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Linear fractional transformations of Stieltjes functions

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Linear fractional transformations of Stieltjes (and inverse Stieljes) functions, which appear naturally in the extension theory of nonnegative symmetric operators with defect one in Hilbert spaces, are investigated.

1 Nevanlinna, Stieltjes, and inverse Stieltjes functions

The class of Nevanlinna functions is intimately connected with selfadjoint operators and relations in Hilbert spaces, and therefore plays a key role in spectral analysis. For instance, the set of Titchmarsh-Weyl coefficients of real trace-normed 2×2 canonical systems on a halfline coincide with the class of Nevanlinna functions. Recall that a scalar function Q is said to be a Nevanlinna function, $Q \in \mathbf{N}$, if it admits an integral representation of the form

$$Q(\lambda) = \alpha + \beta\lambda + \int_{\mathbb{R}} \left(\frac{1}{s-\lambda} - \frac{s}{s^2+1} \right) d\sigma(s), \qquad \int_{\mathbb{R}} \frac{d\sigma(s)}{s^2+1} < \infty, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and σ is a nondecreasing function on $\mathbb{R}$. Various subclasses of Nevanlinna functions have been studied in the past, e.g. Stieltjes and inverse Stieltjes functions in connection with spectral problems for strings in [3–5], and other slightly more general classes in connection with nonnegative symmetric operators in [1]. Recall that a Nevanlinna function Q belongs to the Stieltjes class $\mathbf{S}$ (inverse Stieltjes class $\mathbf{S}^{-1}$) if and only if Q is holomorphic and nonnegative (nonpositive, respectively) on $(-\infty, 0)$. It is clear that $Q \in \mathbf{S}$ if and only if $-1/Q \in \mathbf{S}^{-1}$. Moreover, $Q \in \mathbf{S} \cup \mathbf{S}^{-1}$ if and only if $Q \in \mathbf{N}$ and Q is holomorphic on $(-\infty, 0)$ without zeros there. Alternatively, $Q \in \mathbf{S}$ ($Q \in \mathbf{S}^{-1}$) if and only if $Q(\lambda), \lambda Q(\lambda) \in \mathbf{N}$ ($Q(\lambda), Q(\lambda)/\lambda \in \mathbf{N}$, respectively). The Stieltjes and inverse Stieltjes class can also be characterized via integral representations; cf. [5].

2 Linear fractional transformations of Stieltjes functions

The linear fractional transformations Q_τ, $\tau \in \mathbb{R} \cup \{\infty\}$, of a Nevanlinna function Q are defined by

$$Q_\tau(\lambda) = \frac{Q(\lambda) - \tau}{1 + \tau Q(\lambda)}, \;\; \tau \in \mathbb{R}, \quad \text{and} \quad Q_\infty(\lambda) = -1/Q(\lambda), \;\; \tau = \infty. \tag{2}$$

It is not difficult to see that Q_τ is a Nevanlinna function for all $\tau \in \mathbb{R} \cup \{\infty\}$. Moreover, notice that $(Q_\eta)_\tau = Q_s$ where $s = (\eta + \tau)/(1 - \eta\tau)$ with $\eta, \tau \in \mathbb{R} \cup \{\infty\}$; in particular, the class of functions $\{ Q_\tau : \tau \in \mathbb{R} \cup \{\infty\} \}$ is stable under composition of transformations in (2).

Now assume that Q is holomorphic on $(-\infty, 0)$ except for finitely many points, as is the case for $Q \in \mathbf{S} \cup \mathbf{S}^{-1}$. Then the possibly improper limits of Q at $-\infty$ and 0 exist, they are denoted by b and L:

$$b := \lim_{\lambda \downarrow -\infty} Q(\lambda) \in \mathbb{R} \cup \{-\infty\} \quad \text{and} \quad L := \lim_{\lambda \uparrow 0} Q(\lambda) \in \mathbb{R} \cup \{+\infty\}. \tag{3}$$

Lemma 2.1 *Let Q be a nonconstant Nevanlinna function and let Q_τ be given by* (2), $\tau \in \mathbb{R} \cup \{\infty\}$. *Then:*

