\frac{d^m}{d\lambda^m}\big(A(\lambda)B(\lambda)\big) = \sum_{\substack{p+q=m\\ p,q\geqslant 0}} \binom{m}{p} A^{(p)}(\lambda)B^{(q)}(\lambda), \tag{2.6}$$

$$\frac{d^m}{d\lambda^m}\big(A(\lambda)B(\lambda)C(\lambda)\big) = \sum_{\substack{p+q+r=m\\ p,q,r\geqslant 0}} \frac{m!}{p!\,q!\,r!} A^{(p)}(\lambda)B^{(q)}(\lambda)C^{(r)}(\lambda), \tag{2.7}$$

for $\lambda \in U$. If $A(\lambda)^{-1}$ is invertible for every $\lambda \in U$, then relation (2.6) implies the following formula for the derivative of the inverse:

$$\frac{d}{d\lambda}\left(A(\lambda)^{-1}\right) = -A(\lambda)^{-1}A'(\lambda)A(\lambda)^{-1}. \tag{2.8}$$

In the next lemma, we consider higher derivatives of the γ-field and the Weyl function associated with a quasi-boundary-triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$.

LEMMA 2.4. *For all $\lambda \in \rho(A_0)$ and all $k \in \mathbb{N}$, the following hold.*

(i)
$$\frac{d^k}{d\lambda^k}\gamma(\bar{\lambda})^* = k!\gamma(\bar{\lambda})^*(A_0 - \lambda)^{-k}.$$

(ii)
$$\frac{d^k}{d\lambda^k}\overline{\gamma(\lambda)} = k!(A_0 - \lambda)^{-k}\overline{\gamma(\lambda)}.$$

(iii)
$$\overline{\frac{d^k}{d\lambda^k}M(\lambda)} = \frac{d^{k-1}}{d\lambda^{k-1}}\left(\gamma(\bar{\lambda})^*\overline{\gamma(\lambda)}\right) = k!\gamma(\bar{\lambda})^*(A_0 - \lambda)^{-(k-1)}\overline{\gamma(\lambda)}.$$

Proof. (i) We prove the statement by induction. For $k = 1$, we have

$$\begin{aligned}\frac{d}{d\lambda}\gamma(\bar{\lambda})^* &= \lim_{\mu\to\lambda}\frac{1}{\mu-\lambda}(\gamma(\bar{\mu})^* - \gamma(\bar{\lambda})^*)\\ &= \lim_{\mu\to\lambda}\frac{1}{\mu-\lambda}\Gamma_1\left((A_0-\mu)^{-1} - (A_0-\lambda)^{-1}\right)\\ &= \lim_{\mu\to\lambda}\Gamma_1(A_0-\mu)^{-1}(A_0-\lambda)^{-1} = \lim_{\mu\to\lambda}\gamma(\bar{\mu})^*(A_0-\lambda)^{-1}\\ &= \gamma(\bar{\lambda})^*(A_0-\lambda)^{-1},\end{aligned}$$

where we used Proposition 2.3(i). If we assume that the statement is true for $k \in \mathbb{N}$, then

$$\begin{aligned}\frac{d^{k+1}}{d\lambda^{k+1}}\gamma(\bar{\lambda})^* &= k!\frac{d}{d\lambda}(\gamma(\bar{\lambda})^*(A_0-\lambda)^{-k})\\ &= k!\left[\left(\frac{d}{d\lambda}\gamma(\bar{\lambda})^*\right)(A_0-\lambda)^{-k} + \gamma(\bar{\lambda})^*\frac{d}{d\lambda}(A_0-\lambda)^{-k}\right]\\ &= k!\left[\gamma(\bar{\lambda})^*(A_0-\lambda)^{-1}(A_0-\lambda)^{-k} + \gamma(\bar{\lambda})^*k(A_0-\lambda)^{-k-1}\right]\\ &= k!\,(1+k)\gamma(\bar{\lambda})^*(A_0-\lambda)^{-(k+1)},\end{aligned}$$

which proves the statement in (i) by induction.

(ii) This assertion is obtained from (i) by taking adjoints.

(iii) It follows from Proposition 2.3(ii) that, for $f \in \operatorname{dom} M(\lambda) = \operatorname{ran}\Gamma_0$,

$$\frac{d}{d\lambda}M(\lambda)f = \lim_{\mu\to\lambda}\frac{1}{\mu-\lambda}(M(\mu) - M(\lambda))f = \lim_{\mu\to\lambda}\gamma(\bar{\lambda})^*\gamma(\mu)f = \gamma(\bar{\lambda})^*\gamma(\lambda)f.$$

By taking closures, we obtain the claim for $k = 1$. For $k \geqslant 2$, we use (2.6) to get

$$\begin{aligned}\overline{\frac{d^k}{d\lambda^k}M(\lambda)} &= \frac{d^{k-1}}{d\lambda^{k-1}}(\gamma(\bar{\lambda})^*\overline{\gamma(\lambda)}) = \sum_{\substack{p+q=k-1\\ p,q\geqslant 0}}\binom{k-1}{p}\left(\frac{d^p}{d\lambda^p}\gamma(\bar{\lambda})^*\right)\frac{d^q}{d\lambda^q}\overline{\gamma(\lambda)}\\ &= \sum_{\substack{p+q=k-1\\ p,q\geqslant 0}}\binom{k-1}{p}p!\gamma(\bar{\lambda})^*(A_0-\lambda)^{-p}q!(A_0-\lambda)^{-q}\overline{\gamma(\lambda)}\\ &= \sum_{\substack{p+q=k-1\\ p,q\geqslant 0}}(k-1)!\gamma(\bar{\lambda})^*(A_0-\lambda)^{-(k-1)}\overline{\gamma(\lambda)} = k!\gamma(\bar{\lambda})^*(A_0-\lambda)^{-(k-1)}\overline{\gamma(\lambda)},\end{aligned}$$

which completes the proof. $\square$

The following theorem provides a Krein-type formula for the resolvent difference of A_0 and A_1 if A_1 is self-adjoint. The theorem follows from [**4**, Corollary 3.11(i)] with $\Theta = 0$.

THEOREM 2.5. *Let A be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi-boundary-triple for A^* with $A_0 = T \upharpoonright \ker \Gamma_0$, γ-field γ and Weyl function M. Assume that $A_1 = T \upharpoonright \ker \Gamma_1$ is self-adjoint in $\mathcal{H}$. Then*

$$(A_0 - \lambda)^{-1} - (A_1 - \lambda)^{-1} = \gamma(\lambda) M(\lambda)^{-1} \gamma(\bar{\lambda})^*$$

holds for $\lambda \in \rho(A_1) \cap \rho(A_0)$.

Note that the operator $M(\lambda)^{-1}\gamma(\bar{\lambda})^*$ in Theorem 2.5 is bounded by Proposition 2.3(iii).

