ANTITONICITY OF THE INVERSE FOR SELFADJOINT MATRICES, OPERATORS, AND RELATIONS

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Abstract. Let H_1 and H_2 be selfadjoint operators or relations (multivalued operators) acting on a separable Hilbert space and assume that the inequality $H_1 \le H_2$ holds. Then the validity of the inequalities $-H_1^{-1} \le -H_2^{-1}$ and $H_2^{-1} \le H_1^{-1}$ is characterized in terms of the inertia of H_1 and H_2. Such results are known for matrices and boundedly invertible operators. In the present paper those results are extended to selfadjoint, in general unbounded, not necessarily boundedly invertible, operators and, more generally, for selfadjoint relations in separable Hilbert spaces.

1. Introduction

Let H_1 and H_2 be selfadjoint matrices, operators, or relations (multivalued operators) in a separable Hilbert space, which is not necessarily finite-dimensional. This paper is concerned with a question which goes back to K. Löwner: what are the implications of the inequality $H_1 \le H_2$ for the inverses of H_1 and H_2; cf. [1, 15].

Here specific conditions are investigated under which the implication

$$H_1 \le H_2 \quad \Rightarrow \quad H_2^{-1} \le H_1^{-1} \tag{1.1}$$

is true. In the literature such results are often formulated as *antitonicity results*, see e.g. [4, 10, 16, 18]. Of course, the above implication does not hold in general; a simple counterexample is $H_1 = -I$ and $H_2 = I$. In the finite-dimensional setting necessary and sufficient conditions for the implication in (1.1) to hold are given by the following antitonicity theorem, see [4, 16]. Recall that the inertia of the selfadjoint matrix H_i, $i = 1, 2$, is the ordered triplet, $\mathsf{i}(H_i) = \{\mathsf{i}_i^+, \mathsf{i}_i^-, \mathsf{i}_i^0\}$, of the numbers of positive, negative, and zero eigenvalues of H_i.

Theorem 1.1. *Let H_1 and H_2 be invertible selfadjoint matrices in $\mathbb{C}^n$ and assume that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1} \quad \textit{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

The condition that the matrices H_1 and H_2 are invertible means that $\mathsf{i}_1^0 = \mathsf{i}_2^0 = 0$; hence the condition $\mathsf{i}(H_1) = \mathsf{i}(H_2)$ in Theorem 1.1 is equivalent to $\mathsf{i}_1^- = \mathsf{i}_2^-$ and to $\mathsf{i}_1^+ = \mathsf{i}_2^+$. If in Theorem 1.1 the matrices H_1 and H_2 are not invertible, then

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the inverses H_1^{-1} and H_2^{-1} still exist in the sense of linear relations (multivalued mappings). With this interpretation Theorem 1.1 can be generalized to obtain the following two results, which are new and applicable already in the finite-dimensional setting (cf. [6, 7]).

Theorem 1.2. *Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$ and assume that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1} \quad \textit{if and only if} \quad \mathsf{i}_1^- = \mathsf{i}_2^- .$$

Theorem 1.3. *Let H_1 and H_2 be selfadjoint relations in $\mathbb{C}^n$ and assume that $H_1 \le H_2$. Then*

$$-H_1^{-1} \le -H_2^{-1} \quad \textit{if and only if} \quad \mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0.$$

Clearly, when the selfadjoint relations H_1 and H_2 are invertible matrices, then Theorem 1.2 and 1.3 coincide with Theorem 1.1. However, in the case of non-invertible matrices H_1 and H_2 the above statements are new extensions of Theorem 1.1. Note that, since H_1^{-1} and H_2^{-1} are selfadjoint relations, the condition $-H_1^{-1} \le -H_2^{-1}$ is in general different from the condition $H_2^{-1} \le H_1^{-1}$.

From either of the above theorems also other previously known antitonicity results in the matrix literature can be derived as special cases. For example the main antitonicity result for the Moore-Penrose inverse H^+ of a selfadjoint matrix H, see [4, Theorem 2], can be obtained as a direct consequence of Theorem 1.2.

Corollary 1.4. *Let H_1 and H_2 be selfadjoint matrices in $\mathbb{C}^n$ and assume that $H_1 \le H_2$. Then*

$$H_2^+ \le H_1^+ \quad \textit{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

It should be emphasized that both inequalities $H_2^{-1} \le H_1^{-1}$ and $-H_1^{-1} \le -H_2^{-1}$ occur naturally in the study of limits of monotone matrix functions, and they have different geometrical implications; see [7]. Such inequalities between selfadjoint relations have interesting applications, for instance, in the area of differential equations: they appear in the study of the square-integrability of solutions of definite canonical systems of differential equations; see [6] and the references therein.

The objective of this paper is to prove antitonicity results analogous to Theorem 1.2 and Theorem 1.3 for selfadjoint operators or relations H_1 and H_2 in a separable, not necessarily finite-dimensional, Hilbert space. The results and their proofs can be read with the finite-dimensional case in mind; in fact, the proofs of the main two antitonicity theorems, Theorem 3.2 and Theorem 3.4 below, do not essentially simplify in the finite-dimensional setting. As a preparation some facts on selfadjoint relations in Hilbert spaces are in given Section 2. In particular, the notion of ordering for selfadjoint relations which are bounded from below and the concept of inertia are introduced. Section 3 contains the main results of the paper: the two infinite-dimensional variants of Theorem 1.2 and 1.3. The important ingredients in their proofs are an infinite-dimensional version of Theorem 1.1, which has been independently established in [18, 9] (cf. [10], and see also [3]), combined with suitable perturbations arguments, and a general limit result on monotone operator functions. Various consequences of the two main antitonicity results are discussed, among them an infinite-dimensional version of the antitonicity result for Moore-Penrose inverses in Corollary 1.4.