(i) *If Q is holomorphic on $(-\infty, 0)$, then $b < L$ and Q_τ has precisely one zero and one pole on $(-\infty, 0)$ if and only if*

$$b < \tau < -1/L \leq 0 \quad \text{or} \quad 0 \leq -1/b < \tau < L.$$

(ii) *If Q is holomorphic on $(-\infty, 0)$ except for one point a, then Q_τ is holomorphic on $(-\infty, 0)$ and has no zeros on $(-\infty, 0)$ if and only if*

$$-\infty < L \leq \tau \leq -1/b < 0 \quad \text{or} \quad 0 < -1/L \leq \tau \leq b < \infty. \tag{4}$$

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Proof. (i) Since Q is nonconstant and holomorphic on $(-\infty,0)$ the integral representation (1) yields that Q is strictly increasing on $(-\infty,0)$ and takes on all values between b and L uniquely. Hence (2) shows that Q_τ has a zero on $(-\infty,0)$ for $b<\tau<L$ and a pole on $(-\infty,0)$ for $b<-1/\tau<L$. These inequalities can hold simultaneously only if $b<0<L$ in which case $-1/L\leq 0\leq -1/b$. Now the assertion follows by considering the cases $b<0<-1/\tau<L$ and $b<-1/\tau<0<L$.

(ii) If (4) holds or $Q_\tau\in\mathbf{S}\cup\mathbf{S}^{-1}$ for some τ, then $-\infty<L<b<\infty$. Now proceed as in (i) with the interval (L,b). □

The next results concern the linear fractional transforms Q_τ of a Stieltjes function.

Proposition 2.2 *Let $Q\in\mathbf{S}$ be a nonconstant Stieltjes function and let Q_τ be given by* (2), $\tau\in\mathbb{R}\cup\{\infty\}$. *Then b and L satisfy the inequality $0\leq b<L\leq\infty$ (so that also $-\infty\leq -1/b<-1/L\leq 0$) and the following statements hold:*

(i) *$Q_\tau\in\mathbf{S}$ if and only if $-1/L\leq\tau\leq b$;*

(ii) *$Q_\tau\in\mathbf{S}^{-1}$ if and only if $\tau\leq -1/b$, $\tau\geq L$, or $\tau=\infty$;*

(iii) *Q_τ has a (unique) zero and no poles on $(-\infty,0)$ if and only if $b<\tau<L$;*

(iv) *Q_τ has a (unique) pole and no zeros on $(-\infty,0)$ if and only if $-1/b<\tau<-1/L$.*

In particular, Q (and $-Q^{-1}$) is the only function Q_τ in (2) *belonging to $\mathbf{S}$ ($\mathbf{S}^{-1}$, respectively) if and only if $b=0$ and $L=\infty$.*

Proof. (iii) & (iv) The function Q_τ has a (unique) zero in $(-\infty,0)$ if and only if $b<\tau<L$, and Q_τ has a (unique) pole in $(-\infty,0)$ if and only if $-1/b<\tau<-1/L$ (cf. the proof of Lemma 2.1). Since the inequalities $b<\tau<L$ and $-1/b<\tau<-1/L$ cannot hold simultaneously, (iii) and (iv) follow.

(i) & (ii) ($\Rightarrow$) If $Q_\tau\in\mathbf{S}$ or $Q_\tau\in\mathbf{S}^{-1}$, then, in particular, Q_τ has no zero and is holomorphic on $(-\infty,0)$. Thus, by (iii) & (iv), $\tau\notin(b,L)$ and $\tau\notin(-1/b,-1/L)$. For $-1/L\leq\tau\leq b$ the values of Q_τ on $(-\infty,0)$ are positive and for $\tau\leq -1/b$, $\tau\geq L$, and $\tau=\infty$ the values of Q_τ on $(-\infty,0)$ are negative. ($\Leftarrow$) This implication follows with similar arguments. □

The next theorem shows under which conditions a Nevanlinna function Q possesses a transformation Q_τ in the Stieltjes or inverse Stieltjes class. In view of Lemma 2.1 only Nevanlinna functions that are holomorphic on $(-\infty,0)$, or have at most one pole on $(-\infty,0)$ and satisfy $-\infty<L<0<b<\infty$, have to be considered.

