

Construction of self-adjoint differential operators with prescribed spectral properties

Jussi Behrndt^{1,*} and Andrii Khrabustovskyi^{1,2}

¹ Institute of Applied Mathematics, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria

² Department of Physics, Faculty of Science, University of Hradec Králové, Rokytanského 62, 500 03 Hradec Králové, Czech Republic

Received XXXX, revised XXXX, accepted XXXX

Published online XXXX

Key words Differential operator, Schrödinger operator, essential spectrum, discrete spectrum, Neumann Laplacian, singular potential, boundary condition

Dedicated to the memory of our friend and colleague Hagen Neidhardt

In this expository article some spectral properties of self-adjoint differential operators are investigated. The main objective is to illustrate and (partly) review how one can construct domains or potentials such that the essential or discrete spectrum of a Schrödinger operator of a certain type (e.g. the Neumann Laplacian) coincides with a predefined subset of the real line. Another aim is to emphasize that the spectrum of a differential operator on a bounded domain or bounded interval is not necessarily discrete, that is, eigenvalues of infinite multiplicity, continuous spectrum, and eigenvalues embedded in the continuous spectrum may be present. This *unusual* spectral effect is, very roughly speaking, caused by (at least) one of the following three reasons: The bounded domain has a rough boundary, the potential is singular, or the boundary condition is nonstandard. In three separate explicit constructions we demonstrate how each of these possibilities leads to a Schrödinger operator with prescribed essential spectrum.

Copyright line will be provided by the publisher

1 Introduction

This paper is concerned with spectral theory of self-adjoint differential operators in Hilbert spaces. Before we explain in more detail the topics and results we briefly recall the notions of discrete spectrum and essential spectrum, that play a key role here. Let A be a (typically unbounded) self-adjoint operator in an infinite dimensional complex Hilbert space \mathcal{H} , see also the beginning of Section 4 for more details on the *adjoint* of unbounded operators and the notion *self-adjoint*. The *spectrum* $\sigma(A)$ of A is a closed subset of the real line (which is unbounded if and only if A is unbounded) that consists of all those points λ such that $A - \lambda$ does not admit a bounded inverse. In the case that $A - \lambda$ is not invertible λ is called an *eigenvalue* of A and belongs to the *point spectrum*; in the case that $(A - \lambda)^{-1}$ exists as an unbounded operator the point λ belongs to the *continuous spectrum*. An eigenvalue is *discrete* if it is an isolated point in $\sigma(A)$ and the *eigenspace* $\ker(A - \lambda)$ is finite dimensional (such eigenvalues are often also called *normal* eigenvalues). The set of discrete (or normal) eigenvalues of A is denoted by $\sigma_{\text{disc}}(A)$; the complement of the discrete spectrum in $\sigma(A)$ is called the *essential spectrum* of A and the notation $\sigma_{\text{ess}}(A)$ is used for this set. It is clear that

$$\sigma(A) = \sigma_{\text{disc}}(A) \dot{\cup} \sigma_{\text{ess}}(A)$$

and that $\sigma_{\text{ess}}(A)$ consists of all those spectral points which are in the continuous spectrum, all eigenvalues embedded in the continuous spectrum and all isolated eigenvalues of infinite multiplicity. For the intuition it may be helpful to keep in mind that essential spectrum can only appear in an infinite dimensional Hilbert space, whereas the spectrum of any matrix is necessarily discrete and hence is always present (and the only type of spectrum) of self-adjoint operators in finite dimensional Hilbert spaces. We refer the reader to the monographs [2, 17, 31, 59, 81, 82, 84] for more details on the spectrum of self-adjoint operators.

* Corresponding author E-mail: behrndt@tugraz.at, Phone: +43 316 873 8127, Fax: +43 316 873 8621

The main objective of this expository paper is to illustrate and (partly) review how one can explicitly construct rough domains, singular potentials, or nonstandard boundary conditions with the property that the essential spectrum of a Schrödinger operator coincides with a predefined subset of the real line. The closely related problem of constructing Schrödinger operators with predefined discrete spectrum is also briefly discussed. Very roughly speaking, the results in Section 2 are contained in the well-known papers [8, 29, 53, 54], whereas the main results Theorem 3.3 and Theorem 4.3 in the later sections seem to be new.

More precisely, in Section 2 we treat Laplace operators subject to Neumann boundary conditions (Neumann Laplacians) on bounded domains. It is often believed that self-adjoint Laplace-type operators on bounded domains always have purely discrete spectrum (or, equivalently, a compact resolvent). This is indeed true for Laplace operators subject to Dirichlet boundary conditions (Dirichlet Laplacian), but, in general, not true for Neumann Laplacians. In fact, the discreteness of the spectrum of the Neumann Laplacian is equivalent to the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, and for this a necessary and sufficient criterion was obtained by C.J. Amick [8]; cf. Theorem 2.1. The standard example of a bounded domain for which essential spectrum for Neumann Laplacian appears is a so-called called “rooms-and-passages” domain: a chain of bounded domains (“rooms”) connected through narrow rectangles (“passages”), see Figure 1. Rooms-and-passages domains are widely used in spectral theory and in the theory of Sobolev spaces in order to demonstrate various peculiar effects (see, e.g., [8, 35, 45]). Some spectral properties of such domains were investigated in [26]. We also refer to the comprehensive monograph of V.G. Mazya [71] (see also earlier contributions [67–70]), where rooms-and-passages together with many other tricky domains were treated. In the celebrated paper [54] R. Hempel, L. Seco, and B. Simon constructed a rooms-and-passages domain such that the spectrum of the Neumann Laplacian coincides with a prescribed closed set $\mathfrak{S} \subset [0, \infty)$ with $0 \in \mathfrak{S}$. We review and prove their result in Theorem 2.4; here also the continuous dependence of the eigenvalues of Neumann Laplacians on varying domains discussed in Appendix A plays an important role. We also briefly recall another type of bounded domains – so-called “comb-like” domains – which allow to control the essential spectrum in the case $0 \notin \mathfrak{S}$. Rooms-and-passages domains can also be used in a convenient way to control the discrete spectrum within compact intervals. We demonstrate this in Theorem 2.6, where we establish a slightly weaker version of the following celebrated result by Y. Colin de Verdière [29]: for arbitrary numbers $0 = \lambda_1 < \lambda_2 < \dots < \lambda_m$ there exists a bounded domain $\Omega \subset \mathbb{R}^n$ such that the spectrum of the Neumann Laplacian on Ω is purely discrete and its first m eigenvalues coincide with the above numbers. One of the main ingredients in our proof is a multidimensional version of the intermediate value theorem by R. Hempel, T. Kriecherbauer, and P. Plankensteinner in [53]. In fact, our Theorem 2.6 is also contained in a more general result established in [53], where a domain was constructed in such a way that the essential spectrum and a part of the discrete spectrum of the Neumann Laplacian coincides with prescribed sets.

In Section 3 we show that similar tools and techniques can be used for a class of singular Schrödinger operators describing the motion of quantum particles in potentials being supported at a discrete set. These operators are known as *solvable models* of quantum mechanics [3]. Namely, we will treat differential operators defined by the formal expression

$$-\frac{d^2}{dz^2} + \sum_{k \in \mathbb{N}} \beta_k \langle \cdot, \delta'_{z_k} \rangle \delta'_{z_k},$$

where δ'_{z_k} is the distributional derivative of the delta-function supported at z_k , $\langle \phi, \delta'_{z_k} \rangle$ denotes its action on the test function ϕ and $\beta_k \in \mathbb{R} \cup \{\infty\}$. Such operators are called Schrödinger operator with δ' -interactions (or *point dipole interactions*) and have been studied (also in the multidimensional setting) in numerous papers; here we only refer the reader to [6, 11–14, 16, 24, 25, 27, 28, 34, 36–44, 48, 52, 57, 61–65, 75, 89] and the references therein. We will show in Theorem 3.1 (see also Theorem 3.3) that the points z_k and coefficients β_k can be chosen in such a way that the essential spectrum of the above operator coincides with a predefined closed set. In our proof we make use of well-known convergence results for quadratic forms, which we briefly recall in Appendix B. Some of our arguments are also based and related to results in the recent paper [63] by A. Kostenko and M.M. Malamud.

Finally, in Section 4 we consider a slightly more abstract problem which can also be viewed as a generalization of some of the above problems: for a given densely defined symmetric operator S with infinite deficiency indices, that is, S admits a self-adjoint extension A and $\text{dom}(A)/\text{dom}(S)$ is infinite dimensional, and under the assumption that there exists a self-adjoint extension with discrete spectrum (or, equivalently, compact resolvent), we construct a self-adjoint extensions of S with prescribed essential spectrum (possibly unbounded from below

and above). Here the prescribed essential spectrum is generated via a perturbation argument and a self-adjoint operator Ξ that acts in an infinite dimensional *boundary space* and plays the role of a parameter in a boundary condition. Our result is also related to the series of papers [4, 5, 19–23] by S. Albeverio, J. Brasche, M.M. Malamud, H. Neidhardt, and J. Weidmann in which the existence of self-adjoint extensions with prescribed point spectrum, absolutely continuous spectrum, and singular continuous spectrum in spectral gaps of a fixed underlying symmetric operator was discussed.

Acknowledgements

The authors are most grateful to the anonymous referee for a great many useful comments which improved the manuscript considerably. In particular, the referee kindly provided us with an elegant proof of Proposition A.3 and Corollary A.4. This article is based upon work from COST Action CA18232 MAT-DYN-NET, supported by COST (European Cooperation in Science and Technology), www.cost.eu. J.B. gratefully acknowledges support for the Distinguished Visiting Austrian Chair at Stanford University by the Europe Center and the Freeman Spogli Institute for International Studies. A.K. is supported by the Austrian Science Fund (FWF) under Project No. M 2310-N32.

2 Essential and discrete spectra of Neumann Laplacians

The main objective of this section is to highlight some spectral properties of the Neumann Laplacian on bounded domains. In the following let Ω be a bounded domain in \mathbb{R}^n and assume that $n \geq 2$. As usual the Hilbert space of (equivalence classes of) square integrable complex functions on Ω is denoted by $L^2(\Omega)$, and $H^1(\Omega)$ denotes the first order Sobolev space consisting of functions in $L^2(\Omega)$ that admit weak derivatives (of first order) in $L^2(\Omega)$. An efficient method to introduce the Neumann Laplacian in a mathematically rigorous way is to consider the sesquilinear form \mathfrak{a}_Ω defined by

$$\mathfrak{a}_\Omega[u, v] = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx, \quad \text{dom}(\mathfrak{a}_\Omega) = H^1(\Omega). \quad (2.1)$$

It is clear that this form is densely defined in $L^2(\Omega)$, nonnegative, and one can show that the form is closed, i.e. the form domain $H^1(\Omega)$ equipped with the scalar product $\mathfrak{a}_\Omega[\cdot, \cdot] + (\cdot, \cdot)_{L^2(\Omega)}$ is complete. The well-known first representation theorem (see, e.g. [59, Chapter 6, Theorem 2.1]) associates a unique nonnegative self-adjoint operator A_Ω in $L^2(\Omega)$ to the form \mathfrak{a}_Ω such that the domain inclusion $\text{dom}(A_\Omega) \subset \text{dom}(\mathfrak{a}_\Omega)$ and the equality

$$(A_\Omega u, v)_{L^2(\Omega)} = \mathfrak{a}_\Omega[u, v], \quad u \in \text{dom}(A_\Omega), \quad v \in \text{dom}(\mathfrak{a}_\Omega), \quad (2.2)$$

hold. The operator A_Ω is called *the Neumann Laplacian* on Ω . One can show that

- $A_\Omega u = -\Delta u$, where $-\Delta u$ is understood as a distribution.
- $\text{dom}(A_\Omega) \subset H_{\text{loc}}^2(\Omega) \cap H^1(\Omega)$.
- If $\partial\Omega$ is C^2 -smooth then

$$\text{dom}(A_\Omega) = \{u \in H^2(\Omega) : \partial_n u \upharpoonright_{\partial\Omega} = 0\},$$

where ∂_n denotes the normal derivative on $\partial\Omega$.

Typically the boundary condition $\partial_n u \upharpoonright_{\partial\Omega} = 0$ is referred to as *Neumann boundary condition*, which also justifies the terminology Neumann Laplacian. However, note that some regularity for the boundary of the domain has to be required in order to be able to deal with a normal derivative. For completeness, we note that the assumption of a C^2 -boundary above is not optimal (but almost) for H^2 -regularity of the domain of the Neumann Laplacian.

The rest of this section deals with some spectral properties of Neumann Laplacians. First of all we discuss in a preliminary situation that the Neumann Laplacian may have essential spectrum; since the domain Ω is bounded this may be a bit surprising at first sight. In this context we then recall a well-known result due to R. Hempel,

L. Seco, and B. Simon from [54] how to explicitly construct a bounded rooms-and-passages-type domain (with nonsmooth boundary) such that the essential spectrum of the Neumann Laplacian coincides with a prescribed closed set. Another related topic is to construct Neumann Laplacians on appropriate domains such that finitely many discrete eigenvalues coincide with a given set of points. Here we recall a famous result due to Y. Colin de Verdière from [29], and supplement this theorem with a similar result which is proved with a simple rooms-and-passages-type strategy. Actually, Theorem 2.6 is also a special variant of a more general result by R. Hempel, T. Kriecherbauer, and P. Plankensteiner in [53].

2.1 Neumann Laplacians may have nonempty essential spectrum

Let A_Ω be the self-adjoint Neumann Laplacian in $L^2(\Omega)$ and denote by $\sigma_{\text{ess}}(A_\Omega)$ the essential spectrum of A_Ω . It is well-known that $\sigma_{\text{ess}}(A_\Omega) = \emptyset$ (which is equivalent to the compactness of the resolvent of A_Ω in $L^2(\Omega)$) if and only if

$$\text{the embedding } i_\Omega : H^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact;} \quad (2.3)$$

cf. [87, Satz 21.3]. If the boundary $\partial\Omega$ is sufficiently regular (for example Lipschitz) then (2.3) holds; this result is known as Rellich's embedding theorem. However, in general the embedding i_Ω need not be compact (see, e.g., the classical monograph of R. Courant and D. Hilbert [30]). In fact, C.J. Amick established in [8, Theorem 3] a necessary and sufficient criterion for the compactness of the embedding operator i_Ω , which we recall in the next theorem.

Theorem 2.1 (Amick, 1978) *The embedding i_Ω in (2.3) is compact if and only if*

$$\Gamma_\Omega := \lim_{\varepsilon \rightarrow 0} \sup_{u \in H^1(\Omega)} \frac{\|u\|_{L^2(\Omega^\varepsilon)}^2}{\|u\|_{H^1(\Omega)}^2} = 0, \quad (2.4)$$

where $\Omega^\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$.

Remark 2.2 Necessary and sufficient conditions for $\sigma_{\text{ess}}(A_\Omega) = \emptyset$ have been also obtained in [66]. These conditions are formulated in terms of capacities.

In [8] an example of a bounded domain $\Omega \subset \mathbb{R}^2$ consisting of countably many rooms R_k and passages P_k with $\Gamma_\Omega > 0$ was constructed (see Figure 1).

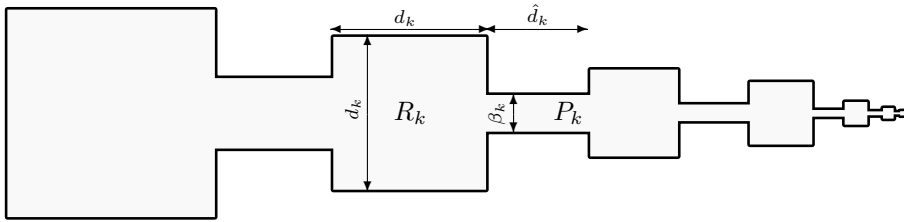


Fig. 1 Rooms-and-passages domain Ω

For the convenience of the reader we wish to recall this construction in the following. Note that we impose slightly different assumptions on the rooms and passages as compared to [8]. Consider some sequences $(d_k)_{k \in \mathbb{N}}$ and $(\hat{d}_k)_{k \in \mathbb{N}}$ of positive numbers such that

$$\sum_{k \in \mathbb{N}} d_k < \infty \quad (2.5)$$

and assume that there is a constant $C_1 > 0$ with the property

$$\hat{d}_k \leq C_1 \min \{d_k; d_{k+1}\}. \quad (2.6)$$

Note that (2.5)-(2.6) imply

$$\sum_{k \in \mathbb{N}} \hat{d}_k < \infty \quad (2.7)$$

and

$$\lim_{k \rightarrow \infty} d_k = 0, \quad \lim_{k \rightarrow \infty} \hat{d}_k = 0. \quad (2.8)$$

One can choose, for example, $d_k = (2k-1)^{-2}$, $\hat{d}_k = (2k)^{-2}$, $k \in \mathbb{N}$; then conditions (2.5) and (2.6) hold with $C_1 \in [\frac{9}{4}, \infty)$. Finally, let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that for all $k \in \mathbb{N}$

$$\beta_k \leq C_2 (\hat{d}_k)^\alpha \quad (2.9)$$

with some $\alpha \geq 3$ and $C_2 > 0$ such that

$$C_2 \leq \frac{1}{C_1} \cdot \left(\max_{k \in \mathbb{N}} \hat{d}_k \right)^{1-\alpha}. \quad (2.10)$$

In the next step define the sequence $(x_k)_{k \in \mathbb{N}}$ by

$$x_k := \sum_{j=1}^k (d_j + \hat{d}_j) - \hat{d}_k, \quad (2.11)$$

and define the rooms R_k and passages P_k by

$$R_k := (x_k - d_k, x_k) \times \left(-\frac{d_k}{2}, \frac{d_k}{2} \right) \quad (2.12)$$

and

$$P_k := [x_k, x_k + \hat{d}_k] \times \left(-\frac{\beta_k}{2}, \frac{\beta_k}{2} \right), \quad (2.13)$$

respectively. Finally, the union of R_k and P_k leads to the desired rooms-and-passages domain

$$\Omega := \bigcup_{k \in \mathbb{N}} (R_k \cup P_k). \quad (2.14)$$

From (2.5), (2.7), and (2.9) it is clear that Ω is bounded. Using (2.6), (2.9), (2.10) and taking into account that $\alpha > 3$ we obtain the estimate

$$\beta_k \leq C_2 (\hat{d}_k)^\alpha \leq C_2 (\hat{d}_k)^{\alpha-1} C_1 \min \{d_k, d_{k+1}\} \leq \min \{d_k, d_{k+1}\}. \quad (2.15)$$

Hence the thickness of the passage P_k is not larger than the sides of the adjacent rooms R_k and R_{k+1} , which also shows that Ω is indeed an open set.