In the following, we deal with extensions of A, which are restrictions of T corresponding to some abstract boundary condition. For a linear operator B in $\mathcal{G}$, we define

$$A_{[B]} f := Tf, \quad \operatorname{dom} A_{[B]} := \{f \in \operatorname{dom} T \colon B\Gamma_1 f = \Gamma_0 f\}. \tag{2.9}$$

In contrast to ordinary boundary triples, self-adjointness of the parameter B does not imply self-adjointness of the corresponding extension $A_{[B]}$ in general. The next theorem provides a useful sufficient condition for this and a variant of Krein's formula, which will be used later; see [**5**, Corollary 6.18 and Theorem 6.19] or [**7**, Corollary 3.11, Theorem 3.13 and Remark 3.14].

THEOREM 2.6. *Let A be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi-boundary-triple for A^* with $A_0 = T \upharpoonright \ker \Gamma_0$, γ-field γ and Weyl function M. Assume that $\operatorname{ran} \Gamma_0 = \mathcal{G}$, that $A_1 = T \upharpoonright \ker \Gamma_1$ is self-adjoint in $\mathcal{H}$ and that $M(\lambda_0) \in \mathfrak{S}_\infty(\mathcal{G})$ for some $\lambda_0 \in \rho(A_0)$.*

If B is a bounded self-adjoint operator in $\mathcal{G}$, then the corresponding extension $A_{[B]}$ is self-adjoint in $\mathcal{H}$ and

$$\begin{aligned}(A_{[B]} - \lambda)^{-1} - (A_0 - \lambda)^{-1} &= \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B\gamma(\bar{\lambda})^* \\ &= \gamma(\lambda) B \big(I - M(\lambda)B\big)^{-1} \gamma(\bar{\lambda})^*\end{aligned}$$

holds for $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ with

$$\big(I - BM(\lambda)\big)^{-1}, \big(I - M(\lambda)B\big)^{-1} \in \mathcal{B}(\mathcal{G}).$$

3. *Elliptic operators on domains with compact boundaries*

In this section, we study self-adjoint realizations of second-order elliptic differential expressions on a bounded or an exterior domain subject to Robin or more general non-local boundary conditions. With the help of quasi-boundary-triple techniques, we express the resolvent power differences of different self-adjoint realizations using Krein-type formulae. Using a detailed analysis of the perturbation term together with smoothing properties of the derivatives of the γ-fields and Weyl function, we then obtain singular value estimates and trace formulae.

3.1. *Self-adjoint elliptic operators with non-local Robin boundary conditions*

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded or unbounded domain with a compact C^∞ boundary $\partial\Omega$. We denote by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\partial\Omega}$ the inner products in the Hilbert spaces $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively. Throughout this section, we consider a formally symmetric, second-order elliptic differential expression

$$(\mathcal{L}f)(x) := -\sum_{j,k=1}^{n} \partial_j (a_{jk} \partial_k f)(x) + a(x) f(x), \quad x \in \Omega,$$

with bounded, infinitely differentiable, real-valued coefficients $a_{jk}, a \in C^\infty(\overline{\Omega})$ that satisfy $a_{jk}(x) = a_{kj}(x)$ for all $x \in \overline{\Omega}$ and $j,k = 1,\dots,n$. We assume that the first partial derivatives of the coefficients a_{jk} are bounded in Ω. Furthermore, $\mathcal{L}$ is assumed to be uniformly elliptic; that is, the condition

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqslant C\sum_{k=1}^{n}\xi_k^2$$

holds for some $C > 0$, all $\xi = (\xi_1,\dots,\xi_n)^T \in \mathbb{R}^n$ and $x \in \overline{\Omega}$.

For a function $f \in C^\infty(\overline{\Omega})$, we denote the trace by $f|_{\partial\Omega}$ and the (oblique) Neumann trace by

$$\partial_{\mathcal{L}} f|_{\partial\Omega} := \sum_{j,k=1}^{n} a_{jk}\nu_j\partial_k f|_{\partial\Omega},$$

with the normal vector field $(\nu_1,\nu_2,\dots,\nu_n)^\top$ pointing outwards from Ω. By continuity, the trace and the Neumann trace can be extended to mappings from $H^s(\Omega)$ to $H^{s-1/2}(\partial\Omega)$ for $s > \frac{1}{2}$ and $H^{s-3/2}(\partial\Omega)$ for $s > \frac{3}{2}$, respectively.

Next we define a quasi-boundary-triple for the adjoint A^* of the minimal operator

$$Af = \mathcal{L}f, \quad \operatorname{dom} A = \{f \in H^2(\Omega) : f|_{\partial\Omega} = \partial_{\mathcal{L}} f|_{\partial\Omega} = 0\}$$

associated with $\mathcal{L}$ in $L^2(\Omega)$. Recall that A is a closed, densely defined, symmetric operator with equal infinite deficiency indices and that

$$A^* f = \mathcal{L}f, \quad \operatorname{dom} A^* = \{f \in L^2(\Omega) : \mathcal{L}f \in L^2(\Omega)\}$$

is the maximal operator associated with $\mathcal{L}$; see, for example, [**1**, **3**]. As the operator T appearing in the definition of a quasi-boundary-triple, we choose

$$Tf = \mathcal{L}f, \quad \operatorname{dom} T = H^{3/2}_{\mathcal{L}}(\Omega) := \{f \in H^{3/2}(\Omega)\colon \mathcal{L}f \in L^2(\Omega)\},$$

and we consider the boundary mappings

$$\begin{aligned} \Gamma_0\colon \operatorname{dom} T &\longrightarrow L^2(\partial\Omega), \quad \Gamma_0 f := \partial_{\mathcal{L}} f|_{\partial\Omega},\\ \Gamma_1\colon \operatorname{dom} T &\longrightarrow L^2(\partial\Omega), \quad \Gamma_1 f := f|_{\partial\Omega}. \end{aligned}$$

Note that the trace and the Neumann trace can be extended to mappings from $H^{3/2}_{\mathcal{L}}(\Omega)$ into $L^2(\partial\Omega)$. With this choice of T and Γ_0 and Γ_1, we have the following proposition.

PROPOSITION 3.1. *The triple $\{L^2(\partial\Omega),\Gamma_0,\Gamma_1\}$ is a quasi-boundary-triple for A^* with the Neumann and Dirichlet operators as self-adjoint operators corresponding to the kernels of the boundary mappings*

$$\begin{aligned} A_{\mathrm{N}} &:= T \upharpoonright \ker\Gamma_0, \quad \operatorname{dom} A_{\mathrm{N}} = \{f \in H^2(\Omega)\colon \partial_{\mathcal{L}} f|_{\partial\Omega} = 0\},\\ A_{\mathrm{D}} &:= T \upharpoonright \ker\Gamma_1, \quad \operatorname{dom} A_{\mathrm{D}} = \{f \in H^2(\Omega)\colon f|_{\partial\Omega} = 0\}. \end{aligned} \tag{3.1}$$

The ranges of the boundary mappings are

$$\operatorname{ran}\Gamma_0 = L^2(\partial\Omega) \quad \text{and} \quad \operatorname{ran}\Gamma_1 = H^1(\partial\Omega),$$

and the γ-field and Weyl function associated with $\{L^2(\partial\Omega),\Gamma_0,\Gamma_1\}$ are given by

$$\gamma(\lambda)\varphi = f_\lambda \quad \text{and} \quad M(\lambda)\varphi = f_\lambda|_{\partial\Omega}, \quad \lambda \in \rho(A_{\mathrm{N}}),$$

for $\varphi \in L^2(\partial\Omega)$, where $f_\lambda \in H^{3/2}_{\mathcal{L}}(\Omega)$ is the unique solution of the boundary value problem $\mathcal{L}u = \lambda u$, $\partial_{\mathcal{L}} u|_{\partial\Omega} = \varphi$.