Recall that a symmetric scalar function Q which is meromorphic on $\mathbb{C}\setminus\mathbb{R}$, is said to belong to the class of generalized Nevanlinna functions with $\kappa\in\mathbb{N}$ negative squares, $Q\in\mathbf{N}_\kappa$, if its Nevanlinna kernel has κ negative squares; see, e.g. [6]. Note that, if $Q\in\mathbf{S}$ ($Q\in\mathbf{S}^{-1}$) then $\lambda Q(\lambda)\in\mathbf{N}$ and, moreover, $Q(\lambda)/\lambda\in\mathbf{N}_1$ ($Q(\lambda)/\lambda\in\mathbf{N}$ and $\lambda Q(\lambda)\in\mathbf{N}_1$).

Theorem 2.3 *Let Q be a nonconstant Nevanlinna function which is holomorphic on $(-\infty,0)$ except for possibly one point, in which case it is assumed that $-\infty<L<0<b<\infty$. Then the following statements are equivalent:*

(i) *there exists $\eta\in\mathbb{R}\cup\{\infty\}$ such that $Q_\eta\in\mathbf{S}$, or equivalently, there exists $\eta\in\mathbb{R}\cup\{\infty\}$ such that $Q_\eta\in\mathbf{S}^{-1}$;*

(ii) *if Q_τ, $\tau\in\mathbb{R}\cup\{\infty\}$, in* (2) *has a zero (pole) on $(-\infty,0)$, then it does not have a pole (zero) on $(-\infty,0)$;*

(iii) *if Q is holomorphic and has a zero on $(-\infty,0)$, then $-\infty<-1/L\leq b<0$; and if Q is not holomorphic on $(-\infty,0)$, then $-\infty<L\leq -1/b<0$;*

(iv) *$\lambda Q_\tau(\lambda)\in\mathbf{N}\cup\mathbf{N}_1$ and $Q_\tau(\lambda)/\lambda\in\mathbf{N}\cup\mathbf{N}_1$ for all $\tau\in\mathbb{R}\cup\{\infty\}$.*

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2.2.
(ii) $\Rightarrow$ (iii) If Q has a zero on $(-\infty,0)$ and $-\infty<b<0<L<\infty$, then Q_∞ has a pole on $(-\infty,0)$ and the corresponding limits $L_\infty=-1/L$ and $b_\infty=-1/b$ satisfy $-\infty<L_\infty<0<b_\infty<\infty$; and conversely. On the other hand, if Q is holomorphic on $(-\infty,0)$ and $b<-1/L\leq 0$, then by Lemma 2.1 the transformation Q_η, $b<\eta<-1/L$, has a pole and a zero on $(-\infty,0)$. This contradiction together with the inequalities $-\infty<b<0<L<\infty$ implies that $-\infty<-1/L\leq b<0$. These inequalities are equivalent to $-\infty<L_\infty\leq -1/b_\infty<0$ for the limits of Q_∞. Hence, (iii) holds.
(iii) $\Rightarrow$ (i) If Q has neither a zero nor a pole on $(-\infty,0)$, then $Q\in\mathbf{S}\cup\mathbf{S}^{-1}$. If Q is not holomorphic on $(-\infty,0)$ and $-\infty<L\leq -1/b<0$, then Lemma 2.1 shows that $Q_L\in\mathbf{S}\cup\mathbf{S}^{-1}$. If Q is holomorphic with a zero on $(-\infty,0)$ and $-\infty<-1/L\leq b<0$, then Q_∞ has a pole and the corresponding limits $L_\infty=-1/L$ and $b_\infty=-1/b$ satisfy $-\infty<L_\infty\leq -1/b_\infty<0$. Thus, $(Q_\infty)_{L_\infty}\in\mathbf{S}\cup\mathbf{S}^{-1}$ again by Lemma 2.1.
(ii) $\Leftrightarrow$ (iv) This follows from the fact that $\lambda Q(\lambda), Q(\lambda)/\lambda\in\mathbf{N}\cup\mathbf{N}_1$ if and only if Q is holomorphic on $(-\infty,0)$ except for at most one point in which case it does not have a zero on $(-\infty,0)$; cf. [2, Theorem 4.5 & Remark 4.7]. □

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