It will now be illustrated that for this particular domain Ω the quantity Γ_Ω in (2.4) is positive, so that the embedding in (2.3) is not compact. In particular, the essential spectrum of the Neumann Laplacian in $L^2(\Omega)$ is not empty. For this purpose consider the piecewise linear functions u_k , $k = 2, 3, \dots$, defined by

$$u_k(\mathbf{x}) = \begin{cases} \frac{1}{d_k}, & \mathbf{x} = (x, y) \in R_k, \\ \frac{x_k + \hat{d}_k - x}{d_k \hat{d}_k}, & \mathbf{x} = (x, y) \in P_k, \\ \frac{x_{k-1} - x}{d_k (x_{k-1} - x_k + d_k)}, & \mathbf{x} = (x, y) \in P_{k-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $x_{k-1} - x_k + d_k = -\hat{d}_{k-1}$ by (2.11). It is easy to see that the function u_k belongs to $H^1(\Omega)$. Next we evaluate its L^2 -norm. One computes

$$\begin{aligned} \|u_k\|_{L^2(\Omega)}^2 &= \|u_k\|_{L^2(R_k)}^2 + \|u_k\|_{L^2(P_{k-1})}^2 + \|u_k\|_{L^2(P_k)}^2 \\ &= 1 + \frac{1}{3(d_k)^2} \left(\beta_{k-1} \hat{d}_{k-1} + \beta_k \hat{d}_k \right) \leq 1 + \frac{C_2}{3(d_k)^2} \left((\hat{d}_{k-1})^{\alpha+1} + (\hat{d}_k)^{\alpha+1} \right). \end{aligned} \quad (2.16)$$

We also have (cf. (2.6))

$$\hat{d}_{k-1} \leq C_1 d_k \quad \text{and} \quad \hat{d}_k \leq C_1 d_k. \quad (2.17)$$

Using (2.17) and taking into account that $\lim_{k \rightarrow \infty} d_k = 0$ and $\alpha \geq 3$, we obtain from (2.16):

$$\|u_k\|_{L^2(\Omega)}^2 = 1 + o(1) \quad \text{as } k \rightarrow \infty. \quad (2.18)$$

Now we estimate the L^2 -norm of ∇u_k . Using (2.9) and (2.17) we get

$$\begin{aligned} \|\nabla u_k\|_{L^2(\Omega)}^2 &= \|\nabla u_k\|_{L^2(P_{k-1})}^2 + \|\nabla u_k\|_{L^2(P_k)}^2 = \frac{1}{(d_k)^2} \left(\frac{\beta_{k-1}}{\hat{d}_{k-1}} + \frac{\beta_k}{\hat{d}_k} \right) \\ &\leq \frac{C_2}{(d_k)^2} \left((\hat{d}_{k-1})^{\alpha-1} + (\hat{d}_k)^{\alpha-1} \right) \leq 2C_1 C_2 (d_k)^{\alpha-3}. \end{aligned} \quad (2.19)$$

Moreover, it is clear that for any $\varepsilon > 0$ there exists $k(\varepsilon) \in \mathbb{N}$ such that $\text{supp}(u_k) \subset \Omega^\varepsilon$ for all $k \geq k(\varepsilon)$. This, (2.18), (2.19), and (2.8) yield $\Gamma_\Omega > 0$ (recall that $\alpha \geq 3$). As an immediate consequence we conclude the following corollary.

Corollary 2.3 *Let Ω be the bounded rooms-and-passages domain in (2.14) and let A_Ω be the self-adjoint Neumann Laplacian in $L^2(\Omega)$. Then*

$$\sigma_{\text{ess}}(A_\Omega) \neq \emptyset$$

The natural question that arises in the context of Corollary 2.3 is what form the essential spectrum of the Neumann Laplacian A_Ω may have. This topic is discussed in the next subsection.

2.2 Neumann Laplacian with prescribed essential spectrum

In the celebrated paper [54] R. Hempel, L. Seco, and B. Simon have shown, using rooms-and-passages-type domains of a similar form as above, that the essential spectrum of the Neumann Laplacian can be a rather arbitrary closed subset of the real line. Below we briefly describe their construction.

We fix sequences $(d_k)_{k \in \mathbb{N}}$ and $(\hat{d}_k)_{k \in \mathbb{N}}$ of positive numbers satisfying

$$\sum_{k \in \mathbb{N}} (d_k + \hat{d}_k) < \infty, \quad (2.20)$$

and of course one then has (2.8). Let the domain $\Omega \subset \mathbb{R}^2$ consist of countably many rooms R_k and passages P_k . Furthermore, in each room we insert an additional ‘‘wall’’ $W_k^{\alpha_k}$ (see Figure 2), and the resulting modified room is then denoted by $R_k^{\alpha_k} = R_k \setminus W_k^{\alpha_k}$, so that

$$\Omega = \bigcup_{k \in \mathbb{N}} (R_k^{\alpha_k} \cup P_k) = \bigcup_{k \in \mathbb{N}} ((R_k \setminus W_k^{\alpha_k}) \cup P_k). \quad (2.21)$$

Here R_k and P_k are defined by (2.12) and (2.13) with β_k satisfying

$$0 < \beta_k \leq \min \{d_k, d_{k+1}\}; \quad (2.22)$$

cf. (2.15). The walls $W_k^{\alpha_k}$ are given by

$$W_k^{\alpha_k} := \left\{ \mathbf{x} = (x, y) \in \mathbb{R}^2 : x = x_k - \frac{d_k}{2}, |y| \in \left[\frac{\alpha_k}{2}, \frac{d_k}{2} \right] \right\},$$

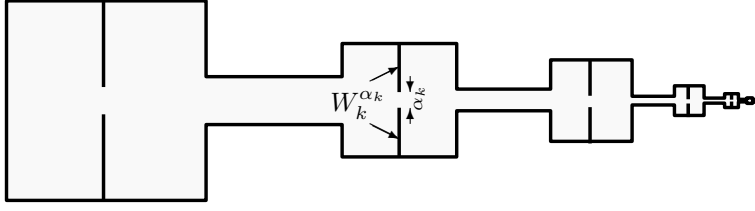


Fig. 2 Rooms-and-passages domain with additional walls

where it is assumed that the sequence $(\alpha_k)_{k \in \mathbb{N}}$ satisfies

$$0 < \alpha_k \leq d_k. \quad (2.23)$$

Theorem 2.4 (Hempel-Seco-Simon, 1991) *Let $\mathfrak{S} \subset [0, \infty)$ be an arbitrary closed set such that $0 \in \mathfrak{S}$. Then there exist sequences $(d_k)_{k \in \mathbb{N}}$, $(\hat{d}_k)_{k \in \mathbb{N}}$, $(\alpha_k)_{k \in \mathbb{N}}$, $(\beta_k)_{k \in \mathbb{N}}$ satisfying (2.20), (2.22), and (2.23) such that*

$$\sigma_{\text{ess}}(A_\Omega) = \mathfrak{S}. \quad (2.24)$$

Proof. This sketch of the proof from [54] consists of three steps. In the first step, where we skip a perturbation argument, it is shown that the original problem to ensure (2.24) for the Neumann Laplacian A_Ω on the domain Ω can be reduced to show the same property for a “decoupled” Neumann Laplacian A_{dec} . The spectrum of this operator can be described explicitly, which is done in the second step. Finally, in a third step the parameters are adjusted in such a way that (2.24) holds.

Step 1. Let $(d_k)_{k \in \mathbb{N}}$, $(\hat{d}_k)_{k \in \mathbb{N}}$, $(\alpha_k)_{k \in \mathbb{N}}$, and $(\beta_k)_{k \in \mathbb{N}}$ be some sequences that satisfy (2.20), (2.22), and (2.23). Let Ω be the corresponding domain in (2.21) and let A_Ω be the Neumann Laplacian on Ω defined via the quadratic form as in (2.1)–(2.2). In the following we denote by $A_{R_k^{\alpha_k}}$ the Neumann Laplacian on the domain $R_k^{\alpha_k} = R_k \setminus W_k^{\alpha_k}$, also defined via the quadratic form

$$\mathfrak{a}_{R_k^{\alpha_k}}[u, v] = \int_{R_k^{\alpha_k}} \nabla u \cdot \overline{\nabla v} \, dx, \quad \text{dom}(\mathfrak{a}_{R_k^{\alpha_k}}) = \text{H}^1(R_k^{\alpha_k}), \quad (2.25)$$

in the same way as in (2.1)–(2.2). Informally speaking, the functions in the domain of this operator satisfy Neumann boundary conditions on the boundary of the room R_k and, in addition, Neumann boundary conditions on both sides of the additional wall $W_k^{\alpha_k}$. Furthermore, we will make use of self-adjoint Laplacians on the interiors \mathring{P}_k of the passages P_k with mixed Dirichlet and Neumann boundary conditions. More precisely, $A_{\mathring{P}_k}^{\text{DN}}$ denotes the self-adjoint Laplacian defined on a subspace of $\text{H}^1(\mathring{P}_k)$, where it is assumed that the functions in the domain satisfy Neumann boundary conditions on $\{\mathbf{x} = (x, y) \in \partial \mathring{P}_k : y = \pm \beta_k/2\}$ and Dirichlet boundary conditions on the remaining part $\{\mathbf{x} = (x, y) \in \partial \mathring{P}_k : x = x_k \vee x = x_k + \hat{d}_k\}$ of the boundary. Now consider the “decoupled” operator

$$A_{\text{dec}} = \bigoplus_{k \in \mathbb{N}} (A_{R_k^{\alpha_k}} \oplus A_{\mathring{P}_k}^{\text{DN}}) \quad (2.26)$$

as an orthogonal sum of the self-adjoint operators $A_{R_k^{\alpha_k}}$ and $A_{\mathring{P}_k}^{\text{DN}}$ in the space

$$\text{L}^2(\Omega) = \bigoplus_{k \in \mathbb{N}} (\text{L}^2(R_k^{\alpha_k}) \oplus \text{L}^2(\mathring{P}_k)).$$

Then one can show that the resolvent difference $(A_{\text{dec}} + \text{I})^{-1} - (A_\Omega + \text{I})^{-1}$ of the Neumann Laplacian A_Ω and the decoupled operator A_{dec} is a compact operator provided $\beta_k \rightarrow 0$ sufficiently fast as $k \rightarrow \infty$. We fix such a sequence $(\beta_k)_{k \in \mathbb{N}}$; then from Weyl’s theorem (see, e.g., [82, Theorem XIII.14]) one concludes

$$\sigma_{\text{ess}}(A_\Omega) = \sigma_{\text{ess}}(A_{\text{dec}})$$

and hence it remains to show that

$$\sigma_{\text{ess}}(A_{\text{dec}}) = \mathfrak{S}.$$

Step 2. First we shall explain how the eigenvalues of the Neumann Laplacian on $R_k^{\alpha_k}$ depend on the size of the wall $W_k^{\alpha_k}$ inside R_k . In this step of the proof the value $\alpha_k = 0$ is also allowed (in this case the room $R_k^{\alpha_k}$ decouples: it becomes a union of two disjoint rectangles). We denote the eigenvalues of $A_{R_k^{\alpha_k}}$ (counted with multiplicities) and ordered as a nondecreasing sequence by $(\lambda_j(R_k^{\alpha_k}))_{j \in \mathbb{N}}$. It is not difficult to check that the corresponding forms $\mathfrak{a}_{R_k^{\alpha_k}}$ in (2.25) are monotone in the parameter α_k , that is, for $0 \leq \alpha_k \leq \tilde{\alpha}_k \leq d_k$ one has

$$\begin{aligned} \text{dom}(\mathfrak{a}_{R_k^{\alpha_k}}) &\supset \text{dom}(\mathfrak{a}_{R_k^{\tilde{\alpha}_k}}), \\ \mathfrak{a}_{R_k^{\alpha_k}}[u, u] &= \mathfrak{a}_{R_k^{\tilde{\alpha}_k}}[u, u] \text{ for all } u \in \text{dom}(\mathfrak{a}_{R_k^{\tilde{\alpha}_k}}), \end{aligned}$$

which means $\mathfrak{a}_{R_k^{\alpha_k}} \leq \mathfrak{a}_{R_k^{\tilde{\alpha}_k}}$ in the sense of ordering of forms. Then it follows from the min-max principle (see, e.g., [31, Section 4.5]) that for each $j \in \mathbb{N}$ the function

$$[0, d_k] \supset \alpha_k \mapsto \lambda_j(R_k^{\alpha_k}) \tag{2.27}$$

is nondecreasing and by Theorem A.1 this function is also continuous. In the present situation it is clear that

$$\lambda_1(R_k^{\alpha_k}) = 0, \tag{2.28}$$

and

$$\lambda_2(R_k^{\alpha_k}) = \begin{cases} 0, & \alpha_k = 0, \\ (\pi/d_k)^2, & \alpha_k = d_k, \end{cases} \tag{2.29}$$

and due to the monotonicity of the function (2.27) one also has

$$\lambda_3(R_k^{\alpha_k}) \geq \lambda_3(R_k^0) = (\pi/d_k)^2. \tag{2.30}$$

Furthermore, if $(\mu_j(\mathring{P}_k))_{j \in \mathbb{N}}$ denote the eigenvalues (counted with multiplicities) of $A_{\mathring{P}_k}^{\text{DN}}$ ordered as a nondecreasing sequence then one verifies that the first eigenvalue $\mu_1(\mathring{P}_k)$ is given by

$$\mu_1(\mathring{P}_k) = (\pi/\hat{d}_k)^2 \tag{2.31}$$

for all $k \in \mathbb{N}$. From the orthogonal sum structure in (2.26) it is clear that

$$\sigma_{\text{ess}}(A_{\text{dec}}) = \text{acc}((\lambda_j(R_k^{\alpha_k}))_{j,k \in \mathbb{N}}) \cup \text{acc}((\mu_j(\mathring{P}_k))_{j,k \in \mathbb{N}}),$$

where the symbol acc denotes the set of accumulation points of a sequence. Observe that the eigenvalues $(\mu_j(\mathring{P}_k))_{j \in \mathbb{N}}$ do not have any finite accumulation point (the smallest eigenvalue satisfies (2.31) and $\lim_{k \rightarrow \infty} \hat{d}_k = 0$ by assumption) and hence we obtain

$$\sigma_{\text{ess}}(A_{\text{dec}}) = \text{acc}((\lambda_j(R_k^{\alpha_k}))_{j,k \in \mathbb{N}}).$$

Step 3. Now we complete the proof by adjusting the parameters in the above construction. Since by assumption \mathfrak{S} is a closed subset of $[0, \infty)$ one can always find a sequence $(s_k)_{k \in \mathbb{N}}$ such that

$$s_k > 0 \text{ and } \text{acc}((s_k)_{k \in \mathbb{N}}) = \begin{cases} \mathfrak{S} \setminus \{0\}, & 0 \text{ is an isolated point of } \mathfrak{S}, \\ \mathfrak{S}, & \text{otherwise.} \end{cases} \tag{2.32}$$

Next, for each $k \in \mathbb{N}$ we fix a number $d_k > 0$ such that

$$s_k < (\pi/d_k)^2. \tag{2.33}$$

In addition, we assume that the numbers d_k are chosen small enough so that

$$\sum_{k \in \mathbb{N}} d_k < \infty.$$

We also fix a sequence of positive numbers $(\hat{d}_k)_{k \in \mathbb{N}}$ such that

$$\sum_{k \in \mathbb{N}} \hat{d}_k < \infty.$$

Using the continuity of the function (2.27) and taking into account (2.29) and (2.33) it is clear that there exists $\alpha_k \in (0, d_k)$ such that the second eigenvalue $\lambda_2(R_k^{\alpha_k})$ of the Neumann Laplacian $A_{R_k^{\alpha_k}}$ satisfies

$$\lambda_2(R_k^{\alpha_k}) = s_k \tag{2.34}$$

for all $k \in \mathbb{N}$. Furthermore, by construction we also have $\lambda_1(R_k^{\alpha_k}) = 0$ and $\lambda_j(R_k^{\alpha_k}) \geq (\pi/d_k)^2$ for $j \geq 3$ (cf. (2.30)), and hence we conclude together with $\lim_{k \rightarrow \infty} d_k = 0$, (2.28), (2.32), (2.34), and $0 \in \mathfrak{S}$ that

$$\text{acc}((\lambda_j(R_k^{\alpha_k}))_{j,k \in \mathbb{N}}) = \text{acc}((\lambda_j(R_k^{\alpha_k}))_{j \leq 2, k \in \mathbb{N}}) = \{0\} \cup \text{acc}((s_k)_{k \in \mathbb{N}}) = \mathfrak{S}.$$

This implies $\sigma_{\text{ess}}(A_{\text{dec}}) = \mathfrak{S}$ and completes the proof. \square

Besides (2.24) it is also shown in [54] that the absolutely continuous spectrum $\sigma_{\text{ac}}(A_\Omega)$ of A_Ω is empty. The argument is as follows: It is verified that the difference $(A_\Omega + \text{I})^{-2} - (A_{\text{dec}} + \text{I})^{-2}$ is a trace class operator, and consequently the absolutely continuous spectra of A_Ω and A_{dec} coincide; cf. [83, page 30, Corollary 3]. Since $\sigma(A_{\text{dec}})$ is pure point one concludes $\sigma_{\text{ac}}(A_\Omega) = \sigma_{\text{ac}}(A_{\text{dec}}) = \emptyset$. Note, that the absolutely continuous spectrum of the Neumann Laplacian on a bounded domain is not always empty. For example, in [86] B. Simon constructed a bounded set Ω having the form of a ‘‘jelly roll’’ such that $\sigma_{\text{ac}}(A_\Omega) = [0, \infty)$.