We remark that the quasi-boundary-triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ in Proposition 3.1 is a generalized boundary triple in the sense of [**16**] since the boundary mapping Γ_0 is surjective.

Proof. The proof of Proposition 3.1 proceeds in the same way as the proof of [**7**, Theorem 4.2], except that here T is defined on the larger space $H^{3/2}_{\mathcal{L}}(\Omega)$. Therefore, we do not repeat the arguments here, but provide only the main references that are necessary to translate the proof of [**7**, Theorem 4.2] to the present situation. The self-adjointness of $A_{\rm D}$ and $A_{\rm N}$ is ensured by Beals [**3**, Theorem 7.1(a)] and Browder [**11**, Theorem 5(iii)]. The trace theorem from [**31**, Chapter 2, Section 7.3] and the corresponding Green identity (see, for example, [**7**, Proof of Theorem 4.2]) yield the asserted properties of the ranges of the boundary mappings Γ_0 and Γ_1 and the abstract Green identity in Definition 2.2. Hence, [**4**, Theorem 2.3] implies that the triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ in Proposition 3.1 is a quasi-boundary-triple for A^*; see [**7**, Theorems 3.2, 4.2 and Proposition 4.3] for further details. □

The space $H^s_{\rm loc}(\overline{\Omega})$, $s \geqslant 0$ consists of all measurable functions f such that, for any bounded open subset $\Omega' \subset \Omega$, the condition $f \upharpoonright \Omega' \in H^s(\Omega')$ holds. Since Ω is a bounded domain or an exterior domain and $\partial\Omega$ is compact, any function in $H^s_{\rm loc}(\overline{\Omega})$ is H^s-smooth up to the boundary $\partial\Omega$. For $f \in H^s_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega)$, $s \geqslant 0$, our assumptions on the coefficients in the differential expression $\mathcal{L}$ imply

$$\begin{aligned} (A_{\rm D} - \lambda)^{-1} f \in H^{s+2}_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega), \quad \lambda \in \rho(A_{\rm D}),\\ (A_{\rm N} - \lambda)^{-1} f \in H^{s+2}_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega), \quad \lambda \in \rho(A_{\rm N}). \end{aligned} \tag{3.2}$$

These smoothing properties can be easily deduced from [**33**, Theorem 4.18], where they are formulated and proved in the language of boundary value problems.

The operators $\gamma(\lambda)$ and $M(\lambda)$ are also called the *Poisson operator* and the *Neumann-to-Dirichlet map* for the differential expression $\mathcal{L} - \lambda$. From Proposition 2.3, various properties of these operators can be deduced. In the next lemma, we collect smoothing properties of these operators, which follow, basically, from Proposition 2.3 and the trace theorem for Sobolev spaces on smooth domains and its generalizations given in [**31**, Chapter 2].

LEMMA 3.2. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition 3.1 with γ-field γ and Weyl function M. Then, for all $s \geqslant 0$, the following statements hold.*

(i) $\operatorname{ran}(\gamma(\lambda) \upharpoonright H^s(\partial\Omega)) \subset H^{s+3/2}_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega)$ *for all* $\lambda \in \rho(A_{\rm N})$.

(ii) $\operatorname{ran}(\gamma(\bar\lambda)^* \upharpoonright H^s_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega)) \subset H^{s+3/2}(\partial\Omega)$ *for all* $\lambda \in \rho(A_{\rm N})$.

(iii) $\operatorname{ran}(M(\lambda) \upharpoonright H^s(\partial\Omega)) \subset H^{s+1}(\partial\Omega)$ *for all* $\lambda \in \rho(A_{\rm N})$.

(iv) $\operatorname{ran}(M(\lambda) \upharpoonright H^s(\partial\Omega)) = H^{s+1}(\partial\Omega)$ *for all* $\lambda \in \rho(A_{\rm D}) \cap \rho(A_{\rm N})$.

Proof. (i) It follows from the decomposition $\operatorname{dom} T = \operatorname{dom} A_{\rm N} \dotplus \ker(T - \lambda)$, $\lambda \in \rho(A_{\rm N})$, and the properties of the Neumann trace [**31**, Chapter 2, Section 7.3] that the restriction of the mapping Γ_0 to

$$\ker(T - \lambda) \cap H^{s+3/2}_{\rm loc}(\overline{\Omega})$$

is a bijection onto $H^s(\partial\Omega)$, $s \geqslant 0$. Hence, by the definition of the γ-field, we obtain

$$\operatorname{ran}(\gamma(\lambda) \upharpoonright H^s(\partial\Omega)) = \ker(T - \lambda) \cap H^{s+3/2}_{\rm loc}(\overline{\Omega}) \subset H^{s+3/2}_{\rm loc}(\overline{\Omega}) \cap L^2(\Omega).$$

(ii) According to Proposition 2.3(i) and the definition of Γ_1, we have

$$\gamma(\overline{\lambda})^* = \Gamma_1 (A_{\rm N} - \lambda)^{-1}.$$

Employing (3.2) and the properties of the Dirichlet trace [**31**, Chapter 2, Section 7.3], we conclude that

$$\operatorname{ran}(\gamma(\bar{\lambda})^* \upharpoonright H^s_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega)) \subset H^{s+3/2}(\partial\Omega)$$

holds for all $s \geqslant 0$.

Assertion (iii) follows from the definition of $M(\lambda)$, item (i), the fact that Γ_1 is the Dirichlet trace operator and properties of the latter.

To verify (iv), let $\psi \in H^{s+1}(\partial\Omega)$. Since $\lambda \in \rho(A_{\mathrm{D}})$, we have the decomposition $\operatorname{dom} T = \operatorname{dom} A_{\mathrm{D}} \dotplus \ker(T-\lambda)$ and there exists a unique function $f_\lambda \in \ker(T-\lambda) \cap H^{s+3/2}_{\mathrm{loc}}(\overline{\Omega})$ such that $f_\lambda|_{\partial\Omega} = \psi$. Hence,

$$\Gamma_0 f_\lambda = \varphi \in H^s(\partial\Omega) \quad \text{and} \quad M(\lambda)\varphi = \psi,$$

that is, $H^{s+1}(\partial\Omega) \subset \operatorname{ran}(M(\lambda) \upharpoonright H^s(\partial\Omega))$, and (iii) implies the assertion. □

In the next proposition, we list some weak Schatten–von Neumann ideal properties of the derivatives of the γ-field and Weyl function, which follow from Lemma 2.4, elliptic regularity and Lemma 2.1.

Proposition 3.3. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition* 3.1 *with γ-field γ and Weyl function M. Then the following statements hold.*

(i) *For all $\lambda \in \rho(A_{\mathrm{N}})$ and $k \in \mathbb{N}_0$,*

$$\begin{aligned} \frac{d^k}{d\lambda^k}\gamma(\lambda) &\in \mathfrak{S}_{(n-1)/(2k+3/2),\infty}(L^2(\partial\Omega), L^2(\Omega)), \\ \frac{d^k}{d\lambda^k}\gamma(\bar{\lambda})^* &\in \mathfrak{S}_{(n-1)/(2k+3/2),\infty}(L^2(\Omega), L^2(\partial\Omega)). \end{aligned} \tag{3.3}$$

(ii) *For all $\lambda \in \rho(A_{\mathrm{N}})$ and $k \in \mathbb{N}_0$,*

$$\frac{d^k}{d\lambda^k} M(\lambda) \in \mathfrak{S}_{(n-1)/(2k+1),\infty}(L^2(\partial\Omega)).$$

Proof. (i) Let $\lambda \in \rho(A_{\mathrm{N}})$ and $k \in \mathbb{N}_0$. It follows from (3.2) that $\operatorname{ran}((A_{\mathrm{N}}-\lambda)^{-k}) \subset H^{2k}_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega)$ and hence from Lemma 3.2(ii) that

$$\operatorname{ran}(\gamma(\bar{\lambda})^*(A_{\mathrm{N}}-\lambda)^{-k}) \subset H^{2k+3/2}(\partial\Omega).$$

Thus, Lemma 2.1 with $\mathcal{K} = L^2(\Omega)$, $\Sigma = \partial\Omega$, $r_1 = 0$ and $r_2 = 2k + \frac{3}{2}$ implies

$$\gamma(\bar{\lambda})^*(A_{\mathrm{N}}-\lambda)^{-k} \in \mathfrak{S}_{(n-1)/(2k+3/2),\infty}(L^2(\Omega), L^2(\partial\Omega)). \tag{3.4}$$

By taking the adjoint in (3.4) and replacing λ by $\bar{\lambda}$, we obtain

$$(A_{\mathrm{N}}-\lambda)^{-k}\gamma(\lambda) \in \mathfrak{S}_{(n-1)/(2k+3/2),\infty}(L^2(\partial\Omega), L^2(\Omega)). \tag{3.5}$$

Now from Lemma 2.4(i) and (ii) and (3.4) and (3.5), we obtain (3.3).

(ii) For $k = 0$, we observe that $\operatorname{ran} M(\lambda) \subset H^1(\partial\Omega)$ by Lemma 3.2(iii). Therefore, Lemma 2.1 with $\mathcal{K} = L^2(\partial\Omega)$, $\Sigma = \partial\Omega$, $r_1 = 0$ and $r_2 = 1$ implies $M(\lambda) \in \mathfrak{S}_{n-1,\infty}(L^2(\partial\Omega))$. For $k \geqslant 1$, we have

$$\frac{d^k}{d\lambda^k} M(\lambda) = k!\gamma(\bar{\lambda})^*(A_{\mathrm{N}}-\lambda)^{-(k-1)}\gamma(\lambda),$$

from Lemma 2.4(iii). Hence, (3.4) and (3.5) imply

$$\frac{d^k}{d\lambda^k} M(\lambda) \in \mathfrak{S}_{(n-1)/(2(k-1)+3/2),\infty} \cdot \mathfrak{S}_{(n-1)/(3/2),\infty} = \mathfrak{S}_{(n-1)/(2k+1),\infty},$$

where the last equality follows from (2.3). □

As a consequence of Theorem 2.5, we obtain a factorization for the resolvent difference of self-adjoint operators A_{N} and A_{D}.

Corollary 3.4. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition* 3.1 *with γ-field γ and Weyl function M. Then*

$$(A_{\mathrm{N}} - \lambda)^{-1} - (A_{\mathrm{D}} - \lambda)^{-1} = \gamma(\lambda) M(\lambda)^{-1} \gamma(\bar{\lambda})^*$$

holds for $\lambda \in \rho(A_{\mathrm{D}}) \cap \rho(A_{\mathrm{N}})$.

Next we define a family of realizations of $\mathcal{L}$ in $L^2(\Omega)$ with general Robin-type boundary conditions of the form

$$A_{[B]} f := \mathcal{L} f, \quad \operatorname{dom} A_{[B]} := \{ f \in H^{3/2}_{\mathcal{L}}(\Omega) \colon B f|_{\partial\Omega} = \partial_{\mathcal{L}} f|_{\partial\Omega} \}, \tag{3.6}$$

where B is a bounded self-adjoint operator in $L^2(\partial\Omega)$. In terms of the quasi-boundary-triple in Proposition 3.1, the operator $A_{[B]}$ coincides with the one in (2.9), which is also equal to the restriction

$$T \upharpoonright \ker(B\Gamma_1 - \Gamma_0).$$

The following corollary is a consequence of Theorem 2.6 since $\operatorname{ran} \Gamma_0 = L^2(\partial\Omega)$, A_{D} is self-adjoint and $M(\lambda)$ is compact for $\lambda \in \rho(A_{\mathrm{N}})$ by Proposition 3.3(ii).

Corollary 3.5. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition* 3.1 *with γ-field γ and Weyl function M, and let B be a bounded self-adjoint operator in $L^2(\partial\Omega)$. Then the corresponding operator $A_{[B]}$ in* (3.6) *is self-adjoint in $L^2(\Omega)$ and*

$$(A_{[B]} - \lambda)^{-1} - (A_{\mathrm{N}} - \lambda)^{-1} = \gamma(\lambda)\big(I - BM(\lambda)\big)^{-1} B \gamma(\bar{\lambda})^* \tag{3.7}$$

$$= \gamma(\lambda) B \big(I - M(\lambda)B\big)^{-1} \gamma(\bar{\lambda})^* \tag{3.8}$$

holds for $\lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{N}})$ with

$$\big(I - BM(\lambda)\big)^{-1}, \big(I - M(\lambda)B\big)^{-1} \in \mathcal{B}\left(L^2(\partial\Omega)\right). \tag{3.9}$$

Note that the operators in (3.9) can be viewed as Robin-to-Neumann maps.

3.2. *Operator ideal properties and traces of resolvent power differences*

In this subsection, we prove the main results of this note: estimates for the singular values of resolvent power differences of two self-adjoint realizations of the differential expression $\mathcal{L}$ subject to Dirichlet, Neumann and non-local Robin boundary conditions.

The first theorem on the difference of the resolvent powers of the Dirichlet and Neumann operators is partially known from [**9**, **26**, **32**], where the proof is based on variational principles, pseudo-differential methods or a reduction to higher-order operators. Here, we give an elementary, direct proof using our approach. In the case of first powers of the resolvents, the trace formula in item Theorem 3.6(ii) is contained in [**2**, **7**]. An equivalent formula can also be found in [**14**], where it is used for the analysis of the Laplace–Beltrami operator on coupled manifolds.