In [54] R. Hempel, L. Seco, and B. Simon also constructed a domain for which (2.24) holds without the restriction $0 \in \mathfrak{S}$. For this purpose so-called comb-like domains are used, see Figure 3. To construct the comb one attaches a sequence of ‘‘teeth’’ $(T_k^{\alpha_k})_{k \in \mathbb{N}}$ to a fixed rectangle Q ; the tooth $T_k^{\alpha_k}$ is obtained from a rectangle T_k by removing an internal wall $W_k^{\alpha_k}$. The teeth have bounded lengths, shrinking widths, and are stacked together without gaps.

The analysis is similar to the rooms-and-passages case. One can prove that the Neumann Laplacian A_Ω on such a comb-like domain Ω is a compact perturbation of the decoupled operator

$$A_{\text{dec}} = \left(\bigoplus_{k \in \mathbb{N}} A_{T_k^{\alpha_k}}^{\text{DN}} \right) \oplus A_Q,$$

where A_Q is the Neumann Laplacian on Q and $A_{T_k^{\alpha_k}}^{\text{DN}}$ is the Laplace operator on the tooth $T_k^{\alpha_k}$ subject to Neumann boundary conditions on $\partial T_k^{\alpha_k} \setminus \partial Q$ and Dirichlet boundary conditions on $\partial T_k^{\alpha_k} \cap \partial Q$. The walls $W_k^{\alpha_k}$ are adjusted in such a way that the lowest eigenvalue of $A_{T_k^{\alpha_k}}^{\text{DN}}$ coincides with a predefined number s_k , while the next eigenvalues tend to ∞ as $k \rightarrow \infty$ and do not contribute to essential spectrum.

The important and somewhat surprising element of the rooms-and-passage and comb-like domain constructions is the form of the decoupled operators with mixed Dirichlet and Neumann boundary conditions. The fact that one chooses Dirichlet conditions on the common part of the boundaries of the passages P_k and modified rooms $R_k^{\alpha_k}$, and similarly on the common part of the boundaries of the modified teeth $T_k^{\alpha_k}$ and the rectangle Q is due to the following well-known effect (see, e.g., [9, 10, 58]): the spectrum of the Neumann Laplacian on $Q \cup T^\varepsilon$, where Q is a fixed domain and T^ε is an attached ‘‘handle’’ of fixed length L and width ε , converges to the direct sum of the Neumann Laplacian on Q and the one-dimensional Dirichlet Laplacian on $(0, L)$ as $\varepsilon \rightarrow 0$.

Finally, we note that A.A. Kiselev and B.S. Pavlov [60] obtained Theorem 2.4 for (a kind of) Neumann Laplacian on a bounded set consisting of an array of two-dimensional domains connected by intervals.

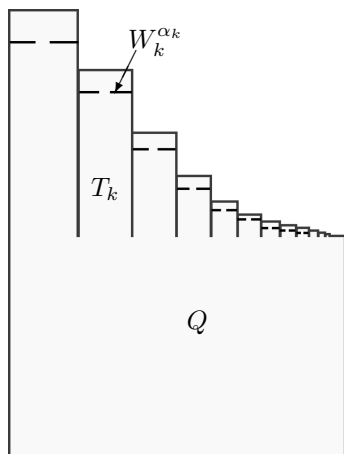


Fig. 3 Comb-like domain

2.3 Neumann Laplacian with prescribed discrete spectrum

In this section we are interested in the discrete spectrum of Neumann Laplacians. First we recall a result by Y. Colin de Verdière from [29].

Theorem 2.5 (Colin de Verdière, 1987) *Let $n \in \mathbb{N} \setminus \{1\}$ and assume that*

$$0 = \lambda_1 < \lambda_2 < \dots < \lambda_m, \quad m \in \mathbb{N}, \quad (2.35)$$

are fixed numbers. Then there exists a bounded domain $\Omega \subset \mathbb{R}^n$ such that the spectrum of the Neumann Laplacian A_Ω on Ω is purely discrete and

$$\lambda_k(\Omega) = \lambda_k, \quad k = 1, \dots, m,$$

where $(\lambda_k(\Omega))_{k \in \mathbb{N}}$ denotes the sequence of the eigenvalues of A_Ω numbered in increasing order with multiplicities taken into account.

In fact, the main result in [29] concerns Riemannian manifolds: for arbitrary λ_k , $k = 1, \dots, m$, satisfying (2.35) and an arbitrary compact connected manifold one can construct a Riemannian metric on this manifold in such a way that the first m eigenvalues of the corresponding Laplace-Beltrami operator coincide with the numbers λ_k , $k = 1, \dots, m$. The idea of the proof of this theorem in [29] is to first construct a suitable differential operator A_Γ on a metric graph Γ such that the first m eigenvalues of A_Γ coincide with λ_k , $k = 1, \dots, m$. Then the graph Γ is “blown” up to a tubular thin domain Ω in such a way that the first m eigenvalues of the Neumann Laplacian A_Ω on Ω are asymptotically close to the first m eigenvalues of A_Γ provided the cross-section of Ω tends to zero. For dimensions $n \geq 3$ the above theorem is extended in [29] by allowing nonsimple eigenvalues. More precisely, one can construct a domain in \mathbb{R}^n , $n \geq 3$, such that the first m eigenvalues of the Neumann Laplacian on this domain coincide with the predefined numbers

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_m.$$

A similar theorem for the Dirichlet Laplacian (at least for $n = 2$) can not be expected. In fact, by a well-known result of L.E. Payne, G. Pólya, and H.F. Weinberger from [78] the ratio between the k -th and the $(k - 1)$ -th Dirichlet eigenvalue (for compact domains in \mathbb{R}^2) is bounded from above (by a domain independent constant) and hence the eigenvalues can not be placed arbitrarily on $(0, \infty)$.

Theorem 2.5 was later improved by R. Hempel, T. Kriecherbauer, and P. Plankensteiner in [53], where a bounded domain Ω was constructed such that the essential spectrum and a finite part of the discrete spectrum of A_Ω coincide with predefined sets. In their construction comb-like domains were used; see Figure 3.

The next theorem may be viewed as a variant of Theorem 2.5, although it is actually a slightly weaker version. In fact, we present this result here since it can be proved with a similar rooms-and-passages domain strategy as

Theorem 2.4. An important ingredient is a multidimensional version of the intermediate value theorem from [53]; cf. Lemma 2.7 below.

Theorem 2.6 *Let $n \in \mathbb{N} \setminus \{1\}$ and assume that*

$$0 < \nu_1 < \nu_2 < \cdots < \nu_m, \quad m \in \mathbb{N},$$

are fixed numbers. Then there exists a bounded domain $\Omega \subset \mathbb{R}^n$ such that the spectrum of the Neumann Laplacian A_Ω on Ω is purely discrete and

$$\lambda_{m+k}(\Omega) = \nu_k, \quad k = 1, \dots, m,$$

where $(\lambda_l(\Omega))_{l \in \mathbb{N}}$ denotes the sequence of the eigenvalues of A_Ω numbered in increasing order with multiplicities taken into account.

Proof. We consider the case $n = 2$; the construction and the arguments in dimensions $n \geq 3$ are similar and left to the reader. The proof consists of several steps and in principle the strategy is similar to the one in the proof of Theorem 2.4. More precisely, we fix an arbitrary open and bounded interval $\mathcal{I} \subset (0, \infty)$ containing all the points ν_k . First we consider the decoupled domain consisting of m pairwise disjoint rooms $R_k^{\alpha_k} = R_k \setminus W_k^{\alpha_k}$ stacked in a row (as before R_k is a square and $W_k^{\alpha_k}$ is an internal wall in it). The first eigenvalue of the Neumann Laplacian on $R_k^{\alpha_k}$ is zero. Moreover, under a suitable choice of d_k and α_k the second eigenvalue coincides with ν_k and the third eigenvalue is larger than $\sup \mathcal{I}$. Consequently the first m eigenvalues of the Neumann Laplacian on $\cup_{k=1}^m R_k^{\alpha_k}$ are zero, the next m eigenvalues are ν_1, \dots, ν_m , and all further eigenvalues are contained in $(\sup \mathcal{I}, \infty)$. Afterwards we connect the rooms by small windows of length ε (see Figure 4); the resulting domain is denoted by Ω . If ε is small the spectrum changes slightly. Namely,

- the eigenvalues $\lambda_1(\Omega), \dots, \lambda_m(\Omega)$ remain in $[0, \inf \mathcal{I})$,
- the eigenvalues $\lambda_{m+1}(\Omega), \dots, \lambda_{2m}(\Omega)$ remain in small neighborhoods of ν_1, \dots, ν_m , respectively, moreover they still belong to \mathcal{I} ,
- the rest of the spectrum remains in $(\sup \mathcal{I}, \infty)$.

It remains to establish the coincidence of $\lambda_{k+m}(\Omega)$ and ν_k as $k = 1, \dots, m$ (so far they are only close as $\varepsilon > 0$ is sufficiently small), which is done by using a multidimensional version of mean value theorem from [53]. Roughly speaking, the shift of the eigenvalues that appears after inserting the small windows can be compensated by varying the constants $\alpha_1, \dots, \alpha_m$ appropriately. In the following we implement this strategy.

Step 1. Fix some positive numbers d_1, \dots, d_m and let $\alpha_1, \dots, \alpha_m$, and ε be nonnegative numbers satisfying

$$\alpha_k \in [0, d_k] \quad \text{and} \quad \varepsilon \in [0, \min_{k=1, \dots, m} d_k]. \quad (2.36)$$

In the following we shall use the notation $\alpha := \{\alpha_1, \dots, \alpha_m\}$. We introduce the domain $\Omega^{\alpha, \varepsilon}$ consisting of m modified rooms $R_k^{\alpha_k}$ being stacked in a row and connected through a small window P_k^ε of length ε . Each room $R_k^{\alpha_k}$ is obtained by removing from a square with side length d_k two additional walls of length $(d_k - \alpha_k)/2$; see Figure 4. More precisely, let $x_k := \sum_{j=1}^k d_j$, define the rooms by

$$R_k^{\alpha_k} = \left((x_k - d_k, x_k) \times \left(-\frac{d_k}{2}, \frac{d_k}{2} \right) \right) \setminus W_k^{\alpha_k},$$

where the walls are

$$W_k^{\alpha_k} = \left\{ (x, y) \in \mathbb{R}^2 : x = x_k - \frac{d_k}{2}, |y| \in \left[\frac{\alpha_k}{2}, \frac{d_k}{2} \right] \right\}$$

and the windows are $P_k^\varepsilon = \{x_k\} \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. Then the domain $\Omega^{\alpha, \varepsilon}$ that we will use is defined as

$$\Omega^{\alpha, \varepsilon} = \left(\bigcup_{k=1}^m R_k^{\alpha_k} \right) \cup \left(\bigcup_{k=1}^{m-1} P_k^\varepsilon \right).$$

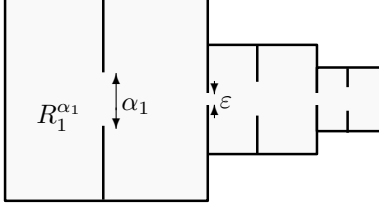


Fig. 4 Domain $\Omega^{\alpha, \varepsilon}$ for $m = 3$

The Neumann Laplacian on $\Omega^{\alpha, \varepsilon}$ will be denoted by $A_{\Omega^{\alpha, \varepsilon}}$ and it is important to observe that for $\varepsilon = 0$ this self-adjoint operator in $L^2(\Omega^{\alpha, \varepsilon})$ decouples in a finite orthogonal sum of Neumann Laplacians $A_{R_k^{\alpha_k}}$ on the rooms $R_k^{\alpha_k}$, that is, one has

$$A_{\Omega^{\alpha, 0}} = \bigoplus_{k=1}^m A_{R_k^{\alpha_k}} \quad (2.37)$$

with respect to the corresponding space decomposition

$$L^2(\Omega^{\alpha, 0}) = \bigoplus_{k=1}^m L^2(R_k^{\alpha_k}).$$

Note that $L^2(\Omega^{\alpha, 0}) = L^2(\Omega^{\alpha, \varepsilon})$ and hence $A_{\Omega^{\alpha, \varepsilon}}$ and $A_{\Omega^{\alpha, 0}}$ act in the same space for all ε in (2.36).

It is clear from (2.37) that the spectrum of the decoupled operator $A_{\Omega^{\alpha, 0}}$ is the union of the spectra of the Neumann Laplacians $A_{R_k^{\alpha_k}}$, $k = 1, \dots, m$. We recall (see Step 2 of the proof of Theorem 2.4) that the functions

$$[0, d_k] \ni \alpha_k \mapsto \lambda_j(R_k^{\alpha_k}), \quad j \in \mathbb{N}, \quad (2.38)$$

are continuous and nondecreasing. Moreover, one has

$$\lambda_1(R_k^{\alpha_k}) = 0, \quad \lambda_2(R_k^{\alpha_k}) = \begin{cases} 0, & \alpha_k = 0, \\ (\pi/d_k)^2, & \alpha_k = d_k, \end{cases} \quad (2.39)$$

and

$$\lambda_3(R_k^{\alpha_k}) \geq \lambda_3(R_k^0) = (\pi/d_k)^2. \quad (2.40)$$

Step 2. Now we approach the main part of the proof, where the parameters will be properly adjusted. Let $0 < \nu_1 < \dots < \nu_m$ be as in the assumptions of the theorem and fix an open interval \mathcal{I} such that

$$0 < \inf \mathcal{I} < \nu_1 \quad \text{and} \quad \nu_m < \sup \mathcal{I}. \quad (2.41)$$

Assume that the numbers d_1, \dots, d_m satisfy

$$\sup \mathcal{I} < \min_k (\pi/d_k)^2. \quad (2.42)$$

Furthermore, let us choose a constant $\gamma > 0$ such that the intervals

$$[\nu_k - \gamma, \nu_k + \gamma], \quad k = 1, \dots, m,$$

are pairwise disjoint and

$$\bigcup_{k=1}^m [\nu_k - \gamma, \nu_k + \gamma] \subset \mathcal{I} \quad (2.43)$$

holds. Next, we introduce the sets

$$L_k = \{\alpha_k \in [0, d_k] : \lambda_2(R_k^{\alpha_k}) \in [\nu_k - \gamma, \nu_k + \gamma]\}. \quad (2.44)$$

Using the continuity and monotonicity of the function in (2.38) and taking into account (2.39), (2.41)-(2.43) we conclude that each L_k is a nonempty compact interval. We set

$$\alpha_k^- = \min L_k, \quad \alpha_k^+ = \max L_k, \quad \text{and} \quad \mathcal{D} = \prod_{k=1}^m [\alpha_k^-, \alpha_k^+].$$

It is clear from (2.39), (2.41), and (2.43) that $\alpha_k^\pm > 0$. From now on we assume that

$$\alpha = \{\alpha_1, \dots, \alpha_m\} \in \mathcal{D}.$$

As usual we denote the eigenvalues of the decoupled operator $A_{\Omega^{\alpha,0}}$ in (2.37) by $(\lambda_j(\Omega^{\alpha,0}))_{j \in \mathbb{N}}$, counted with multiplicities and ordered as a nondecreasing sequence. It follows from $\lambda_1(R_k^{\alpha_k}) = 0$ for $k = 1, \dots, m$, that

$$\lambda_k(\Omega^{\alpha,0}) = 0 \quad \text{for} \quad k = 1, \dots, m.$$

Furthermore, (2.40), (2.42), (2.43), and (2.44) imply that the $m+k$ -th eigenvalues of the orthogonal sum $A_{\Omega^{\alpha,0}}$ coincides with the second eigenvalue $\lambda_2(R_k^{\alpha_k})$ of the Neumann Laplacian $A_{R_k^{\alpha_k}}$ as $k = 1, \dots, m$:

$$\lambda_{m+k}(\Omega^{\alpha,0}) = \lambda_2(R_k^{\alpha_k}) \in \overline{B_\gamma(\nu_k)} \quad \text{for} \quad k = 1, \dots, m. \quad (2.45)$$

Moreover, it is clear from (2.40) and (2.42) that

$$\lambda_k(\Omega^{\alpha,0}) > \sup \mathcal{I} \quad \text{for} \quad k = 2m+1, 2m+2, \dots$$

We introduce the functions $f_k^0 : \mathcal{D} \rightarrow \mathbb{R}$ by

$$f_k^0(\alpha_1, \alpha_2, \dots, \alpha_m) = \lambda_{k+m}(\Omega^{\alpha,0}), \quad k = 1, \dots, m. \quad (2.46)$$

It is important to note that due to (2.45) the value $\lambda_{k+m}(\Omega^{\alpha,0})$ of the function f_k^0 depends only on the k -th variable α_k . Using this and taking into account that the mapping (2.38) is nondecreasing we get

$$\begin{aligned} f_k^0(\alpha_1^+, \dots, \alpha_{k-1}^+, \alpha_k^-, \alpha_{k+1}^+, \dots, \alpha_m^+) \\ = \nu_k - \gamma < \nu_k < \nu_k + \gamma \\ = f_k^0(\alpha_1^-, \dots, \alpha_{k-1}^-, \alpha_k^+, \alpha_{k+1}^-, \dots, \alpha_m^-). \end{aligned} \quad (2.47)$$

Step 3. Let $(\lambda_j(\Omega^{\alpha,\varepsilon}))_{j \in \mathbb{N}}$ be the eigenvalues of the Neumann Laplacian $A_{\Omega^{\alpha,\varepsilon}}$ on $\Omega^{\alpha,\varepsilon}$ counted with multiplicities and ordered as a nondecreasing sequence.

For $\varepsilon \geq 0$ we introduce the functions $f_k^\varepsilon : \mathcal{D} \rightarrow \mathbb{R}$ by

$$f_k^\varepsilon(\alpha_1, \alpha_2, \dots, \alpha_m) = \lambda_{k+m}(\Omega^{\alpha,\varepsilon}), \quad k = 1, \dots, m. \quad (2.48)$$

Of course, for $\varepsilon = 0$ these functions coincide with the functions in (2.46). Observe that, in contrast to f_k^0 , for $\varepsilon > 0$ the values $\lambda_{k+m}(\Omega^{\alpha,\varepsilon})$ of f_k^ε in general do not depend only on the k -th variable. It is important to note that Theorem A.1 and Remark A.2 show that the functions

$$\varepsilon \mapsto \lambda_j(\Omega^{\alpha,\varepsilon}), \quad j \in \mathbb{N},$$

are continuous for each fixed α . Hence it follows together with (2.47) that

$$\begin{aligned} f_k^\varepsilon(\alpha_1^+, \dots, \alpha_{k-1}^+, \alpha_k^-, \alpha_{k+1}^+, \dots, \alpha_m^+) < \nu_k \\ < f_k^\varepsilon(\alpha_1^-, \dots, \alpha_{k-1}^-, \alpha_k^+, \alpha_{k+1}^-, \dots, \alpha_m^-) \end{aligned} \quad (2.49)$$

for $\varepsilon > 0$ sufficiently small. From now on we fix $\varepsilon > 0$ for which (2.49) holds.

To proceed further we need the following multidimensional version of the intermediate value theorem, which was established in [53, Lemma 3.5].

Lemma 2.7 (Hempel-Kriecherbauer-Plankensteiner, 1997) *Let $a_k < b_k$, $k = 1, \dots, m$, and $\mathcal{D} := \prod_{k=1}^m [a_k, b_k]$. Assume that $f : \mathcal{D} \rightarrow \mathbb{R}^m$ is continuous and that each component function f_k is nondecreasing in each of its arguments. Let us suppose that $F_k^- < F_k^+$, $k = 1, \dots, m$, where*

$$\begin{aligned} F_k^- &= f_k(b_1, b_2, \dots, b_{k-1}, a_k, b_{k+1}, \dots, b_m), \\ F_k^+ &= f_k(a_1, a_2, \dots, a_{k-1}, b_k, a_{k+1}, \dots, b_m). \end{aligned}$$

Then for any $F \in \prod_{k=1}^m [F_k^-, F_k^+]$ there exists $x \in \mathcal{D}$ such that $f(x) = F$.

We apply this lemma to the function $f^\varepsilon = (f_1^\varepsilon, \dots, f_m^\varepsilon)$ defined by (2.48). By Theorem A.1 and Remark A.2 $f^\varepsilon : \mathcal{D} \rightarrow \mathbb{R}^m$ is continuous. Moreover, by the min-max principle each component f_k^ε of f^ε is nondecreasing in each of its arguments. Using this and (2.49) we conclude that f^ε satisfies all assumptions in Lemma 2.7 and hence there exists $\alpha \in \mathcal{D}$ such that

$$f_k^\varepsilon(\alpha) = \nu_k.$$

With this choice of $\alpha = \{\alpha_1, \dots, \alpha_m\}$ and d_1, \dots, d_m fixed as in the beginning of Step 2 (see (2.42)) it follows that $\lambda_{m+k}(\Omega^{\alpha, \varepsilon}) = \nu_k$ for $k = 1, \dots, m$. This completes the proof of Theorem 2.6. \square

3 Singular Schrödinger operators with δ' -interactions

In this section we show that the methods used to control the spectrum of the Neumann Laplacian in the previous section can also be applied to singular Schrödinger operators describing the motion of quantum particles in potentials being supported at a discrete (finite or infinite) set of points. These operators are often referred to as *solvable models* in quantum mechanics, since their mathematical and physical quantities (e.g., their spectrum) can be determined explicitly. We refer to the monograph [3] for an introduction to this topic. We also note that in the mathematical literature such operators are often called *Schrödinger operators with point interactions*.

The classical example of a Schrödinger operator with δ -interactions is the following formal expression

$$-\frac{d^2}{dz^2} + \sum_{k \in \mathbb{N}} \alpha_k \delta_{z_k},$$

where δ_{z_k} are Dirac delta-functions supported at the points $z_k \in \mathbb{R}$ and $\alpha_k \in \mathbb{R} \cup \{\infty\}$. In the present paper we treat the closely related model of a Schrödinger operator with δ' -interactions (or *point dipole interactions*) defined by the formal expression

$$-\frac{d^2}{dz^2} + \sum_{k \in \mathbb{N}} \beta_k \langle \cdot, \delta'_{z_k} \rangle \delta'_{z_k}, \quad (3.1)$$

where δ'_{z_k} is the distributional derivative of the delta-function supported at $z_k \in \mathbb{R}$, $\langle \phi, \delta'_{z_k} \rangle$ denotes its action on the test function ϕ , and $\beta_k \in \mathbb{R} \cup \{\infty\}$. The above formal expression can be realized as a self-adjoint operator in L^2 with the action $-\frac{d^2}{dz^2}$ and domain consisting of local H^2 -functions u that satisfy

$$u'(z_k - 0) = u'(z_k + 0), \quad u(z_k + 0) - u(z_k - 0) = \beta_k u'(z_k \pm 0)$$

(the case $\beta = \infty$ stands for a decoupling with Neumann conditions at $z_k \pm 0$). The existence of this model was pointed out by A. Grossmann, R. Høegh-Krohn, M. Mebkhout in [52], the first rigorous mathematical treatment of δ' -interactions is due to F. Gesztesy and H. Holden in [48]. Among the numerous subsequent contributions we mention the more recent papers [62, 63] by A. Kostenko and M.M. Malamud, in which also the more elaborate case $|z_k - z_{k-1}| \rightarrow 0$ as $|k| \rightarrow \infty$ was treated. In these papers self-adjointness, lower semiboundedness and spectral properties of the underlying operators have been studied in detail.

Our goal and strategy is similar to [54] in the context of the Neumann Laplacian: We wish to construct an operator of the form (3.1) with predefined essential spectrum; cf. Theorem 2.4. At this point we present the main result of this section on a formal level without giving a precise definition of the underlying operator. This will be done during its proof; cf. Theorem 3.3 for a more precise formulation of Theorem 3.1.

Theorem 3.1 *Let $\mathfrak{S} \subset [0, \infty)$ be an arbitrary closed set such that $0 \in \mathfrak{S}$. Then there exists a bounded interval $(a, b) \subset \mathbb{R}$, a sequence of points $(z_k)_{k \in \mathbb{N}}$ in (a, b) , and a sequence of positive numbers $(\beta_k)_{k \in \mathbb{N}}$ such that the operator A_β in $L^2(a, b)$ defined by the formal expression (3.1) satisfies*

$$\sigma_{\text{ess}}(A_\beta) = \mathfrak{S}.$$

Proof. For the construction of the self-adjoint Schrödinger operator with δ' -interactions we use a similar idea as in the construction of the rooms-and-passages domains in the previous section. Here we split the sequence of points $(z_k)_{k \in \mathbb{N}}$ in (3.1) in two interlacing subsequences $(x_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$, where the point y_k is in the middle of (x_{k-1}, x_k) , and instead of β_k we denote the interaction strengths at the points x_k by p_k and at the points y_k by q_k . Instead of A_β we shall write $A_{p,q}$ for the corresponding Schrödinger operator with δ' -interactions, see Step 3 of the proof. The intervals (x_{k-1}, x_k) will play the role of the rooms, the interactions at the points x_k will play the role of the passages, and the interactions at the points y_k will play the role of the additional walls inside the rooms.

As in the proof of Theorem 2.4 we fix a sequence $(s_k)_{k \in \mathbb{N}}$ such that

$$s_k > 0 \text{ and } \text{acc}((s_k)_{k \in \mathbb{N}}) = \begin{cases} \mathfrak{S} \setminus \{0\}, & 0 \text{ is an isolated point of } \mathfrak{S}, \\ \mathfrak{S}, & \text{otherwise,} \end{cases} \quad (3.2)$$

and for each $k \in \mathbb{N}$ we choose numbers $d_k > 0$ that satisfy

$$s_k < (\pi/d_k)^2. \quad (3.3)$$

Moreover, we can assume that the d_k are chosen sufficiently small, so that

$$\sum_{k \in \mathbb{N}} d_k < \infty \quad (3.4)$$

and hence

$$\lim_{k \rightarrow \infty} d_k = 0 \quad (3.5)$$

holds. Finally, we set

$$x_0 = 0, \quad x_k = x_{k-1} + d_k, \quad y_k = \frac{x_{k-1} + x_k}{2}, \quad \mathcal{I}_k = (x_{k-1}, x_k), \quad k \in \mathbb{N},$$

and we consider the interval (a, b) , where

$$a = x_0 = 0 \quad \text{and} \quad b = \sum_{k \in \mathbb{N}} d_k.$$

The proof consists of four steps. In the first step we discuss the spectral properties of the Schrödinger operator $\mathbf{A}_{q_k, \mathcal{I}_k}$ on the interval \mathcal{I}_k with a δ' -interaction of strength $q_k > 0$ at y_k and Neumann boundary conditions at the endpoints of \mathcal{I}_k . In the second step we consider the direct sum of these operators:

$$A_{\infty, q} = \bigoplus_{k \in \mathbb{N}} \mathbf{A}_{q_k, \mathcal{I}_k}.$$

Note that the Neumann conditions at $x_k \pm 0$ can be regarded as δ' -interaction with *infinite* strength. Thus $A_{\infty, q}$ corresponds to the Schrödinger operator on (a, b) with δ' -interactions of strengths q_k at the points y_k and δ' -interactions of strengths ∞ at the points x_k . The interaction strengths q_k will be adjusted in such a way that the essential spectrum of $A_{\infty, q}$ coincides with \mathfrak{S} . In fact, $\sigma_{\text{ess}}(A_{\infty, q})$ is the union of the point 0 and all accumulation points of a sequence formed by the *second* eigenvalues of $\mathbf{A}_{q_k, \mathcal{I}_k}$. In the third step we perturb the decoupled operator $A_{\infty, q}$ linking the intervals \mathcal{I}_{k+1} and \mathcal{I}_k by a δ' -interaction of a sufficiently large strength $p_k > 0$ for all

$k \in \mathbb{N}$; the corresponding operator is denoted by $A_{p,q}$. We will prove in the last step that the essential spectra of $A_{p,q}$ and $A_{\infty,q}$ coincide if the interaction strengths p_k tend to ∞ for $k \rightarrow \infty$ sufficiently fast.

Step 1. Let $q_k \in (0, \infty]$ and let $\mathbf{a}_{q_k, \mathcal{I}_k}$ be the sesquilinear form in $L^2(\mathcal{I}_k)$ defined by

$$\begin{aligned} \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}] &= \int_{\mathcal{I}_k} \mathbf{u}' \cdot \overline{\mathbf{v}'} dx \\ &\quad + \frac{1}{q_k} (\mathbf{u}(y_k + 0) - \mathbf{u}(y_k - 0)) \overline{(\mathbf{v}(y_k + 0) - \mathbf{v}(y_k - 0))}, \\ \text{dom}(\mathbf{a}_{q_k, \mathcal{I}_k}) &= \mathbf{H}^1(\mathcal{I}_k \setminus \{y_k\}); \end{aligned}$$

for $q_k = \infty$ we use the convention $\infty^{-1} = 0$. The form $\mathbf{a}_{q_k, \mathcal{I}_k}$ is densely defined, nonnegative, and closed in $L^2(\mathcal{I}_k)$. Hence by the first representation theorem there is a unique nonnegative self-adjoint operator $\mathbf{A}_{q_k, \mathcal{I}_k}$ in $L^2(\mathcal{I}_k)$ such that $\text{dom}(\mathbf{A}_{q_k, \mathcal{I}_k}) \subset \text{dom}(\mathbf{a}_{q_k, \mathcal{I}_k})$ and

$$(\mathbf{A}_{q_k, \mathcal{I}_k} \mathbf{u}, \mathbf{v})_{L^2(\mathcal{I}_k)} = \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}], \quad \mathbf{u} \in \text{dom}(\mathbf{A}_{q_k, \mathcal{I}_k}), \quad \mathbf{v} \in \text{dom}(\mathbf{a}_{q_k, \mathcal{I}_k}).$$

Integration by parts shows that $\text{dom}(\mathbf{A}_{q_k, \mathcal{I}_k})$ consists of all those functions $\mathbf{u} \in \mathbf{H}^2(\mathcal{I}_k \setminus \{y_k\})$ that satisfy the δ' -jump condition

$$\mathbf{u}'(y_k - 0) = \mathbf{u}'(y_k + 0) = \frac{1}{q_k} (\mathbf{u}(y_k + 0) - \mathbf{u}(y_k - 0))$$

at the point y_k and Neumann boundary conditions

$$\mathbf{u}'(x_{k-1}) = \mathbf{u}'(x_k) = 0$$

at the endpoints of \mathcal{I}_k . Furthermore, the action of $\mathbf{A}_{q_k, \mathcal{I}_k}$ is given by

$$(\mathbf{A}_{q_k, \mathcal{I}_k} \mathbf{u}) \upharpoonright_{(x_{k-1}, y_k)} = -(\mathbf{u} \upharpoonright_{(x_{k-1}, y_k)})'', \quad (\mathbf{A}_{q_k, \mathcal{I}_k} \mathbf{u}) \upharpoonright_{(y_k, x_k)} = -(\mathbf{u} \upharpoonright_{(y_k, x_k)})''.$$

The spectrum of the self-adjoint operator $\mathbf{A}_{q_k, \mathcal{I}_k}$ is purely discrete. We use the notation $\{\lambda_j(\mathbf{A}_{q_k, \mathcal{I}_k})\}_{j \in \mathbb{N}}$ for the corresponding eigenvalues counted with multiplicities and ordered as a nondecreasing sequence. Some properties of these eigenvalues are collected in the next lemma. Here we will also make use of the Neumann Laplacian on \mathcal{I}_k , defined as usual via the form

$$\mathbf{a}_{0, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}] = (\mathbf{u}', \mathbf{v}')_{L^2(\mathcal{I}_k)}, \quad \text{dom}(\mathbf{a}_{0, \mathcal{I}_k}) = \mathbf{H}^1(\mathcal{I}_k), \quad (3.6)$$

and we shall denote this operator by $\mathbf{A}_{0, \mathcal{I}_k}$. To avoid confusion we emphasize that the form domain $\mathbf{H}^1(\mathcal{I}_k)$ of the Neumann Laplacian $\mathbf{A}_{0, \mathcal{I}_k}$ is smaller than the form domain $\mathbf{H}^1(\mathcal{I}_k \setminus \{y_k\})$ of the operators $\mathbf{A}_{q_k, \mathcal{I}_k}$ with $q_k \in (0, \infty]$. Furthermore, we mention already here that the self-adjoint operator $\mathbf{A}_{\infty, \mathcal{I}_k}$ is the direct sum of the Neumann Laplacians on the intervals (x_{k-1}, y_k) and (y_k, x_k) .

Lemma 3.2 *For each $j \in \mathbb{N}$*

$$\begin{aligned} &\text{the function } (0, \infty] \ni q_k \mapsto \lambda_j(\mathbf{A}_{q_k, \mathcal{I}_k}) \text{ is} \\ &\text{monotonically decreasing and continuous,} \end{aligned} \quad (3.7)$$

and one has

$$\lim_{q_k \rightarrow +0} \lambda_j(\mathbf{A}_{q_k, \mathcal{I}_k}) = \lambda_j(\mathbf{A}_{0, \mathcal{I}_k}). \quad (3.8)$$

Proof of Lemma 3.2. The monotonicity of the function (3.7) follows from the min-max principle and the monotonicity of the function $q_k \mapsto \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{u}]$ for each fixed $\mathbf{u} \in \mathbf{H}^1(\mathcal{I}_k)$. To prove the continuity of the

function (3.7) consider $q_k, \widehat{q}_k \in (0, \infty]$, $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\mathcal{I}_k)$, and set $\mathbf{u} = (\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f}$ and $\mathbf{v} = (\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{g}$. Then we have

$$\begin{aligned} & \left((\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f} - (\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f}, \mathbf{g} \right)_{\mathbf{L}^2(\mathcal{I}_k)} \\ &= \left(\mathbf{u}, (\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k} + \mathbf{I}) \mathbf{v} \right)_{\mathbf{L}^2(\mathcal{I}_k)} - \left((\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I}) \mathbf{u}, \mathbf{v} \right)_{\mathbf{L}^2(\mathcal{I}_k)} \\ &= \mathbf{a}_{\widehat{q}_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}] - \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}] \\ &= \left(\frac{1}{\widehat{q}_k} - \frac{1}{q_k} \right) (\mathbf{u}(y_k + 0) - \mathbf{u}(y_k - 0)) \overline{(\mathbf{v}(y_k + 0) - \mathbf{v}(y_k - 0))} \end{aligned} \quad (3.9)$$

With $\mathcal{I}_k^+ = (y_k, x_k)$ and $\mathcal{I}_k^- = (x_{k-1}, y_k)$ one has the standard trace estimate (see, e.g. [18, Lemma 1.3.8])

$$|\mathbf{u}(y_k \pm 0)|^2 \leq \frac{d_k}{2} \|\mathbf{u}'\|_{\mathbf{L}^2(\mathcal{I}_k^\pm)}^2 + \frac{4}{d_k} \|\mathbf{u}\|_{\mathbf{L}^2(\mathcal{I}_k^\pm)}^2, \quad \mathbf{u} \in \mathbf{H}^1(\mathcal{I}_k^\pm),$$

and with $C_k = \max\{d_k, 8d_k^{-1}\}$ we estimate

$$\begin{aligned} |\mathbf{u}(y_k + 0) - \mathbf{u}(y_k - 0)|^2 &\leq 2|\mathbf{u}(y_k + 0)|^2 + 2|\mathbf{u}(y_k - 0)|^2 \\ &\leq d_k \|\mathbf{u}'\|_{\mathbf{L}^2(\mathcal{I}_k \setminus \{y_k\})}^2 + 8d_k^{-1} \|\mathbf{u}\|_{\mathbf{L}^2(\mathcal{I}_k)}^2 \\ &\leq C_k (\mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{u}] + \|\mathbf{u}\|_{\mathbf{L}^2(\mathcal{I}_k)}^2) \\ &= C_k \left((\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I}) \mathbf{u}, \mathbf{u} \right)_{\mathbf{L}^2(\mathcal{I}_k)} \\ &= C_k \left(\mathbf{f}, (\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f} \right)_{\mathbf{L}^2(\mathcal{I}_k)} \\ &\leq C_k \|\mathbf{f}\|_{\mathbf{L}^2(\mathcal{I}_k)} \|(\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f}\|_{\mathbf{L}^2(\mathcal{I}_k)} \\ &\leq C_k \|\mathbf{f}\|_{\mathbf{L}^2(\mathcal{I}_k)}^2, \end{aligned} \quad (3.10)$$

where we have used $q_k > 0$ in the third estimate. In the same way we get $|\mathbf{v}(y_k + 0) - \mathbf{v}(y_k - 0)|^2 \leq C_k \|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{I}_k)}^2$. Hence (3.9) leads to the estimate

$$\begin{aligned} & \left| \left((\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f} - (\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k} + \mathbf{I})^{-1} \mathbf{f}, \mathbf{g} \right)_{\mathbf{L}^2(\mathcal{I}_k)} \right| \\ &\leq C_k \left| \frac{1}{\widehat{q}_k} - \frac{1}{q_k} \right| \|\mathbf{f}\|_{\mathbf{L}^2(\mathcal{I}_k)} \|\mathbf{g}\|_{\mathbf{L}^2(\mathcal{I}_k)}, \end{aligned}$$

and from this we conclude

$$\|(\mathbf{A}_{q_k, \mathcal{I}_k} + \mathbf{I})^{-1} - (\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k} + \mathbf{I})^{-1}\| \rightarrow 0 \text{ as } \widehat{q}_k \rightarrow q_k. \quad (3.11)$$

It is well-known (see, e.g., [59, 81] or [79, Corollary A.15]) that the norm-resolvent convergence in (3.11) implies the convergence of the eigenvalues, namely for each $j \in \mathbb{N}$ we obtain

$$\lambda_j(\mathbf{A}_{\widehat{q}_k, \mathcal{I}_k}) \rightarrow \lambda_j(\mathbf{A}_{q_k, \mathcal{I}_k}) \quad \text{as } \widehat{q}_k \rightarrow q_k,$$

and hence the function in (3.7) is continuous.