THEOREM 3.6. *Let A_{D} and A_{N} be the self-adjoint Dirichlet and Neumann realizations of $\mathcal{L}$ in (3.1), respectively, and let M be the Weyl function from Proposition 3.1. Then the following statements hold.*

(i) *For all $m \in \mathbb{N}$ and $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$,*

$$(A_{\mathrm{N}} - \lambda)^{-m} - (A_{\mathrm{D}} - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/2m,\infty}(L^2(\Omega)). \tag{3.10}$$

(ii) *If $m > (n-1)/2$, then the resolvent power difference in (3.10) is a trace class operator and, for all $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$,*

$$\operatorname{tr}\left((A_{\mathrm{N}} - \lambda)^{-m} - (A_{\mathrm{D}} - \lambda)^{-m}\right) = \frac{1}{(m-1)!}\operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\left(M(\lambda)^{-1}M'(\lambda)\right)\right).$$

Proof. (i) The proof of the first item is carried out in two steps.

Step 1. Let us introduce the operator function

$$S(\lambda) := M(\lambda)^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}}).$$

Note that the product is well defined since $\operatorname{ran}(\gamma(\bar{\lambda})^*) \subset H^1(\partial\Omega) = \operatorname{dom}(M(\lambda)^{-1})$. Since A_{D} is self-adjoint, it follows from Proposition 2.3(iii) that $S(\lambda)$ is a bounded operator from $L^2(\Omega)$ to $L^2(\partial\Omega)$ for $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$. We prove by induction the following smoothing property for the derivatives of S:

$$u \in H^s_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \Longrightarrow S^{(k)}(\lambda)u \in H^{s+2k+1/2}(\partial\Omega), \quad s \geqslant 0,\ k \in \mathbb{N}_0. \tag{3.11}$$

Since $\gamma(\bar{\lambda})^*$ maps $H^s_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega)$ into $H^{s+3/2}(\partial\Omega)$ for $s \geqslant 0$ by Lemma 3.2(ii) and $M(\lambda)^{-1}$ maps $H^{s+3/2}(\partial\Omega)$ into $H^{s+1/2}(\partial\Omega)$ by Lemma 3.2(iv), relation (3.11) is true for $k = 0$. Now let $l \in \mathbb{N}_0$ and assume that (3.11) is true for every $k = 0, 1, \ldots, l$. By (2.6), (2.8) and Lemma 2.4(i) and (iii), we have

$$\begin{aligned}
S'(\lambda)u &= \frac{d}{d\lambda}(M(\lambda)^{-1})\gamma(\bar{\lambda})^*u + M(\lambda)^{-1}\frac{d}{d\lambda}\gamma(\bar{\lambda})^*u \\
&= -M(\lambda)^{-1}M'(\lambda)M(\lambda)^{-1}\gamma(\bar{\lambda})^*u + M(\lambda)^{-1}\gamma(\bar{\lambda})^*(A_{\mathrm{N}} - \lambda)^{-1}u \\
&= -M(\lambda)^{-1}\gamma(\bar{\lambda})^*\gamma(\lambda)M(\lambda)^{-1}\gamma(\bar{\lambda})^*u + S(\lambda)(A_{\mathrm{N}} - \lambda)^{-1}u \\
&= S(\lambda)(A_{\mathrm{N}} - \lambda)^{-1}u - S(\lambda)\gamma(\lambda)S(\lambda)u
\end{aligned}$$

for all $u \in L^2(\Omega)$. Hence, with the help of (2.6), (2.7) and Lemma 2.4(ii), we obtain

$$\begin{aligned}
S^{(l+1)}(\lambda) &= \frac{d^l}{d\lambda^l}\left(S(\lambda)(A_{\mathrm{N}} - \lambda)^{-1} - S(\lambda)\gamma(\lambda)S(\lambda)\right) \\
&= \sum_{\substack{p+q=l \\ p,q\geqslant 0}} \binom{l}{p} S^{(p)}(\lambda)\frac{d^q}{d\lambda^q}(A_{\mathrm{N}} - \lambda)^{-1} \\
&\quad - \sum_{\substack{p+q+r=l \\ p,q,r\geqslant 0}} \frac{l!}{p!\,q!\,r!} S^{(p)}(\lambda)\gamma^{(q)}(\lambda)S^{(r)}(\lambda) \\
&= \sum_{\substack{p+q=l \\ p,q\geqslant 0}} \frac{l!}{p!} S^{(p)}(\lambda)(A_{\mathrm{N}} - \lambda)^{-(q+1)} \\
&\quad - \sum_{\substack{p+q+r=l \\ p,q,r\geqslant 0}} \frac{l!}{p!\,r!} S^{(p)}(\lambda)(A_{\mathrm{N}} - \lambda)^{-q}\gamma(\lambda)S^{(r)}(\lambda).
\end{aligned} \tag{3.12}$$

By the induction hypothesis, the smoothing property (3.2) and Lemma 3.2(i), we have, for $s \geqslant 0$ and $p, q \geqslant 0$, $p+q=l$,

$$\begin{aligned} u \in H^s_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \\ \Longrightarrow (A_{\mathrm{N}}-\lambda)^{-(q+1)} u \in H^{s+2q+2}_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \\ \Longrightarrow S^{(p)}(\lambda)(A_{\mathrm{N}}-\lambda)^{-(q+1)} u \in H^{s+2q+2+2p+1/2}(\partial\Omega) = H^{s+2(l+1)+1/2}(\partial\Omega), \end{aligned}$$

and, for $s \geqslant 0$ and $p, q, r \geqslant 0$, $p+q+r=l$,

$$\begin{aligned} u \in H^s_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \\ \Longrightarrow S^{(r)}(\lambda) u \in H^{s+2r+1/2}(\partial\Omega) \\ \Longrightarrow \gamma(\lambda) S^{(r)}(\lambda) u \in H^{s+2r+1/2+3/2}_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \\ \Longrightarrow (A_{\mathrm{N}}-\lambda)^{-q} \gamma(\lambda) S^{(r)}(\lambda) u \in H^{s+2r+2+2q}_{\mathrm{loc}}(\overline{\Omega}) \cap L^2(\Omega) \\ \Longrightarrow S^{(p)}(\lambda)(A_{\mathrm{N}}-\lambda)^{-q} \gamma(\lambda) S^{(r)}(\lambda) u \in H^{s+2r+2+2q+2p+1/2}(\partial\Omega) = H^{s+2(l+1)+1/2}(\partial\Omega), \end{aligned}$$

which, together with (3.12), shows (3.11) for $k = l+1$ and hence, by induction, for all $k \in \mathbb{N}_0$. Therefore, an application of Lemma 2.1 yields

$$S^{(k)}(\lambda) \in \mathfrak{S}_{(n-1)/(2k+1/2),\infty}(L^2(\Omega), L^2(\partial\Omega)), \quad k \in \mathbb{N}_0, \ \lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}}). \tag{3.13}$$