It remains to prove (3.8). For this we will use Theorem B.2 from Appendix B. Note first that the set

$$\left\{ \mathbf{u} \in \mathbf{H}^1(\mathcal{I}_k \setminus \{y_k\}) = \text{dom}(\mathbf{a}_{q_k, \mathcal{I}_k}) : \sup_{q_k > 0} \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{u}] < \infty \right\}$$

coincides with the form domain $\text{dom}(\mathbf{a}_{0, \mathcal{I}_k}) = \mathbf{H}^1(\mathcal{I}_k)$ of the Neumann Laplacian in (3.6). Moreover, for each \mathbf{u}, \mathbf{v} from this set one has

$$\lim_{q_k \rightarrow 0} \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}] = \mathbf{a}_{0, \mathcal{I}_k}[\mathbf{u}, \mathbf{v}].$$

Since the spectra of the operators $\mathbf{A}_{q_k, \mathcal{I}_k}$ and $\mathbf{A}_{0, \mathcal{I}_k}$ are purely discrete Theorem B.2 shows (3.8). \square

Now we return to the spectral properties of the self-adjoint operators $\mathbf{A}_{q_k, \mathcal{I}_k}$. In particular, the eigenvalues of the Neumann Laplacian $\mathbf{A}_{0, \mathcal{I}_k}$ on \mathcal{I}_k and the direct sum of the Neumann Laplacians $\mathbf{A}_{\infty, \mathcal{I}_k}$ on (x_{k-1}, y_k) and (y_k, x_k) can be easily calculated. For our purposes it suffices to note that

$$\lambda_1(\mathbf{A}_{0, \mathcal{I}_k}) = \lambda_1(\mathbf{A}_{\infty, \mathcal{I}_k}) = 0, \quad (3.12)$$

$$\lambda_2(\mathbf{A}_{0, \mathcal{I}_k}) = (\pi/d_k)^2, \quad \lambda_2(\mathbf{A}_{\infty, \mathcal{I}_k}) = 0, \quad (3.13)$$

$$\lambda_3(\mathbf{A}_{\infty, \mathcal{I}_k}) = (2\pi/d_k)^2. \quad (3.14)$$

It follows from (3.7), (3.12), and (3.14) that for any $q_k \in (0, \infty]$ we have

$$\lambda_1(\mathbf{A}_{q_k, \mathcal{I}_k}) = 0, \quad \lambda_3(\mathbf{A}_{q_k, \mathcal{I}_k}) \geq (2\pi/d_k)^2. \quad (3.15)$$

Also, using (3.7), (3.8), (3.13) and taking into account that $0 < s_k < (\pi/d_k)^2$ from (3.3) we conclude that there exists $q_k > 0$ such that

$$\lambda_2(\mathbf{A}_{q_k, \mathcal{I}_k}) = s_k, \quad k \in \mathbb{N}. \quad (3.16)$$

From now on we fix $q_k > 0$ for which (3.16) holds.

Step 2. Now we consider the direct sum

$$A_{\infty, q} = \bigoplus_{k \in \mathbb{N}} \mathbf{A}_{q_k, \mathcal{I}_k} \quad (3.17)$$

of the nonnegative self-adjoint operators $\mathbf{A}_{q_k, \mathcal{I}_k}$ in the space

$$\mathbb{L}^2(a, b) = \bigoplus_{k=1}^{\infty} \mathbb{L}^2(\mathcal{I}_k).$$

In a more explicit form $A_{\infty, q}$ is given by

$$\begin{aligned} (A_{\infty, q} u) \upharpoonright_{\mathcal{I}_k} &= \mathbf{A}_{q_k, \mathcal{I}_k} \mathbf{u}_k, \\ \text{dom}(A_{\infty, q}) &= \left\{ u \in \mathbb{L}^2(a, b) : \mathbf{u}_k \in \text{dom}(\mathbf{A}_{q_k, \mathcal{I}_k}), \right. \\ &\quad \left. \sum_{k \in \mathbb{N}} \|\mathbf{A}_{q_k, \mathcal{I}_k} \mathbf{u}_k\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 < \infty \right\}, \end{aligned}$$

where $\mathbf{u}_k := u \upharpoonright_{\mathcal{I}_k}$ stands for the restriction of the function u onto the interval \mathcal{I}_k . Note that the corresponding sesquilinear form $\mathbf{a}_{\infty, q}$ associated with $A_{\infty, q}$ is

$$\begin{aligned} \mathbf{a}_{\infty, q}[u, v] &= \sum_{k \in \mathbb{N}} \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}_k, \mathbf{v}_k], \\ \text{dom}(\mathbf{a}_{\infty, q}) &= \left\{ u \in \mathbb{L}^2(a, b) : \mathbf{u}_k \in \text{dom}(\mathbf{a}_{q_k, \mathcal{I}_k}), \sum_{k \in \mathbb{N}} \mathbf{a}_{q_k, \mathcal{I}_k}[\mathbf{u}_k, \mathbf{u}_k] < \infty \right\}. \end{aligned}$$

It is clear that the operator $A_{\infty, q}$ in (3.17) is self-adjoint and nonnegative in $\mathbb{L}^2(a, b)$. Furthermore, it is not difficult to check that

$$\sigma_{\text{ess}}(A_{\infty, q}) = \text{acc}((\lambda_j(\mathbf{A}_{q_k, \mathcal{I}_k}))_{j, k \in \mathbb{N}})$$

holds. Taking into account that $0 \in \mathfrak{S}$ and using (3.2), (3.5), (3.15), (3.16), we arrive at

$$\sigma_{\text{ess}}(A_{\infty, q}) = \{0\} \cup \text{acc}((s_k)_{k \in \mathbb{N}}) = \mathfrak{S}.$$

Step 3. In this step we perturb the decoupled operator $A_{\infty,q}$ linking the intervals \mathcal{I}_{k+1} and \mathcal{I}_k by a δ' -interaction of sufficiently large strength $p_k > 0$ for all $k \in \mathbb{N}$. The corresponding self-adjoint operator will be denoted by $A_{p,q}$. More precisely, for $p_k > 0, k \in \mathbb{N}$, we consider the sesquilinear form $\mathfrak{a}_{p,q}$

$$\begin{aligned} \mathfrak{a}_{p,q}[u, v] &= \sum_{k \in \mathbb{N}} \mathfrak{a}_{q_k, \mathcal{I}_k}[\mathbf{u}_k, \mathbf{v}_k] \\ &\quad + \sum_{k \in \mathbb{N}} \frac{1}{p_k} (u(x_k + 0) - u(x_k - 0)) \overline{(v(x_k + 0) - v(x_k - 0))}, \\ \text{dom}(\mathfrak{a}_{p,q}) &= \left\{ u \in L^2(a, b) : \mathbf{u}_k \in \text{dom}(\mathfrak{a}_{q_k, \mathcal{I}_k}), \mathfrak{a}_{p,q}[u, u] < \infty \right\}, \end{aligned}$$

in $L^2(a, b)$. This form is nonnegative and densely defined in $L^2(a, b)$. Moreover, the form is closed by [63, Lemma 2.6] and the corresponding nonnegative self-adjoint operator $A_{p,q}$ is given by

$$\begin{aligned} (A_{p,q}u) \upharpoonright_{(a,b) \setminus \mathcal{Z}} &= -(u \upharpoonright_{(a,b) \setminus \mathcal{Z}})'' , \\ \text{dom}(A_{p,q}) &= \left\{ u \in H^2((a, b) \setminus \mathcal{Z}) : u'(a) = 0, \right. \\ &\quad \left. u'(y_k + 0) = u'(y_k - 0) = \frac{1}{q_k} (u(y_k + 0) - u(y_k - 0)), \right. \\ &\quad \left. u'(x_k + 0) = u'(x_k - 0) = \frac{1}{p_k} (u(x_k + 0) - u(x_k - 0)) \right\}, \end{aligned}$$

where $\mathcal{Z} = \{x_k : k \in \mathbb{N}\} \cup \{y_k : k \in \mathbb{N}\}$; cf. [63, Lemma 2.6 and Proposition 2.1]. Now consider

$$\rho_k := \max \left\{ \frac{1}{p_k d_k}, \frac{1}{p_k d_{k+1}} \right\}, \quad k \in \mathbb{N},$$

and assume that

$$\rho_k \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{3.18}$$

Step 4. In this step we verify

$$\sigma_{\text{ess}}(A_{p,q}) = \sigma_{\text{ess}}(A_{\infty,q}). \tag{3.19}$$

by showing that the difference of resolvents

$$T_{p,q} := (A_{p,q} + \mathbf{I})^{-1} - (A_{\infty,q} + \mathbf{I})^{-1}$$

is a compact operator. Then (3.19) is an immediate consequence of the Weyl theorem, see, e.g. [82, Theorem XIII.14]. We note that in a similar situation a related perturbation result and the invariance of the essential spectrum was shown in [63, Theorem 1.3].

In the following let $\kappa_n = \sup_{k \in [n, \infty) \cap \mathbb{N}} \rho_k$. Then it follows from (3.18) that

$$\kappa_n < \infty \text{ for each } n \in \mathbb{N} \quad \text{and} \quad \kappa_n \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.20}$$

In a first step we claim that

$$\text{dom}(\mathfrak{a}_{\infty,q}) = \text{dom}(\mathfrak{a}_{p,q}). \tag{3.21}$$

In fact, the inclusion $\text{dom}(\mathfrak{a}_{p,q}) \subset \text{dom}(\mathfrak{a}_{\infty,q})$ follows directly from the definition of the above form domains. To prove the reverse inclusion we have to show that

$$\sum_{k \in \mathbb{N}} \frac{1}{p_k} |u(x_k + 0) - u(x_k - 0)|^2 < \infty \tag{3.22}$$

for $u \in \text{dom}(\mathfrak{a}_{\infty,q})$. Using the standard trace estimates (see, e.g. [18, Lemma 1.3.8])

$$\begin{aligned} |u(x_k + 0)|^2 &\leq d_{k+1} \|u'\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2 + \frac{2}{d_{k+1}} \|u\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2, \\ |u(x_k - 0)|^2 &\leq d_k \|u'\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 + \frac{2}{d_k} \|u\|_{\mathbb{L}^2(\mathcal{I}_k)}^2, \end{aligned}$$

and taking into account that $\sup_{k \in \mathbb{N}} d_k < b - a$ and $q_k > 0$ we obtain

$$\begin{aligned} &\sum_{k \in \mathbb{N}} \frac{1}{p_k} |u(x_k + 0) - u(x_k - 0)|^2 \\ &\leq 2 \sum_{k \in \mathbb{N}} \frac{1}{p_k} |u(x_k + 0)|^2 + 2 \sum_{k \in \mathbb{N}} \frac{1}{p_k} |u(x_k - 0)|^2 \\ &\leq 2 \sum_{k \in \mathbb{N}} \frac{1}{p_k} \left(d_{k+1} \|u'\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2 + \frac{2}{d_{k+1}} \|u\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2 \right) \\ &\quad + 2 \sum_{k \in \mathbb{N}} \frac{1}{p_k} \left(d_k \|u'\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 + \frac{2}{d_k} \|u\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 \right) \\ &\leq 2\kappa_1 \sum_{k \in \mathbb{N}} d_{k+1}^2 \|u'\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2 + 4\kappa_1 \sum_{k \in \mathbb{N}} \|u\|_{\mathbb{L}^2(\mathcal{I}_{k+1})}^2 \\ &\quad + 2\kappa_1 \sum_{k \in \mathbb{N}} d_k^2 \|u'\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 + 4\kappa_1 \sum_{k \in \mathbb{N}} \|u\|_{\mathbb{L}^2(\mathcal{I}_k)}^2 \\ &\leq 4\kappa_1 (b - a)^2 \|u'\|_{\mathbb{L}^2(a,b)}^2 + 8\kappa_1 \|u\|_{\mathbb{L}^2(a,b)}^2 \\ &\leq 4\kappa_1 (b - a)^2 \mathfrak{a}_{\infty,q}[u, u] + 8\kappa_1 \|u\|_{\mathbb{L}^2(a,b)}^2, \end{aligned} \tag{3.23}$$

and thus (3.22) holds. We have shown (3.21).

Now let $f, g \in \mathbb{L}^2(a, b)$ be arbitrary and consider the functions

$$\begin{aligned} u &= (A_{p,q} + \mathbf{I})^{-1} f \in \text{dom}(A_{p,q}) \subset \text{dom}(\mathfrak{a}_{p,q}), \\ v &= (A_{\infty,q} + \mathbf{I})^{-1} g \in \text{dom}(A_{\infty,q}) \subset \text{dom}(\mathfrak{a}_{\infty,q}). \end{aligned}$$

Using (3.21) and the fact that $(A_{\infty,q} + \mathbf{I})^{-1}$ is a self-adjoint operator we get

$$\begin{aligned} (T_{p,q} f, g)_{\mathbb{L}^2(a,b)} &= ((A_{p,q} + \mathbf{I})^{-1} f - (A_{\infty,q} + \mathbf{I})^{-1} f, g)_{\mathbb{L}^2(a,b)} \\ &= (u, (A_{\infty,q} + \mathbf{I})v)_{\mathbb{L}^2(a,b)} - ((A_{p,q} + \mathbf{I})u, v)_{\mathbb{L}^2(a,b)} \\ &= \mathfrak{a}_{\infty,q}[u, v] - \mathfrak{a}_{p,q}[u, v] \\ &= - \sum_{k \in \mathbb{N}} \frac{1}{p_k} (u(x_k + 0) - u(x_k - 0)) \overline{(v(x_k + 0) - v(x_k - 0))}. \end{aligned} \tag{3.24}$$

Next we introduce the operators $\Gamma_p, \Gamma_\infty : \mathbb{L}^2(a, b) \rightarrow l^2(\mathbb{N})$ defined by

$$\begin{aligned} (\Gamma_p f)_k &:= \frac{1}{\sqrt{p_k}} [((A_{p,q} + \mathbf{I})^{-1} f)(x_k + 0) - ((A_{p,q} + \mathbf{I})^{-1} f)(x_k - 0)], \\ (\Gamma_\infty g)_k &:= \frac{1}{\sqrt{p_k}} [((A_{\infty,q} + \mathbf{I})^{-1} g)(x_k + 0) - ((A_{\infty,q} + \mathbf{I})^{-1} g)(x_k - 0)], \end{aligned}$$

on their natural domains

$$\begin{aligned} \text{dom}(\Gamma_p) &= \{f \in \mathbb{L}^2(a, b) : \Gamma_p f \in l^2(\mathbb{N})\}, \\ \text{dom}(\Gamma_\infty) &= \{g \in \mathbb{L}^2(a, b) : \Gamma_\infty g \in l^2(\mathbb{N})\}. \end{aligned}$$

Note that $\text{dom}(\Gamma_p)$ coincides with the whole $L^2(a, b)$; this follows immediately from the $\text{ran}(A_{p,q} + I)^{-1} \subset \text{dom}(a_{p,q})$. Let us prove that the operator Γ_p is compact. For this purpose we introduce the finite rank operators

$$\Gamma_p^n : L^2(a, b) \rightarrow l^2(\mathbb{N}), \quad (\Gamma_p^n f)_k = \begin{cases} (\Gamma_p f)_k, & k \leq n, \\ 0, & k > n. \end{cases}$$

Let $f \in L^2(a, b) = \text{dom}(\Gamma_p)$ and $u = (A_{p,q} + I)^{-1}f$. Using the same arguments as in the proof of (3.23) and (3.10) we obtain

$$\begin{aligned} \|\Gamma_p^n f - \Gamma_p f\|_{l^2(\mathbb{N})}^2 &= \sum_{k:k>n} \frac{1}{p_k} |u(x_k + 0) - u(x_k - 0)|^2 \\ &\leq 4\kappa_{n+1}(b-a)^2 \left(\mathfrak{a}_{p,q}[u, u] + \|u\|_{L^2(a,b)}^2 \right) + 8\kappa_{n+1} \|u\|_{L^2(a,b)}^2 \\ &= 4\kappa_{n+1}(b-a)^2 (f, u)_{L^2(a,b)} + 8\kappa_{n+1} \|u\|_{L^2(a,b)}^2 \\ &\leq (4\kappa_{n+1}(b-a)^2 + 8\kappa_{n+1}) \|f\|_{L^2(a,b)}^2 \end{aligned} \quad (3.25)$$

and hence it follows from (3.20) that $\|\Gamma_p^n - \Gamma_p\|_{l^2(\mathbb{N})} \rightarrow 0$ as $n \rightarrow \infty$. Since Γ_p^n are finite rank operators we conclude that the operator Γ_p is compact. Furthermore, it is easy to see that Γ_∞ is a bounded operator defined on $L^2(a, b)$. Indeed, for $g \in L^2(a, b)$ and $v = (A_{\infty,q} + I)^{-1}g$ one verifies in the same way as in (3.23) and (3.25) that

$$\begin{aligned} \|\Gamma_\infty g\|_{l^2(\mathbb{N})}^2 &= \sum_{k \in \mathbb{N}} (p_k)^{-1} |v(x_k + 0) - v(x_k - 0)|^2 \\ &\leq (4\kappa_1(b-a)^2 + 8\kappa_1) \|g\|_{L^2(a,b)}^2. \end{aligned}$$

Now (3.24) can be rewritten in the form

$$(T_{p,q} f, g)_{L^2(a,b)} = -(\Gamma_p f, \Gamma_\infty g)_{l^2(\mathbb{N})}, \quad f, g \in L^2(a, b),$$

and hence we have $T_{p,q} = -(\Gamma_\infty)^* \Gamma_p$. Since Γ_p is compact and Γ_∞ is bounded (thus $(\Gamma_\infty)^*$ is also bounded) we conclude that $T_{p,q}$ is compact. \square

For the convenience of the reader we now formulate Theorem 3.1 in a more precise form.