Step 2. Using Krein's formula from Corollary 3.4 and (2.6), we can write, for $m \in \mathbb{N}$ and $\lambda \in \rho(A_{\mathrm{N}}) \cap \rho(A_{\mathrm{D}})$,

$$\begin{aligned} (A_{\mathrm{N}}-\lambda)^{-m} - (A_{\mathrm{D}}-\lambda)^{-m} &= \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{d\lambda^{m-1}} \left((A_{\mathrm{N}}-\lambda)^{-1} - (A_{\mathrm{D}}-\lambda)^{-1}\right) \\ &= \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{d\lambda^{m-1}} (\gamma(\lambda) S(\lambda)) \\ &= \frac{1}{(m-1)!} \sum_{\substack{p+q=m-1 \\ p,q \geqslant 0}} \binom{m-1}{p} \gamma^{(p)}(\lambda) S^{(q)}(\lambda). \end{aligned} \tag{3.14}$$

Since, by Proposition 3.3(i), (3.13) and (2.3),

$$\gamma^{(p)}(\lambda) S^{(q)}(\lambda) \in \mathfrak{S}_{\frac{n-1}{2p+3/2},\infty} \cdot \mathfrak{S}_{\frac{n-1}{2q+1/2},\infty} = \mathfrak{S}_{\frac{n-1}{2(p+q)+2},\infty} = \mathfrak{S}_{\frac{n-1}{(2m)},\infty}, \tag{3.15}$$

for p, q with $p+q = m-1$, we obtain (3.10).

(ii) If $m > (n-1)/2$, then $(n-1)/2m < 1$ and, by (2.2) and (3.15), each term in the sum in (3.14) is a trace class operator and, by a similar argument, also $S^{(q)}(\lambda)\gamma^{(p)}(\lambda)$. Hence, the operator in (3.10) is a trace class operator, and we can apply the trace to (3.14) and use (2.4), (2.5) and Lemma 2.4(iii) to obtain

$$\begin{aligned} &(m-1)! \operatorname{tr}((A_{\mathrm{N}}-\lambda)^{-m} - (A_{\mathrm{D}}-\lambda)^{-m}) \\ &\quad = \operatorname{tr}\left(\sum_{\substack{p+q=m-1 \\ p,q \geqslant 0}} \binom{m-1}{p} \gamma^{(p)}(\lambda) S^{(q)}(\lambda) \right) \end{aligned}$$

$$= \sum_{\substack{p+q=m-1\\ p,q\geqslant 0}} \binom{m-1}{p} \operatorname{tr}(\gamma^{(p)}(\lambda)S^{(q)}(\lambda)) = \sum_{\substack{p+q=m-1\\ p,q\geqslant 0}} \binom{m-1}{p} \operatorname{tr}(S^{(q)}(\lambda)\gamma^{(p)}(\lambda))$$
$$= \operatorname{tr}\left(\sum_{\substack{p+q=m-1\\ p,q\geqslant 0}} \binom{m-1}{p} S^{(q)}(\lambda)\gamma^{(p)}(\lambda) \right) = \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}} (S(\lambda)\gamma(\lambda)) \right)$$
$$= \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}} \left(M(\lambda)^{-1}\gamma(\bar{\lambda})^*\gamma(\lambda) \right) \right) = \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}} \left(M(\lambda)^{-1}M'(\lambda) \right) \right),$$

which completes the proof. □

In the following theorem, which contains the main result of this note, we prove weak Schatten–von Neumann estimates for resolvent power differences of two self-adjoint realizations $A_{[B_1]}$ and $A_{[B_2]}$ of $\mathcal{L}$ with Robin or more general non-local boundary conditions. In this situation, the estimates are better than for the pair of Dirichlet and Neumann realizations in Theorem 3.6. For the first powers of the resolvents, this has already been observed in [**6**, **7**, **28**]. In the special important case when the resolvent power difference is a trace class operator, we express its trace as the trace of a certain operator acting on the boundary $\partial\Omega$, which is given in terms of the Weyl function and the operators B_1 and B_2 in the boundary conditions; cf. [**7**, Corollary 4.12] for the case of first powers and [**8**, **21**] for one-dimensional Schrödinger operators and other finite-dimensional situations. We also mention that the special case of classical Robin boundary conditions, where B_1 and B_2 are multiplication operators with real-valued L^∞ functions, is contained in the theorem.

Theorem 3.7. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition 3.1 with Weyl function M and let A_{N} be the self-adjoint Neumann operator in (3.1). Moreover, let B_1 and B_2 be bounded self-adjoint operators in $L^2(\partial\Omega)$, define $A_{[B_1]}$ and $A_{[B_2]}$ as in (3.6) and set*

$$t := \begin{cases} \dfrac{n-1}{s} & \text{if } B_1 - B_2 \in \mathfrak{S}_{s,\infty}(L^2(\partial\Omega)) \text{ for some } s > 0,\\ 0 & \text{otherwise.} \end{cases}$$

Then the following statements hold.

(i) *For all $m \in \mathbb{N}$ and $\lambda \in \rho(A_{[B_1]}) \cap \rho(A_{[B_2]})$,*

$$(A_{[B_1]} - \lambda)^{-m} - (A_{[B_2]} - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/(2m+t+1),\infty}(L^2(\Omega)). \tag{3.16}$$

(ii) *If $m > (n-t)/2 - 1$, then the resolvent power difference in (3.16) is a trace class operator and, for all $\lambda \in \rho(A_{[B_1]}) \cap \rho(A_{[B_2]}) \cap \rho(A_{\mathrm{N}})$,*

$$\operatorname{tr}\left((A_{[B_1]} - \lambda)^{-m} - (A_{[B_2]} - \lambda)^{-m} \right) = \frac{1}{(m-1)!} \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}} (U(\lambda)M'(\lambda)) \right), \tag{3.17}$$

where $U(\lambda) := (I - B_1M(\lambda))^{-1}(B_1 - B_2)(I - M(\lambda)B_2)^{-1}$.

Proof. (i) In order to shorten notation and to avoid the distinction of several cases, we set

$$\mathfrak{A}_r := \begin{cases} \mathfrak{S}_{(n-1)/r,\infty}(L^2(\partial\Omega)) & \text{if } r > 0,\\ \mathcal{B}(L^2(\partial\Omega)) & \text{if } r = 0. \end{cases}$$

It follows from (2.3) and the fact that $\mathfrak{S}_{p,\infty}(L^2(\partial\Omega))$, $p > 0$ is an ideal in $\mathcal{B}(L^2(\partial\Omega))$ that

$$\mathfrak{A}_{r_1} \cdot \mathfrak{A}_{r_2} = \mathfrak{A}_{r_1+r_2}, \quad r_1, r_2 \geqslant 0. \tag{3.18}$$

Moreover, the assumption on the difference of B_1 and B_2 yields

$$B_1 - B_2 \in \mathfrak{A}_t. \tag{3.19}$$

The proof of item (i) is divided into three steps.

Step 1. Let B be a bounded self-adjoint operator in $L^2(\partial\Omega)$ and set

$$T(\lambda) := (I - BM(\lambda))^{-1}, \quad \lambda \in \rho(A_{[B]}) \cap \rho(A_{\mathrm{N}}),$$

where $T(\lambda) \in \mathcal{B}(L^2(\partial\Omega))$ by Corollary 3.5. We show

$$T^{(k)}(\lambda) \in \mathfrak{A}_{2k+1}, \quad k \in \mathbb{N}, \tag{3.20}$$

by induction. Relation (2.8) implies

$$T'(\lambda) = T(\lambda)BM'(\lambda)T(\lambda), \tag{3.21}$$

which is in $\mathfrak{A}_3$ by Proposition 3.3(ii). Let $l \in \mathbb{N}$ and assume that (3.20) is true for every $k = 1, \dots, l$, which implies in particular that

$$T^{(k)}(\lambda) \in \mathfrak{A}_{2k}, \quad k = 0, \dots, l. \tag{3.22}$$

Then

$$T^{(l+1)}(\lambda) = \frac{d^l}{d\lambda^l}\big(T(\lambda)BM'(\lambda)T(\lambda)\big) = \sum_{\substack{p+q+r=l\\ p,q,r\geqslant 0}} \frac{l!}{p!\,q!\,r!} T^{(p)}(\lambda)BM^{(q+1)}(\lambda)T^{(r)}(\lambda),$$

by (3.21) and (2.7). Relation (3.22), the boundedness of B, Proposition 3.3(ii) and (3.18) imply

$$T^{(p)}(\lambda)BM^{(q+1)}(\lambda)T^{(r)}(\lambda) \in \mathfrak{A}_{2p} \cdot \mathfrak{A}_{2(q+1)+1} \cdot \mathfrak{A}_{2r} = \mathfrak{A}_{2(l+1)+1},$$

since $p + q + r = l$. This shows (3.20) for $k = l + 1$ and hence, by induction, for all $k \in \mathbb{N}$. Since $T(\lambda) \in \mathcal{B}(L^2(\partial\Omega))$, we have

$$T^{(k)}(\lambda) \in \mathfrak{A}_{2k}, \quad k \in \mathbb{N}_0,\ \lambda \in \rho(A_{\mathrm{N}}), \tag{3.23}$$

and by similar considerations also

$$\frac{d^k}{d\lambda^k}\big(I - M(\lambda)B\big)^{-1} \in \mathfrak{A}_{2k}, \quad k \in \mathbb{N}_0,\ \lambda \in \rho(A_{\mathrm{N}}). \tag{3.24}$$

Step 2. With B_1, B_2 as in the statement of the theorem, set

$$T_1(\lambda) := (I - B_1M(\lambda))^{-1} \quad \text{and} \quad T_2(\lambda) := (I - M(\lambda)B_2)^{-1}$$

for $\lambda \in \rho(A_{[B_1]}) \cap \rho(A_{[B_2]}) \cap \rho(A_{\mathrm{N}})$. We can write $U(\lambda) = T_1(\lambda)(B_1 - B_2)T_2(\lambda)$ and hence

$$U^{(k)}(\lambda) = \frac{d^k}{d\lambda^k}\big(T_1(\lambda)(B_1 - B_2)T_2(\lambda)\big) = \sum_{\substack{p+q=k\\ p,q\geqslant 0}} \binom{k}{p} T_1^{(p)}(\lambda)(B_1 - B_2)T_2^{(q)}(\lambda).$$

By (3.19), (3.23) and (3.24), each term in the sum satisfies

$$T_1^{(p)}(\lambda)(B_1 - B_2)T_2^{(q)}(\lambda) \in \mathfrak{A}_{2p} \cdot \mathfrak{A}_t \cdot \mathfrak{A}_{2q} = \mathfrak{A}_{2k+t},$$

and hence

$$U^{(k)}(\lambda) \in \mathfrak{A}_{2k+t}, \quad k \in \mathbb{N}_0,\ \lambda \in \rho(A_{\mathrm{N}}). \tag{3.25}$$

Step 3. By applying (3.7) to $A_{[B_1]}$ and (3.8) to $A_{[B_2]}$ and taking the difference, we obtain that, for $\lambda \in \rho(A_{[B_1]}) \cap \rho(A_{[B_2]}) \cap \rho(A_{\mathrm{N}})$,

$$\begin{aligned}
&(A_{[B_1]} - \lambda)^{-1} - (A_{[B_2]} - \lambda)^{-1} \\
&\quad = \gamma(\lambda) \Big[\big(I - B_1 M(\lambda)\big)^{-1} B_1 - B_2 \big(I - M(\lambda) B_2\big)^{-1} \Big] \gamma(\bar{\lambda})^* \\
&\quad = \gamma(\lambda) \Big[\big(I - B_1 M(\lambda)\big)^{-1} B_1 \big(I - M(\lambda) B_2\big) \big(I - M(\lambda) B_2\big)^{-1} \\
&\qquad - \big(I - B_1 M(\lambda)\big)^{-1} \big(I - B_1 M(\lambda)\big) B_2 \big(I - M(\lambda) B_2\big)^{-1} \Big] \gamma(\bar{\lambda})^* \\
&\quad = \gamma(\lambda) \Big[\big(I - B_1 M(\lambda)\big)^{-1} (B_1 - B_2) \big(I - M(\lambda) B_2\big)^{-1} \Big] \gamma(\bar{\lambda})^* = \gamma(\lambda) U(\lambda) \gamma(\bar{\lambda})^*.
\end{aligned}$$

Taking derivatives we get, for $m \in \mathbb{N}$,

$$\begin{aligned}
&(A_{[B_1]} - \lambda)^{-m} - (A_{[B_2]} - \lambda)^{-m} \\
&\quad = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{d\lambda^{m-1}} \big((A_{[B_1]} - \lambda)^{-1} - (A_{[B_2]} - \lambda)^{-1} \big) \\
&\quad = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{d\lambda^{m-1}} \big(\gamma(\lambda) U(\lambda) \gamma(\bar{\lambda})^* \big) \\
&\quad = \frac{1}{(m-1)!} \sum_{\substack{p+q+r=m-1 \\ p,q,r \geqslant 0}} \frac{(m-1)!}{p!\,q!\,r!} \gamma^{(p)}(\lambda) U^{(q)}(\lambda) \frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^*.
\end{aligned} \tag{3.26}$$

By Proposition 3.3(i) and (3.25), each term in the sum satisfies

$$\gamma^{(p)}(\lambda) U^{(q)}(\lambda) \frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^* \in \mathfrak{S}_{\frac{n-1}{2p+3/2},\infty} \cdot \mathfrak{S}_{\frac{n-1}{2q+t},\infty} \cdot \mathfrak{S}_{\frac{n-1}{2r+3/2},\infty} = \mathfrak{S}_{\frac{n-1}{2m+t+1},\infty}, \tag{3.27}$$

which proves (3.16).