Theorem 3.3 *Let $\mathfrak{S} \subset [0, \infty)$ be an arbitrary closed set such that $0 \in \mathfrak{S}$ and choose the sequences $(s_k)_{k \in \mathbb{N}}$ and $(d_k)_{k \in \mathbb{N}}$ as in (3.2)–(3.4). Let $(q_k)_{k \in \mathbb{N}}$ and $(p_k)_{k \in \mathbb{N}}$ be sequences such that (3.16) and (3.18) hold. Then the self-adjoint Schrödinger operator $A_{p,q}$ with δ' -interactions of strengths $(q_k)_{k \in \mathbb{N}}$ and $(p_k)_{k \in \mathbb{N}}$ at the points $(y_k)_{k \in \mathbb{N}}$ and $(x_k)_{k \in \mathbb{N}}$, respectively, satisfies*

$$\sigma_{\text{ess}}(A_{p,q}) = \mathfrak{S}.$$

At the end of this section we note that there exist many other methods for the construction of Schrödinger operators with predefined spectral properties. For example, F. Gesztesy, W. Karwowski, and Z. Zhao constructed in [49, 50] a smooth potential V (which is a limit of suitably chosen N -soliton solutions of the Korteweg-de Vries equation as $N \rightarrow \infty$) such that the Schrödinger operator $H = -\frac{d^2}{dx^2} + V$ has purely absolutely continuous spectrum \mathbb{R}_+ and a prescribed sequence of points in \mathbb{R}_- is contained in the set of eigenvalues of H .

4 Essential spectra of self-adjoint extensions of symmetric operators

The aim of this slightly more abstract section is to discuss some possible spectral properties of self-adjoint extensions of a given symmetric operator in a separable Hilbert space. In a similar context the existence of self-adjoint extensions with prescribed point spectrum, absolutely continuous spectrum, and singular continuous spectrum in spectral gaps of a fixed underlying symmetric operator was discussed in [4, 5, 19–23], see also [74] for a related result on prescribed eigenvalue asymptotics or, e.g., the earlier contributions [7, 46, 47, 55]. Our

main observation here is the fact that for a symmetric operator with infinite deficiency indices one can construct self-adjoint extensions with prescribed essential spectrum; cf. Theorem 4.3 below.

In the following let \mathcal{H} be a separable (infinite dimensional) complex Hilbert space with scalar product (\cdot, \cdot) . Recall that a linear operator S in \mathcal{H} is said to be *symmetric* if

$$(Sf, g) = (f, Sg), \quad f, g \in \text{dom}(S).$$

We point out that a symmetric operator is in general not self-adjoint. More precisely, if the domain $\text{dom}(S)$ of S is dense in \mathcal{H} then the adjoint S^* of the operator S is given by

$$S^*h = k, \\ \text{dom}(S^*) = \{h \in \mathcal{H} : \exists k \in \mathcal{H} \text{ such that } (Sf, h) = (f, k) \text{ for all } f \in \text{dom}(S)\},$$

and the fact that S is symmetric is equivalent to the inclusion $S \subset S^*$ in the sense that $\text{dom}(S) \subset \text{dom}(S^*)$ and $S^*f = Sf$ for all $f \in \text{dom}(S)$. However, this is obviously a weaker property than the more natural physical property of *self-adjointness*, that is, $S = S^*$. A symmetric operator is not necessarily closed (although closable) and the spectrum of a symmetric operator which is not self-adjoint may cover the whole complex plane (or at least the upper or lower complex halfplane). We also point out that the closure \overline{S} of a symmetric operator S need not be self-adjoint and it need not have self-adjoint extensions in \mathcal{H} . A symmetric operator S is called *essentially self-adjoint* if its closure \overline{S} is self-adjoint, that is, $\overline{S} = S^*$ – however, we shall not deal with essential self-adjoint operators here. We emphasize that from a spectral theoretic point of view a symmetric operator (or an essentially self-adjoint operator) which is not self-adjoint is not suitable as an observable in the description of a physical quantum system.

It is an important issue to understand in which situations a symmetric operator admits self-adjoint extensions and how these self-adjoint extensions can be described. These questions have already been discussed in the classical contribution [76] by J. von Neumann. For completeness we recall that a self-adjoint operator A in \mathcal{H} is an extension of a densely defined symmetric operator S if $S \subset A$; since A is self-adjoint this is equivalent to $A \subset S^*$. We start by recalling the so-called first von Neumann formula in the next theorem.

Theorem 4.1 *Let S be a densely defined closed symmetric operator in \mathcal{H} . Then the domain of the adjoint operator S^* admits the direct sum decomposition*

$$\text{dom}(S^*) = \text{dom}(S) \dot{+} \ker(S^* - i) \dot{+} \ker(S^* + i). \quad (4.1)$$

Note that $S^*f_i = if_i$ for all $f_i \in \ker(S^* - i)$ and similarly $S^*f_{-i} = -if_{-i}$ for all $f_{-i} \in \ker(S^* + i)$. The spaces $\ker(S^* - i)$ and $\ker(S^* + i)$ are usually called *defect spaces* of S and their dimensions are the *deficiency indices* of S . It will turn out that the deficiency indices and isometric operators in between the defect spaces are particularly important in the theory of self-adjoint extensions. One can show that the dimension of $\ker(S^* - \lambda_+)$ does not depend on $\lambda_+ \in \mathbb{C}^+$ and the dimension of $\ker(S^* - \lambda_-)$ does not depend on $\lambda_- \in \mathbb{C}^-$. However, for fixed $\lambda_+ \in \mathbb{C}^+$ and $\lambda_- \in \mathbb{C}^-$ and hence, in particular, for $\lambda_+ = i$ and $\lambda_- = -i$, the dimensions of $\ker(S^* - \lambda_+)$ and $\ker(S^* - \lambda_-)$ may be different. According to the second von Neumann formula both dimensions coincide if and only if S admits self-adjoint extensions in \mathcal{H} .

Theorem 4.2 *Let S be a densely defined closed symmetric operator in \mathcal{H} . Then there exist self-adjoint extensions A of S in \mathcal{H} if and only if*

$$\dim(\ker(S^* - i)) = \dim(\ker(S^* + i)).$$

If, in this case $U : \ker(S^ - i) \rightarrow \ker(S^* + i)$ is a unitary operator and $\text{dom}(S^*)$ is decomposed as in (4.1), then the operator A defined by*

$$A(f_S + f_i + f_{-i}) = Sf_S + if_i - if_{-i}, \\ \text{dom}(A) = \{f = f_S + f_i + f_{-i} \in \text{dom}(S^*) : f_{-i} = Uf_i\}, \quad (4.2)$$

is a self-adjoint extension of S and, vice versa, for any self-adjoint extension A of S there exists a unitary operator $U : \ker(S^ - i) \rightarrow \ker(S^* + i)$ such that (4.2) holds.*

It is intuitively clear that for a densely defined symmetric operator with infinite deficiency indices (and hence equal deficiency indices as \mathcal{H} is assumed to be separable) there is a lot of flexibility for the unitary operators in between the infinite dimensional defect subspaces. This flexibility also allows to construct self-adjoint extensions with various different spectral properties.

Now we wish to consider the following particular situation. Let again S be a densely defined closed symmetric operator in \mathcal{H} with equal infinite deficiency indices and assume that there exists a self-adjoint extension of S such that the resolvent is a compact operator. In this situation we shall construct another self-adjoint extension A of S with prescribed essential spectrum in Theorem 4.3 below. In this context we also mention the following variant of a remarkable result due to J. Brasche, which can be viewed as the final outcome of a long and fruitful investigation originally initiated by H. Neidhardt and his coauthors: For a densely defined closed symmetric operator S in \mathcal{H} with equal infinite deficiency indices and a (finite or infinite) gap there exists a self-adjoint extension with arbitrary spectrum within this gap; cf. [19, Theorem 27] for more details.

Theorem 4.3 *Let S be a densely defined closed symmetric operator in \mathcal{H} with equal infinite deficiency indices and assume that there exists a self-adjoint extension of S with compact resolvent. Let \mathcal{G} be a separable infinite dimensional Hilbert space and let Ξ be a self-adjoint operator in \mathcal{G} with $\mathbb{R} \cap \rho(\Xi) \neq \emptyset$. Then there exists a self-adjoint extension A of S in \mathcal{H} such that*

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(\Xi). \quad (4.3)$$

In particular, for any closed set $\Sigma \subset \mathbb{R}$ with $\Sigma \neq \mathbb{R}$ there exists a self-adjoint extension A of S in \mathcal{H} such that $\sigma_{\text{ess}}(A) = \Sigma$.

Proof. Let A_0 be a self-adjoint extension of S in \mathcal{H} such that the resolvent $(A_0 - \lambda)^{-1}$ is a compact operator for some, and hence for all, $\lambda \in \rho(A_0)$. Let us fix some point $\mu \in \mathbb{R} \cap \rho(A_0) \cap \rho(\Xi)$. Note that this is possible since we have assumed $\mathbb{R} \cap \rho(\Xi) \neq \emptyset$ and the spectrum of A_0 is a discrete subset of the real line due to the compactness assumption. In the present situation the spaces $\ker(S^* - \lambda_+)$ and $\ker(S^* - \lambda_-)$ for $\lambda_{\pm} \in \mathbb{C}^{\pm}$ are both infinite dimensional and one can show that here also the space $\ker(S^* - \mu)$ is infinite dimensional; this follows, e.g., from the direct sum decomposition

$$\text{dom}(S^*) = \text{dom}(A_0) \dot{+} \ker(S^* - \mu) \quad (4.4)$$

and the fact that S^* is an infinite dimensional extension of A_0 . Moreover, it is no restriction to assume that the Hilbert space \mathcal{G} in the assumptions of the theorem coincides with $\ker(S^* - \mu)$ since any two separable infinite dimensional Hilbert spaces can be identified via a unitary operator. Now observe the orthogonal sum decomposition

$$\mathcal{H} = \ker(S^* - \mu) \oplus \text{ran}(S - \mu)$$

and with respect to this decomposition we consider the bounded everywhere defined operator

$$R_{\mu} := (A_0 - \mu)^{-1} + \begin{bmatrix} (\Xi - \mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.5)$$

We claim that R_{μ}^{-1} is a well-defined operator. In fact, if $R_{\mu}h = 0$ for some $h \in \mathcal{H}$ then (4.5) implies

$$(A_0 - \mu)^{-1}h = - \begin{bmatrix} (\Xi - \mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix} h,$$

and since the left-hand side belongs to $\text{dom}(A_0)$ and the right-hand side is nonzero only in $\ker(S^* - \mu)$ it follows from the direct sum decomposition (4.4) that $h = 0$. This confirms that R_{μ}^{-1} , and hence also

$$A := R_{\mu}^{-1} + \mu$$

is a well-defined operator. It is clear from $R_{\mu} = (A - \mu)^{-1}$ that $\mu \in \rho(A)$ and A is self-adjoint in \mathcal{H} since the same is obviously true for R_{μ} in (4.5). In order to determine the essential spectrum of A recall the Weyl theorem

(see, e.g., [82, Theorem XIII.14]) which states that compact perturbations in resolvent sense do not change the essential spectrum. In the present situation we have that

$$(A - \mu)^{-1} - \begin{bmatrix} (\Xi - \mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = R_\mu - \begin{bmatrix} (\Xi - \mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = (A_0 - \mu)^{-1}$$

is a compact operator and hence the essential spectrum $\sigma_{\text{ess}}((A - \mu)^{-1})$ coincides with the essential spectrum of the diagonal block matrix operator, that is,

$$\sigma_{\text{ess}} \left(\begin{bmatrix} (\Xi - \mu)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right) = \sigma_{\text{ess}}((\Xi - \mu)^{-1}) \cup \{0\}.$$

This implies (4.3). \square

From the construction of the operator A in the proof of Theorem 4.3 the following representation can be concluded:

$$\begin{aligned} A(f_0 + f_\mu) &= A_0 f_0 + \mu f_\mu, \\ \text{dom}(A) &= \{f_0 + f_\mu \in \text{dom}(A_0) \dot{+} \ker(S^* - \mu) : \\ &\quad (\Xi - \mu)f_\mu = P_\mu(A_0 - \mu)f_0\}; \end{aligned} \quad (4.6)$$

here P_μ denotes the orthogonal projection in \mathcal{H} onto $\ker(S^* - \mu)$. In fact, since (4.5) is the resolvent $(A - \mu)^{-1}$ of A it follows that the elements $f \in \text{dom}(A)$ have the form $f = R_\mu h$, $h \in \mathcal{H}$. Due to the direct sum decomposition (4.4) we have $R_\mu h = f = f_0 + f_\mu$ with some $f_0 \in \text{dom}(A_0)$ and some $f_\mu \in \ker(S^* - \mu)$, and when comparing with (4.5) it follows that $f_0 = (A_0 - \mu)^{-1}h$ and $f_\mu = (\Xi - \mu)^{-1}P_\mu h$. Hence it is clear that $f = R_\mu h \in \text{dom}(A)$ satisfies the condition

$$(\Xi - \mu)f_\mu = P_\mu(A_0 - \mu)f_0 \quad (4.7)$$

in (4.6). On the other hand, if $f = f_0 + f_\mu \in \text{dom}(A_0) \dot{+} \ker(S^* - \mu)$ is such that (4.7) holds then one can verify in a similar way that there exists $h \in \mathcal{H}$ such that $f = R_\mu h$, and hence $f \in \text{dom}(A)$. Summing up we have shown the representation (4.6).

Finally we note that the explicit form (4.6) of A comes via a restriction of the adjoint operator S^* and the decomposition (4.4); the domain of A is described by an abstract boundary condition depending on the choice of the operator Ξ . This abstract result can of course be formulated in various explicit situations, e.g., for infinitely many δ' -interactions as in Section 3 or for the Laplacian on a bounded domain as in Section 2, where the boundary condition in (4.6) can be specified further.

Furthermore, the self-adjoint extensions A and A_0 can be described in the formalism of von Neumann's second formula in Theorem 4.2. If one fixes a unitary operator $U_0 : \ker(S^* - i) \rightarrow \ker(S^* + i)$ for the representation of A_0 in (4.2) then the unitary operator $U : \ker(S^* - i) \rightarrow \ker(S^* + i)$ corresponding to the self-adjoint extension A can be expressed in terms of U_0 and the parameter Ξ . The technical details are left to the reader.

A Continuous dependence of the eigenvalues on varying domains

In this appendix we establish an auxiliary result on the continuous dependence of the eigenvalues of the Neumann Laplacian on varying domains, which is useful and convenient for the proofs of Theorem 2.4 and Theorem 2.6.

For our purposes it is sufficient to consider the following geometric setting: Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume that also the subdomains

$$\Omega_\pm = \Omega \cap \{(x, y) \in \mathbb{R}^2 : \pm x > 0\}$$

are (bounded, nontrivial) Lipschitz domains. Furthermore, we assume that the set

$$\Gamma = \Omega \cap \{(x, y) \in \mathbb{R}^2 : x = 0\} = \partial\Omega_- \cap \partial\Omega_+$$

is a (compact) interval with the endpoints $(0, A)$ and $(0, B)$ in \mathbb{R}^2 , where $A < B$. For $a, b \in [A, B]$ fixed such that $a \leq b$ we introduce the domain $\Omega_{a,b}$ by

$$\Omega_{a,b} = \Omega_- \cup \Omega_+ \cup \Gamma_{a,b}, \text{ where } \Gamma_{a,b} = \{0\} \times (a, b) \quad (\text{A.1})$$

(see Figure 5, left). Note that $\Omega_{A,B} = \Omega$ and $\Omega_{a,a} = \Omega_- \cup \Omega_+ = \Omega \setminus \Gamma$. We denote by $Y \subset [A, B] \times [A, B]$ the set of all admissible pairs $\{a, b\}$, that is,

$$Y = \{\{a, b\} : A \leq a \leq b \leq B\}.$$

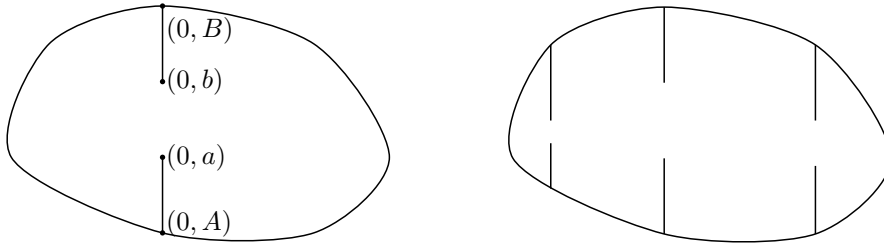


Fig. 5 Domain $\Omega_{a,b}$ with one wall (left) and $m = 3$ walls (right)

Since the domain $\Omega_{a,b}$ in (A.1) has the *cone property* (see, e.g. [1, Chapter IV, 4.3]) it follows from Rellich's theorem [1, Theorem 6.2] that the embedding $H^1(\Omega_{a,b}) \hookrightarrow L^2(\Omega_{a,b})$ is compact. Therefore, the spectrum of the Neumann Laplacian $A_{\Omega_{a,b}}$ on $\Omega_{a,b}$ is purely discrete. We denote by $(\lambda_k(\Omega_{a,b}))_{k \in \mathbb{N}}$ the sequence of eigenvalues of $A_{\Omega_{a,b}}$ numbered in nondecreasing order with multiplicities taken into account.

Theorem A.1 For each $k \in \mathbb{N}$ the function $\{a, b\} \mapsto \lambda_k(\Omega_{a,b})$ is continuous on Y .

Remark A.2 Theorem A.1 remains valid for more general domains $\Omega_{a,b}$ obtained from Ω by adding $m > 1$ walls in the same way – see Figure 5 (right, here $m = 3$). In this case $a = \{a_1, \dots, a_m\}$, $b = \{b_1, \dots, b_m\}$ with

$$A_j \leq a_j \leq b_j \leq B_j, \quad j = 1, \dots, m, \quad (\text{A.2})$$

and $\{a, b\} \mapsto \lambda_k(\Omega_{a,b})$ is continuous on $\{\{a, b\} \in \mathbb{R}^{2m} : (\text{A.2}) \text{ holds}\}$.

For the proof of Theorem A.1 we will make use of a variant of an abstract result from [56] (originally formulated and proved for operators in *varying* Hilbert spaces), which is here obtained as Corollary A.4 below. The next proposition and its elegant proof were communicated to us by an anonymous referee.

Proposition A.3 Let B_n , $n \in \mathbb{N}$, and B be compact normal operators in a Hilbert space \mathcal{H} . Assume that the following conditions hold:

- (i) $\forall f \in \mathcal{H} : B_n f \rightarrow Bf$ as $n \rightarrow \infty$;
- (ii) for any bounded sequence $(f_n)_{n \in \mathbb{N}}$ in \mathcal{H} there exists $u \in \mathcal{H}$ and a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $B_{n_k} f_{n_k} \rightarrow u$ in \mathcal{H} as $k \rightarrow \infty$.

Then

$$\|B_n - B\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{A.3})$$

Proof. Let $A_n := B_n - B$, $n \in \mathbb{N}$, and note that, due to the compactness of the operator B ,

$$\text{(ii) holds also with } A_n \text{ instead of } B_n. \quad (\text{A.4})$$

Let $\mu_n \in \sigma(A_n)$ be such that $|\mu_n|$ coincides with the spectral radius of A_n . Due to the compactness of A_n , μ_n is an eigenvalue of A_n . Moreover, since A_n is also a normal operator, one has (see, e.g., [59, Ch. 5, Eq. 2.4])

$$\|A_n\| = |\mu_n|. \quad (\text{A.5})$$

To prove (A.3) it suffices to show $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. Assume the opposite: there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N} : \quad |\mu_{n_k}| \geq C_1 \quad \text{for some } C_1 > 0. \quad (\text{A.6})$$

It follows from (i) and the Uniform Boundedness Principle (see, e.g. [81, Theorem III.9]) that the sequence of norms $(\|A_n\|)_{n \in \mathbb{N}}$ is bounded. Hence

$$\exists C_2 > 0 : \quad |\mu_{n_k}| \leq C_2, \quad k \in \mathbb{N}. \quad (\text{A.7})$$

Let $f_{n_k}, k \in \mathbb{N}$ be an eigenfunction of A_{n_k} corresponding to μ_{n_k} and normalized by

$$\|f_{n_k}\|_{\mathcal{H}} = 1. \quad (\text{A.8})$$

Due to (A.4), (A.7), (A.8) there is a subsequence (for simplicity, still denoted by $(n_k)_{k \in \mathbb{N}}$) and $\mu \in \mathbb{C}, u \in \mathcal{H}$ such that

$$\mu_{n_k} \rightarrow \mu \text{ as } k \rightarrow \infty, \quad (\text{A.9})$$

$$A_{n_k} f_{n_k} \rightarrow u \text{ strongly in } \mathcal{H} \text{ as } k \rightarrow \infty. \quad (\text{A.10})$$

Due to (A.6) $\mu \neq 0$. Taking into account that $A_{n_k} f_{n_k} = \mu_{n_k} f_{n_k}$, we conclude from (A.9)-(A.10) that

$$f_{n_k} \rightarrow f := \mu^{-1}u \text{ strongly in } \mathcal{H} \text{ as } k \rightarrow \infty. \quad (\text{A.11})$$

Finally, using (A.6), (A.8), (A.5), and (A.7) we get

$$\begin{aligned} 0 < C_1 &\leq |\mu_{n_k}| = \|\mu_{n_k} f_{n_k}\|_{\mathcal{H}} = \|A_{n_k}(f_{n_k} - f) + A_{n_k}f\|_{\mathcal{H}} \\ &\leq |\mu_{n_k}| \|f_{n_k} - f\|_{\mathcal{H}} + \|A_{n_k}f\|_{\mathcal{H}} \\ &\leq C_2 \|f_{n_k} - f\|_{\mathcal{H}} + \|A_{n_k}f\|_{\mathcal{H}}. \end{aligned} \quad (\text{A.12})$$

Due to (i) and (A.11) the right-hand-side of (A.12) tends to 0 as $k \rightarrow \infty$; a contradiction. \square

From Proposition A.3 we immediately get the following useful corollary.

Corollary A.4 *Let $B_n, n \in \mathbb{N}$, and B be compact non-negative operators in a Hilbert space \mathcal{H} satisfying assumptions (i) and (ii) in Proposition A.3. We denote by $(\mu_k(B_n))_{k \in \mathbb{N}}$ and $(\mu_k(B))_{k \in \mathbb{N}}$ the sequences of the eigenvalues of B_n and B , respectively, numbered in nonincreasing order with multiplicities taken into account. Then for each $k \in \mathbb{N}$*

$$\mu_k(B_n) \rightarrow \mu_k(B) \text{ as } n \rightarrow \infty. \quad (\text{A.13})$$

Proof of Theorem A.1. Fix some $\{a, b\} \in Y$ and consider an arbitrary sequence $\{a_n, b_n\} \in Y, n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. We have to show that for each $k \in \mathbb{N}$

$$\lambda_k(\Omega_{a_n, b_n}) \rightarrow \lambda_k(\Omega_{a, b}) \text{ as } n \rightarrow \infty. \quad (\text{A.14})$$

The strategy is to apply Corollary A.4 to the resolvents of the Neumann Laplacians $A_{\Omega_{a_n, b_n}}$ and $A_{\Omega_{a, b}}$. More precisely, we consider the operators

$$B_n = (A_{\Omega_{a_n, b_n}} + \text{I})^{-1} \quad \text{and} \quad B = (A_{\Omega_{a, b}} + \text{I})^{-1},$$

which are compact non-negative operators acting in $\mathcal{H} = L^2(\Omega) = L^2(\Omega_{a, b})$. We show below that these operators satisfy the assumptions (i) and (ii) in Proposition A.3. Then it follows that (A.13) holds for each $k \in \mathbb{N}$ and from

$$\mu_k(B_n) = (\lambda_k(\Omega_{a_n, b_n}) + 1)^{-1} \quad \text{and} \quad \mu_k(B) = (\lambda_k(\Omega_{a, b}) + 1)^{-1}$$

we conclude (A.14).

In order to check condition (i) let $f \in L^2(\Omega)$ and set $u_n = B_n f$, $n \in \mathbb{N}$. For $\phi \in H^1(\Omega_{a_n, b_n})$ it follows from the definition of the Neumann Laplacian $A_{\Omega_{a_n, b_n}}$ that u_n satisfies

$$(\nabla u_n, \nabla \phi)_{L^2(\Omega_{a_n, b_n})} + (u_n, \phi)_{L^2(\Omega_{a_n, b_n})} = ((A_{\Omega_{a_n, b_n}} + \mathbf{I})u_n, \phi)_{L^2(\Omega_{a_n, b_n})} = (f, \phi)_{L^2(\Omega_{a_n, b_n})}. \quad (\text{A.15})$$

In particular, using (A.15) for $\phi = u_n$ we get

$$\begin{aligned} \|u_n\|_{H^1(\Omega_{a_n, b_n})}^2 &= \|\nabla u_n\|_{L^2(\Omega_{a_n, b_n})}^2 + \|u_n\|_{L^2(\Omega_{a_n, b_n})}^2 = (f, u_n)_{L^2(\Omega_{a_n, b_n})} \\ &\leq \|f\|_{L^2(\Omega_{a_n, b_n})} \|u_n\|_{L^2(\Omega_{a_n, b_n})} \leq \|f\|_{L^2(\Omega_{a_n, b_n})} \|u_n\|_{H^1(\Omega_{a_n, b_n})}, \end{aligned}$$

and therefore

$$\|u_n\|_{H^1(\Omega_{a_n, b_n})} \leq \|f\|_{L^2(\Omega_{a_n, b_n})}. \quad (\text{A.16})$$

We set $u_n^\pm = u_n \upharpoonright_{\Omega_\pm}$. Below we shall use the same \pm -superscript notation for restrictions of other functions onto Ω_\pm . It follows from (A.16) that $(u_n^\pm)_{n \in \mathbb{N}}$ is a bounded sequence in $H^1(\Omega_\pm)$ and hence there exist $u^\pm \in H^1(\Omega_\pm)$ and a subsequence $n_k \rightarrow \infty$ such that

$$u_{n_k}^\pm \rightharpoonup u^\pm \text{ in } H^1(\Omega_\pm) \quad (\text{A.17})$$

(as usual the notation \rightharpoonup is used for the *weak* convergence). With the help of Rellich's theorem we conclude from (A.17) that

$$u_{n_k}^\pm \rightarrow u^\pm \text{ in } H^{1-\kappa}(\Omega_\pm), \quad \kappa \in (0, 1]. \quad (\text{A.18})$$

Finally, well-known mapping properties of the trace operator on $H^1(\Omega_\pm)$ (see, e.g., [73, Theorem 3.37]) together with (A.18) lead to

$$\gamma_\Gamma^\pm u_{n_k}^\pm \rightarrow \gamma_\Gamma^\pm u^\pm \text{ in } L^2(\Gamma) \quad (\text{A.19})$$

as $n_k \rightarrow \infty$, where $\gamma_\Gamma^\pm u^\pm$ stands for the restriction of the trace of the function $u^\pm \in H^1(\Omega_\pm)$ onto Γ .

Next we introduce the set of functions

$$\begin{aligned} \widehat{H}^1(\Omega_{a,b}) &= \{u \in H^1(\Omega_{a,b}) : \exists \delta = \delta(u) > 0 \text{ such that} \\ &\quad u = 0 \text{ in } \delta\text{-neighborhoods of } (0, a) \text{ and } (0, b)\}. \end{aligned}$$

It is known that $\widehat{H}^1(\Omega_{a,b})$ is dense in $H^1(\Omega_{a,b})$ (this is due to the fact that the capacity of the set $\{(0, a), (0, b)\}$ is zero; we refer to [80] for more details). Now let $\phi \in \widehat{H}^1(\Omega_{a,b})$. It is clear that for n_k sufficiently large we also have $\phi \in H^1(\Omega_{a_{n_k}, b_{n_k}})$ and hence (A.15) is valid. The identity (A.15) written componentwise reads as

$$\begin{aligned} (\nabla u_{n_k}^-, \nabla \phi^-)_{L^2(\Omega_-)} + (\nabla u_{n_k}^+, \nabla \phi^+)_{L^2(\Omega_+)} + (u_{n_k}^-, \phi^-)_{L^2(\Omega_-)} + (u_{n_k}^+, \phi^+)_{L^2(\Omega_+)} \\ = (f^-, \phi^-)_{L^2(\Omega_-)} + (f^+, \phi^+)_{L^2(\Omega_+)}, \end{aligned}$$

and passing to the limit (we have weak convergence in $H^1(\Omega_\pm)$ by (A.17)) as $n_k \rightarrow \infty$ we get

$$\begin{aligned} (\nabla u^-, \nabla \phi^-)_{L^2(\Omega_-)} + (\nabla u^+, \nabla \phi^+)_{L^2(\Omega_+)} + (u^-, \phi^-)_{L^2(\Omega_-)} + (u^+, \phi^+)_{L^2(\Omega_+)} \\ = (f^-, \phi^-)_{L^2(\Omega_-)} + (f^+, \phi^+)_{L^2(\Omega_+)}. \end{aligned} \quad (\text{A.20})$$

Let us denote

$$u(x) = \begin{cases} u^-(x), & x \in \Omega_-, \\ u^+(x), & x \in \Omega_+. \end{cases}$$

Obviously $u \in L^2(\Omega)$. Using (A.18) with $\kappa = 1$ we obtain

$$u_{n_k} \rightarrow u \text{ in } L^2(\Omega). \quad (\text{A.21})$$

Since $u_{n_k} \in H^1(\Omega_{a_{n_k}, b_{n_k}})$ it is clear that

$$\gamma_{\Gamma_{a_{n_k}, b_{n_k}}}^- u_{n_k}^- = \gamma_{\Gamma_{a_{n_k}, b_{n_k}}}^+ u_{n_k}^+,$$

where $\gamma_{\Gamma_{a_{n_k}, b_{n_k}}}^\pm$ is the restriction of the trace onto $\Gamma_{a_{n_k}, b_{n_k}} = \{0\} \times (a_{n_k}, b_{n_k})$. Therefore, (A.19) implies that $\gamma_{\Gamma_{a', b'}}^- u^- = \gamma_{\Gamma_{a', b'}}^+ u^+$ for any interval $(a', b') \subset (a, b)$ and, consequently,

$$\gamma_{\Gamma_{a, b}}^- u^- = \gamma_{\Gamma_{a, b}}^+ u^+.$$

As $u^\pm \in H^1(\Omega_\pm)$ this implies $u \in H^1(\Omega_{a, b})$ and (A.20) can be written in the form

$$(\nabla u, \nabla \phi)_{L^2(\Omega_{a, b})} + (u, \phi)_{L^2(\Omega_{a, b})} = (f, \phi)_{L^2(\Omega_{a, b})}. \quad (\text{A.22})$$

Since $\widehat{H}^1(\Omega_{a, b})$ is dense in $H^1(\Omega_{a, b})$ this equality holds for any $\phi \in \widehat{H}^1(\Omega_{a, b})$. It is easy to see that (A.22) is equivalent to $u = Bf$. This also shows that the limit function u is independent of the subsequence n_k and hence we conclude that (A.21) holds for any subsequence n_k . Thus,

$$B_n f = u_n \rightarrow u = Bf \text{ in } L^2(\Omega)$$

as $n \rightarrow \infty$. We have verified assumption (i).

To check assumption (ii) in Proposition A.3 let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^2(\Omega)$. The same arguments as in the proof of (i) (cf. (A.16)) show that the sequence $(B_n f_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\Omega \setminus \Gamma)$, and hence contains a weakly convergent subsequence in $H^1(\Omega \setminus \Gamma)$. Since the embedding

$$H^1(\Omega \setminus \Gamma) \hookrightarrow L^2(\Omega \setminus \Gamma) = L^2(\Omega)$$

is compact (again we use Rellich's embedding theorem) we conclude that there is a strongly convergent subsequence in $L^2(\Omega)$, that is, condition (ii) is satisfied. \square

Remark A.5 Besides the continuity of the function $\{a, b\} \mapsto \lambda_k(\Omega_{a, b})$ one can also conclude that it decreases (resp., increases) monotonically with respect to a (resp., with respect to b). This follows easily from the min-max principle (see, e.g., [31, Section 4.5]). Note, that, in general, when one perturbs a fixed domain Ω by removing a subset S_a ($a \in \mathbb{R}$ is a parameter) the monotonicity of the eigenvalues of the Neumann Laplacian in $\Omega_a := \Omega \setminus S_a$ does not follow from the monotonicity of the underlying domains with respect to a , i.e., even if $\Omega_a \subset \Omega_{\bar{a}}$, it does not mean that $\lambda(\Omega_a) \geq \lambda(\Omega_{\bar{a}})$ (see [72, Section 2.3] for more details). This is in contrast to Dirichlet Laplacian, where the monotonicity is always present – see, e.g., [51, 72, 77, 80] for the properties of Dirichlet eigenvalues in so perturbed domains. However, in our configuration monotonicity nevertheless holds for Neumann eigenvalues. This is due to a special structure of the removed set having the form of two walls with zero thickness.

B Convergence results for monotone sequences of quadratic forms

We recall a well-known convergence result for a sequence of monotonically increasing quadratic forms from [85] which is used in the proof of Theorem 3.1.