(ii) If $m > (n-t)/2 - 1$, then $(n-1)/(2m+t+1) < 1$ and, by (2.2) and (3.27), all terms in the sum in (3.26) are trace class operators, and the same is true if we change the order in the product in (3.27). Hence, we can apply the trace to the expression in (3.26) and use (2.4), (2.5) and Lemma 2.4(iii) to obtain

$$\begin{aligned}
&(m-1)! \operatorname{tr}((A_{[B_1]} - \lambda)^{-m} - (A_{[B_2]} - \lambda)^{-m}) \\
&\quad = \operatorname{tr} \left(\sum_{\substack{p+q+r=m-1 \\ p,q,r \geqslant 0}} \frac{(m-1)!}{p!\,q!\,r!} \gamma^{(p)}(\lambda) U^{(q)}(\lambda) \frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^* \right) \\
&\quad = \sum_{\substack{p+q+r=m-1 \\ p,q,r \geqslant 0}} \frac{(m-1)!}{p!\,q!\,r!} \operatorname{tr} \left(\gamma^{(p)}(\lambda) U^{(q)}(\lambda) \frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^* \right) \\
&\quad = \sum_{\substack{p+q+r=m-1 \\ p,q,r \geqslant 0}} \frac{(m-1)!}{p!\,q!\,r!} \operatorname{tr} \left(U^{(q)}(\lambda) \left(\frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^* \right) \gamma^{(p)}(\lambda) \right) \\
&\quad = \operatorname{tr} \left(\sum_{\substack{p+q+r=m-1 \\ p,q,r \geqslant 0}} \frac{(m-1)!}{p!\,q!\,r!} U^{(q)}(\lambda) \left(\frac{d^r}{d\lambda^r} \gamma(\bar{\lambda})^* \right) \gamma^{(p)}(\lambda) \right) \\
&\quad = \operatorname{tr} \left(\frac{d^{m-1}}{d\lambda^{m-1}} \big(U(\lambda) \gamma(\bar{\lambda})^* \gamma(\lambda) \big) \right) = \operatorname{tr} \left(\frac{d^{m-1}}{d\lambda^{m-1}} \big(U(\lambda) M'(\lambda) \big) \right),
\end{aligned}$$

which shows (3.17). □

REMARK 3.8. The statements of Theorem 3.7 remain true if A is an arbitrary closed symmetric operator in a Hilbert space $\mathcal{H}$ and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi-boundary-triple for A^* such that $\operatorname{ran}\Gamma_0 = \mathcal{G}$ and the statements of Proposition 3.3 are true with $L^2(\Omega)$ and $L^2(\partial\Omega)$ replaced by $\mathcal{H}$ and $\mathcal{G}$, respectively.

As a special case of the last theorem, let us consider the situation when $B_1 = B$ and $B_2 = 0$, where B is a bounded self-adjoint operator in $L^2(\partial\Omega)$. This immediately leads to the following corollary.

COROLLARY 3.9. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition 3.1 with Weyl function M and let A_N be the self-adjoint Neumann operator in (3.1). Moreover, let B be a bounded self-adjoint operator in $L^2(\partial\Omega)$, define $A_{[B]}$ as in (3.6) and set*

$$t := \begin{cases} \dfrac{n-1}{s} & \text{if } B \in \mathfrak{S}_{s,\infty}(L^2(\partial\Omega)) \text{ for some } s > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following statements hold.

(i) *For all $m \in \mathbb{N}$ and $\lambda \in \rho(A_{[B]}) \cap \rho(A_N)$,*

$$(A_{[B]} - \lambda)^{-m} - (A_N - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/(2m+t+1),\infty}(L^2(\Omega)).$$

(ii) *If $m > (n-t)/2 - 1$, then the resolvent power difference in (3.28) is a trace class operator and, for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_N)$,*

$$\begin{aligned} &\operatorname{tr}\left((A_{[B]} - \lambda)^{-m} - (A_N - \lambda)^{-m}\right) \\ &\quad = \frac{1}{(m-1)!} \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\left(\left(I - BM(\lambda)\right)^{-1} BM'(\lambda)\right)\right). \end{aligned}$$

The following theorem, where we compare operators with non-local and Dirichlet boundary conditions, is a consequence of Theorems 3.6 and 3.7.

THEOREM 3.10. *Let $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ be the quasi-boundary-triple from Proposition 3.1 with Weyl function M and let A_D be the self-adjoint Dirichlet operator in (3.1). Moreover, let B be a bounded self-adjoint operator in $L^2(\partial\Omega)$ and define $A_{[B]}$ as in (3.6). Then the following statements hold.*

(i) *For all $m \in \mathbb{N}$ and $\lambda \in \rho(A_{[B]}) \cap \rho(A_D)$,*

$$(A_{[B]} - \lambda)^{-m} - (A_D - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/2m,\infty}(L^2(\Omega)). \tag{3.28}$$

(ii) *If $m > (n-1)/2$, then the resolvent power difference in (3.28) is a trace class operator and, for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_D) \cap \rho(A_N)$,*

$$\operatorname{tr}\left((A_{[B]} - \lambda)^{-m} - (A_D - \lambda)^{-m}\right) = \frac{1}{(m-1)!} \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\left(V(\lambda)M'(\lambda)\right)\right), \tag{3.29}$$

where $V(\lambda) := (I - M(\lambda)B)^{-1}M(\lambda)^{-1}$.

Proof. (i) Let us fix $\lambda \in \rho(A_{[B]}) \cap \rho(A_D) \cap \rho(A_N)$. From Theorems 3.6(i) and 3.7(i), it follows that

$$\begin{aligned} X_1(\lambda) &:= (A_N - \lambda)^{-m} - (A_D - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/2m,\infty}, \\ X_2(\lambda) &:= (A_{[B]} - \lambda)^{-m} - (A_N - \lambda)^{-m} \in \mathfrak{S}_{(n-1)/(2m+1),\infty} \subset \mathfrak{S}_{(n-1)/2m,\infty}, \end{aligned}$$

and thus

$$(A_{[B]} - \lambda)^{-m} - (A_{\mathrm{D}} - \lambda)^{-m} = X_1(\lambda) + X_2(\lambda) \in \mathfrak{S}_{(n-1)/2m,\infty}.$$

By analyticity, we can extend this to all points λ in $\rho(A_{[B]}) \cap \rho(A_{\mathrm{D}})$.

(ii) If $m > (n-1)/2$, then $(n-1)/2m < 1$ and hence, by item (i) and (2.2), the operator in (3.28) is a trace class operator. Using Theorem 3.6(ii) and Corollary 3.9(ii), we obtain

$$\begin{aligned} &\operatorname{tr}\left((A_{[B]} - \lambda)^{-m} - (A_{\mathrm{D}} - \lambda)^{-m}\right) = \operatorname{tr}\left(X_1(\lambda) + X_2(\lambda)\right) \\ &\quad = \frac{1}{(m-1)!} \operatorname{tr}\left(\frac{d^{m-1}}{d\lambda^{m-1}}\left[\left(M(\lambda)^{-1} + \left(I - BM(\lambda)\right)^{-1}B\right) M'(\lambda)\right]\right). \end{aligned}$$

Since

$$\begin{aligned} &M(\lambda)^{-1} + \left(I - BM(\lambda)\right)^{-1}B \\ &\quad = \left(I - BM(\lambda)\right)^{-1}\left[\left(I - BM(\lambda)\right) + BM(\lambda)\right] M(\lambda)^{-1} = V(\lambda), \end{aligned}$$

this implies (3.29). □

Note that, for B a multiplication operator by a bounded function β, the statement in (i) of the previous theorem is exactly the estimate (1.2).

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