Consider a family $\{\mathfrak{a}_q\}_{q>0}$ of densely defined closed nonnegative sesquilinear forms in a Hilbert space \mathcal{H} . For simplicity we assume that the domain of \mathfrak{a}_q is the same for all q , and we use the notation $\text{dom}(\mathfrak{a}_q) = \mathcal{H}^1$. Let \mathcal{A}_q be the nonnegative self-adjoint operator associated with the form \mathfrak{a}_q via the first representation theorem. Now assume, in addition, that the family $\{\mathfrak{a}_q\}_{q>0}$ of forms increases monotonically as q decreases, i.e. for any $0 < q < \tilde{q} < \infty$ one has

$$\mathfrak{a}_q[u, u] \geq \mathfrak{a}_{\tilde{q}}[u, u], \quad u \in \mathcal{H}^1. \quad (\text{B.1})$$

We define the limit form \mathfrak{a}_0 as follows:

$$\text{dom}(\mathfrak{a}_0) = \left\{ u \in \mathcal{H}^1 : \sup_{q>0} \mathfrak{a}_q[u, u] < \infty \right\}, \quad \mathfrak{a}_0[u, v] = \lim_{q \rightarrow 0} \mathfrak{a}_q[u, v].$$

One verifies that \mathfrak{a}_0 is a well-defined nonnegative symmetric sesquilinear form (which is not necessarily densely defined) and, in fact, by [85] the limit form \mathfrak{a}_0 is closed. Let us now assume that $\text{dom}(\mathfrak{a}_0)$ is dense in \mathcal{H} , so that one can associate a nonnegative self-adjoint operator \mathcal{A}_0 with \mathfrak{a}_0 via the first representation theorem.¹ According to [85] one then has convergence of the corresponding nonnegative self-adjoint operators in the strong resolvent sense (see also [32, Proposition 4.1] or [15, Theorem 4.2]):

Theorem B.1 (Simon, 1978) *For each $f \in \mathcal{H}$ one has*

$$\|(\mathcal{A}_q + \mathbf{I})^{-1}f - (\mathcal{A}_0 + \mathbf{I})^{-1}f\| \rightarrow 0 \text{ as } q \rightarrow 0. \quad (\text{B.2})$$

Now let us assume, in addition, that the spectra of the self-adjoint operators \mathcal{A}_q and \mathcal{A}_0 are purely discrete. We write $(\lambda_k(\mathcal{A}_q))_{k \in \mathbb{N}}$ and $(\lambda_k(\mathcal{A}_0))_{k \in \mathbb{N}}$ for the eigenvalues of these operators counted with multiplicities and ordered as nondecreasing sequences. In this case one can conclude the following spectral convergence:

Theorem B.2 *For each $k \in \mathbb{N}$ one has*

$$\lambda_k(\mathcal{A}_q) \rightarrow \lambda_k(\mathcal{A}_0) \text{ as } q \rightarrow 0. \quad (\text{B.3})$$

Proof. The discreteness of the spectra of \mathcal{A}_q and \mathcal{A}_0 is equivalent to the compactness of the resolvents $(\mathcal{A}_q + \mathbf{I})^{-1}$ and $(\mathcal{A}_0 + \mathbf{I})^{-1}$. Moreover, (B.1) implies $(\mathcal{A}_q + \mathbf{I})^{-1} \leq (\mathcal{A}_{\tilde{q}} + \mathbf{I})^{-1}$ provided $0 < q < \tilde{q} < \infty$; cf. [85, Proposition 1.1]. Then by [59, Theorem VIII-3.5] the strong convergence in (B.2) becomes even convergence in the operator norm, that is,

$$\|(\mathcal{A}_q + \mathbf{I})^{-1} - (\mathcal{A}_0 + \mathbf{I})^{-1}\| \rightarrow 0 \text{ as } q \rightarrow 0. \quad (\text{B.4})$$

It is well-known (see, e.g., [33, Lemma XI.9.5] or [79, Corollary A.15]) that the norm resolvent convergence (B.4) implies the convergence of the eigenvalues, i.e. (B.3) holds. \square

References

- [1] R.A. Adams, Sobolev Spaces, Academic Press, New York-London, 1975.
- [2] N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover Publications, New York, 1993.
- [3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, Solvable Models in Quantum Mechanics, 2nd edition. With an appendix by Pavel Exner, AMS Chelsea Publishing, Providence, RI, 2005.
- [4] S. Albeverio, J. Brasche, M.M. Malamud, and H. Neidhardt, Inverse spectral theory for symmetric operators with several gaps: scalar-type Weyl functions, J. Funct. Anal. 228 (2005), 144–188.
- [5] S. Albeverio, J. Brasche, and H. Neidhardt, On inverse spectral theory for self-adjoint extensions: mixed types of spectra, J. Funct. Anal. 154 (1998), 130–173.
- [6] S. Albeverio and L. Nizhnik, A Schrödinger operator with a δ' -interaction on a Cantor set and Krein-Feller operators, Math. Nachr. 279 (2006), 467–476.
- [7] Š.A. Alimov and V.A. Il'in, Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators. II. Self-adjoint extension of the Laplace operator with an arbitrary spectrum, Diferentsialnye Uravnenija 7 (1971), 851–882.
- [8] C.J. Amick, Some remarks on Rellich's theorem and the Poincaré inequality, J. Lond. Math. Soc., II. Ser. 18 (1978), 81–93.
- [9] C. Anné, Spectre du Laplacien et écrasement d'anses, Ann. sci. École norm. Sup. 20 (1987), 271–280.
- [10] J.M. Arrieta, Rates of eigenvalues on a dumbbell domain. Simple eigenvalue case, Trans. Am. Math. Soc. 347 (1995), 3503–3531.
- [11] J.E. Avron, P. Exner, and Y. Last, Periodic Schrödinger operators with large gaps and Wannier-Stark ladders, Phys. Rev. Lett. 72 (1994), 896–899.
- [12] D. Barseghyan and A. Khrabustovskiy, Gaps in the spectrum of a periodic quantum graph with periodically distributed δ' -type interactions, J. Phys. A: Math. Theor. 48 (2015), 255201.
- [13] J. Behrndt, P. Exner, and V. Lotoreichik, Schrödinger operators with δ and δ' interactions on Lipschitz surfaces and chromatic numbers of associated partitions, Rev. Math. Phys. 26 (2014), 1450015.

¹ We wish to mention here that in the situation where the limit form \mathfrak{a}_0 is not densely defined one associates a self-adjoint relation (multivalued operator) via the corresponding generalized first representation theorem for nondensely defined closed nonnegative forms; see [32, 85, 88] or the more recent paper [15] for details and related convergence results.

- [14] J. Behrndt, G. Grubb, M. Langer, and V. Lotoreichik, Spectral asymptotics for resolvent differences of elliptic operators with δ and δ' -interactions on hypersurfaces, *J. Spectr. Theory* 5 (2015), 697–729.
- [15] J. Behrndt, S. Hassi, H. de Snoo, and R. Wietsma, Monotone convergence theorems for semibounded operators and forms with applications, *Proc. Roy. Soc. Edinburgh Sect. A* 140 (2010), 927–951.
- [16] J. Behrndt, M. Langer, and V. Lotoreichik, Schrödinger operators with δ and δ' -potentials supported on hypersurfaces, *Ann. Henri Poincaré* 14 (2013), 385–423.
- [17] M.S. Birman and M.Z. Solomjak, *Spectral Theory of Self-adjoint Operators in Hilbert space*, Reidel, Dordrecht, 1987.
- [18] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs 186, AMS Providence, RI, 2013.
- [19] J. Brasche, Spectral theory for self-adjoint extensions. Spectral theory of Schrödinger operators, pp. 51–96, *Contemp. Math.* 340, Aportaciones Mat., Amer. Math. Soc., Providence, RI, 2004.
- [20] J. Brasche, M.M. Malamud, and H. Neidhardt, Selfadjoint extensions with several gaps: finite deficiency indices, *Oper. Theory Adv. Appl.* 162 (2006), 85–101.
- [21] J. Brasche and H. Neidhardt, On the absolutely continuous spectrum of self-adjoint extensions, *J. Funct. Anal.* 131 (1995), 364–385.
- [22] J. Brasche and H. Neidhardt, On the singular continuous spectrum of self-adjoint extensions, *Math. Z.* 222 (1996), 533–542.
- [23] J. Brasche, H. Neidhardt, and J. Weidmann, On the point spectrum of selfadjoint extensions, *Math. Z.* 214 (1993), 343–355.
- [24] J. Brasche and L.P. Nizhnik, One-dimensional Schrödinger operators with general point interactions, *Methods Funct. Anal. Topology* 19 (2013), 4–15.
- [25] J. Brasche and L.P. Nizhnik, One-dimensional Schrödinger operator with δ' -interactions on a set of Lebesgue measure zero, *Oper. Matrices* 7 (2013), 887–904.
- [26] B.M. Brown, W.D. Evans, I.G. Wood, Some spectral properties of Rooms and Passages domains and their skeletons, Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday, 69–85, *Proc. Sympos. Pure Math.*, 87, Amer. Math. Soc., Providence, RI, 2013.
- [27] D. Buschmann, G. Stolz, and J. Weidmann, Onedimensional Schrödinger operators with local point interactions, *J. Reine Angew. Math.* 467 (1995), 169–186.
- [28] G. Cardone and A. Khrabustovskyi, δ' -interaction as a limit of a thin Neumann waveguide with transversal window, *J. Math. Anal. Appl.* 473 (2019), 1320–1342.
- [29] Y. Colin de Verdière, Construction de Laplaciens dont une partie finie du spectre est donnée, *Ann. Sci. École Norm. Sup.* 20 (1987), 599–615.
- [30] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik II*, Springer, Berlin, 1937.
- [31] E.B. Davies, *Spectral Theory and Differential Operators*, Cambridge University Press, Cambridge, 1995.
- [32] V.A. Derkach and M.M. Malamud, The extension theory of hermitian operators and the moment problem, *J. Math. Sciences*, 73 (1995), 141–242.
- [33] N. Dunford and J.T. Schwartz, *Linear Operators. Part II*, John Wiley & Sons, New York, 1988.
- [34] J. Eckhardt, A. Kostenko, M.M. Malamud, and G. Teschl, One-dimensional Schrödinger operators with δ' -interactions on Cantor-type sets, *J. Differential Equations* 257 (2014), 415–449.
- [35] W.D. Evans and D.J. Harris, Sobolev embeddings for generalized ridged domains, *Proc. London Math. Soc.* (3) 54 (1987), 141–175.
- [36] P. Exner, Lattice Kronig-Penney models, *Phys. Rev. Lett.* 74 (1995), 3503–3506.
- [37] P. Exner, The absence of the absolutely continuous spectrum for δ' Wannier- Stark ladders, *J. Math. Phys.* 36 (1995), 4561–4570.
- [38] P. Exner, Contact interactions on graph superlattices, *J. Phys. A* 29 (1996), 87–102.
- [39] P. Exner and M. Jex, Spectral asymptotics of a strong δ' interaction on a planar loop, *J. Phys. A* 46 (2013), 345201.
- [40] P. Exner and M. Jex, Spectral asymptotics of a strong δ' interaction supported by a surface, *Phys. Lett. A* 378 (2014), 2091–2095.
- [41] P. Exner and A. Khrabustovskyi, On the spectrum of narrow Neumann waveguide with periodically distributed traps, *J. Phys. A: Math. Theor.* 48 (2015), 315301.
- [42] P. Exner and A. Khrabustovskyi, Gap control by singular Schrödinger operators in a periodically structured metamaterial, *J. Math. Phys. Anal. Geom.* 14 (2018), 270–285.
- [43] P. Exner and J. Lipovský, Smilansky-Solomyak model with a δ' -interaction, *Phys. Lett. A* 382 (2018), 1207–1213.
- [44] P. Exner and J. Rohleder, Generalized interactions supported on hypersurfaces, *J. Math. Phys.* 57 (2016), 041507.
- [45] L.E. Fraenkel, On regularity of the boundary in the theory of Sobolev spaces, *Proc. London Math. Soc.* (3) 39 (1979), 385–427.
- [46] M.M. Gëhtman, On the question of the spectrum of selfadjoint extensions of a symmetric semibounded operator, *Dokl. Akad. Nauk SSSR* 186 (1969), 1250–1252.
- [47] M.M. Gëhtman, An investigation of the spectrum of certain nonclassical selfadjoint extensions of Laplace's operator, *Funkcional. Anal. i Priložen* 4 (1970), 72.

- [48] F. Gesztesy and H. Holden, A new class of solvable models in quantum mechanics describing point interactions on the line, *J. Phys. A: Math. Gen.* 20 (1987), 5157–5177.
- [49] F. Gesztesy, W. Karwowski, Z. Zhao, New types of soliton solutions, *Bull. Amer. Math. Soc. (N.S.)* 27 (1992), 266–272.
- [50] F. Gesztesy, W. Karwowski, Z. Zhao, Limits of soliton solutions, *Duke Math. J.* 68 (1992), 101–150.
- [51] F. Gesztesy and F. Zhao, Domain perturbations, Brownian motion, capacities, and ground states of Dirichlet Schrödinger operators, *Math. Z.* 215 (1994), 143–150.
- [52] A. Grossmann, R. Høegh-Krohn, and M. Mebkhout, A class of explicitly soluble, local, many-center Hamiltonians for one-particle quantum mechanics in two and three dimensions. I, *J. Math. Phys.* 21 (1980), 2376–2385.
- [53] R. Hempel, T. Kriecherbauer, and P. Plankensteiner, Discrete and Cantor spectrum for Neumann Laplacians of combs, *Math. Nachr.* 188 (1997), 141–168.
- [54] R. Hempel, L.A. Seco, and B. Simon, The essential spectrum of Neumann Laplacians on some bounded singular domains, *J. Funct. Anal.* 102 (1991), 448–483.
- [55] V.A. Il'in, Conditions for the convergence of spectral decompositions that correspond to self-adjoint extensions of elliptic operators. I. Self-adjoint extension of the Laplace operator with a point spectrum, *Differencialnye Uravnenija* 7 (1971), 670–710.
- [56] G.A. Iosif'yan, O.A. Oleinik, and A.S. Shamaev, On the limit behavior of the spectrum of a sequence of operators defined in different Hilbert spaces, *Russian Math. Surveys* 44 (1989), 195–196.
- [57] M. Jex and V. Lotoreichik, On absence of bound states for weakly attractive δ' -interactions supported on non-closed curves in \mathbb{R}^2 , *J. Math. Phys.* 57 (2016), 022101.
- [58] S. Jimbo, The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary condition, *J. Differ. Equations* 77 (1989), 322–350.
- [59] T. Kato, *Perturbation Theory for Linear Operators*, Berlin-Heidelberg-New York, Springer, 1966.
- [60] A.A. Kiselev, B.S. Pavlov, The essential spectrum of the Laplace operator of the Neumann problem in a model domain of complex structure, *Theoret. and Math. Phys.* 99 (1994), 383–395.
- [61] A. Kostenko and M.M. Malamud, Schrödinger operators with δ' -interactions and the Krein-Stieltjes string, *Dokl. Math.* 81 (2010), 342–347.
- [62] A. Kostenko and M.M. Malamud, 1-D Schrödinger operators with local point interactions on a discrete set, *J. Differential Equations* 249 (2010), 253–304.
- [63] A. Kostenko and M.M. Malamud, Spectral theory of semibounded Schrödinger operators with δ' -interactions, *Ann. Henri Poincaré* 15 (2014), 501–541.
- [64] V. Lotoreichik and J. Rohleder, An eigenvalue inequality for Schrödinger operators with δ and δ' -interactions supported on hypersurfaces, *Oper. Theor. Adv. Appl.* 247 (2015), 173–184.
- [65] A. Mantile, A. Posilicano, and M. Sini, Self-adjoint elliptic operators with boundary conditions on not closed hypersurfaces, *J. Differ. Equations* 261 (2016), 1–55.
- [66] V. Mazya, On Neumann's problem for domains with irregular boundaries, *Siberian Math. J.* 9 (1968), 990–1012.
- [67] W. Mazja, *Einbettungssätze für Sobolewsche Räume. Teil 1*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979.
- [68] W. Mazja, *Einbettungssätze für Sobolewsche Räume. Teil 2*, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980.
- [69] V.G. Mazya, *Sobolev spaces (Russian)*, Leningrad. Univ., Leningrad, 1985.
- [70] V.G. Mazya, S.V. Poborchi, *Differentiable functions on bad domains*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
- [71] V. Mazya, *Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition*, Springer, Heidelberg, 2011.
- [72] V.G. Mazya, S.A. Nazarov, and B.A. Plamenevskij, Asymptotic expansions of the eigenvalues of boundary value problems for the Laplace operator in domains with small holes, *Math. USSR, Izv.* 24(1985), 321–345.
- [73] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.
- [74] V.A. Mikhailets, Selfadjoint extensions of operators with a prescribed spectrum asymptotics, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* 5 (1979), 338–341, 398.
- [75] V.A. Mikhailets, Schrödinger operator with point δ' -interactions, *Dokl. Math.* 348 (1996), 727–730.
- [76] J. von Neumann, Allgemeine Eigenwerttheorie hermitescher Funktionaloperatoren, *Math. Annalen* 102 (1930), 49–131.
- [77] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, *Duke Math. J.* 48 (1981), 767–778.
- [78] L.E. Payne, G. Pólya, and H.F. Weinberger, On the ratio of consecutive eigenvalues, *J. Math. and Phys.* 35 (1956), 289–298.
- [79] O. Post, Spectral convergence of quasi-one-dimensional spaces, *Ann. Henri Poincaré* 7 (2006), 933–973.
- [80] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains, *J. Funct. Anal.* 18 (1975), 27–59.
- [81] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*, Academic Press, New York–London, 1972.
- [82] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York–London, 1978.
- [83] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. III. Analysis of Operators*, Academic Press, New York–London, 1979.

- [84] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert space*, Springer, Dordrecht, 2012.
- [85] B. Simon, A canonical decomposition for quadratic forms with applications to monotone convergence theorems, *J. Funct. Anal.* 28 (1978), 377–385.
- [86] B. Simon, The Neumann Laplacian of a jelly roll, *Proc. Amer. Math. Soc.* 114 (1992), 783–785.
- [87] H. Triebel, *Höhere Analysis*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1972.
- [88] J. Weidmann, Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet, *Math. Scand.* 54 (1984) 51–69.
- [89] J. Zhao, G. Shi, and J. Yan, Discreteness of spectrum for Schrödinger operators with δ' -type conditions on infinite regular trees, *Proc. Roy. Soc. Edinburgh Sect. A* 147 (2017), 1091–1117.