2. Ordering and inertia of selfadjoint relations

This section contains an introduction to selfadjoint relations in Hilbert spaces. In particular the notions of ordering and inertia for selfadjoint relations in Hilbert spaces are introduced and investigated.

2.1. **Linear relations.** Let $\mathfrak{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. A *(closed) relation* H in $\mathfrak{H}$ is a (closed) linear subspace of the product space $\mathfrak{H}\times\mathfrak{H}$. As such, H is considered to consist of pairs $\{h,k\}\in\mathfrak{H}\times\mathfrak{H}$, so that H is the graph of a multivalued (linear) operator in $\mathfrak{H}$. The *domain, range, kernel,* and *multivalued part* of a relation H are defined as follows:

$$\begin{aligned}\operatorname{dom} H &= \{h\in\mathfrak{H} : \{h,k\}\in H\}, \quad \operatorname{ran} H = \{k\in\mathfrak{H} : \{h,k\}\in H\},\\ \ker H &= \{h\in\mathfrak{H} : \{h,0\}\in H\}, \quad \operatorname{mul} H = \{k\in\mathfrak{H} : \{0,k\}\in H\}.\end{aligned}$$

Note that, if H is closed then $\ker H$ and $\operatorname{mul} H$ are closed subspaces. A number $\lambda\in\mathbb{C}$ is called an *eigenvalue* of H if $\{h,\lambda h\}\in H$ for some nontrivial $h\in\mathfrak{H}$, which is then called an *eigenvector*. Similarly, ∞ is said to be an *eigenvalue* of H if $\{0,k\}\in H$ or, equivalently, $k\in\operatorname{mul} H$, for some nontrivial $k\in\mathfrak{H}$, which is then called an *eigenvector*. The relation H is an operator precisely when $\operatorname{mul} H=\{0\}$, i.e., when ∞ is not an eigenvalue of H.

Each relation H has an *inverse* H^{-1} and an *adjoint* H^*, which are defined as

$$\begin{aligned}H^{-1} &= \big\{\{k,h\} : \{h,k\}\in H\big\};\\ H^* &= \big\{\,\{h,k\}\in\mathfrak{H}\times\mathfrak{H} : (g,h)=(f,k) \text{ for all } \{f,g\}\in H\,\big\}.\end{aligned}$$

In particular, $\operatorname{dom} H^{-1}=\operatorname{ran} H$ and $\ker H^{-1}=\operatorname{mul} H$. Note that the adjoint is a closed relation in $\mathfrak{H}$ and that it coincides with the usual adjoint when H is a densely defined operator.

For a relation H in $\mathfrak{H}$ and $\lambda\in\mathbb{C}$, the relation $H-\lambda$ is given by

$$H-\lambda=\big\{\,\{h,k-\lambda h\} : \{h,k\}\in H\,\big\}.$$

Its inverse, $(H-\lambda)^{-1}$, is a relation whose kernel and multivalued part coincide with $\operatorname{mul} H$ and $\ker(H-\lambda)$, respectively. Furthermore, it satisfies the following spectral mapping identity:

$$(H-\lambda)^{-1}=-\frac{1}{\lambda}+\frac{1}{\lambda^2}\left(-H^{-1}-\left(-\frac{1}{\lambda}\right)\right)^{-1},\quad \lambda\in\mathbb{C}\setminus\{0\}. \tag{2.1}$$

For a closed relation H the number $\lambda\in\mathbb{C}$ is said to belong to the *resolvent set* of H, $\lambda\in\rho(H)$, if $(H-\lambda)^{-1}$ is an everywhere defined operator. The resolvent set is an open subset of $\mathbb{C}$. For $\lambda\in\rho(H)$ the operator $(H-\lambda)^{-1}$ is called the *resolvent operator* of H (at λ).

2.2. **Selfadjoint relations.** A relation H is said to be *symmetric* if $(k,h)\in\mathbb{R}$ for all $\{h,k\}\in H$. By the polarization formula, H is symmetric precisely when $H\subset H^*$. A relation H is called *selfadjoint* if $H=H^*$; in particular, a selfadjoint relation is automatically closed. A selfadjoint relation H in $\mathfrak{H}$ induces the following orthogonal decompositions of the space:

$$\mathfrak{H}=\overline{\operatorname{dom}}\, H\oplus\operatorname{mul} H \quad\text{and}\quad \mathfrak{H}=\overline{\operatorname{ran}}\, H\oplus\ker H, \tag{2.2}$$

where $\overline{\operatorname{dom}}\, H$ and $\overline{\operatorname{ran}}\, H$ indicate the closures of $\operatorname{dom} H$ and $\operatorname{ran} H$, respectively. This shows that H admits the following orthogonal decomposition:

$$H = H_s \mathbin{\widehat{\oplus}} (\{0\} \times \operatorname{mul} H), \tag{2.3}$$

where $H_s = H \cap (\overline{\operatorname{dom}}\, H \times \overline{\operatorname{dom}}\, H)$, the so-called *orthogonal operator part* of H, is a selfadjoint operator in $\overline{\operatorname{dom}}\, H$ and $\{0\} \times \operatorname{mul} H$ is a selfadjoint relation in $\operatorname{mul} H$. The symbol $\widehat{\oplus}$ in (2.3) indicates the orthogonality of the summands. It follows from (2.3) that the finite spectra of H and of H_s coincide. Hence $\mathbb{C} \setminus \mathbb{R} \subset \rho(H)$ and the selfadjoint operator part H_s is bounded if and only if $\operatorname{dom} H$ is closed. Moreover, if $\operatorname{ran} H$ is closed, then there exists a reduced neighborhood of 0 in $\mathbb{R}$ which belongs to $\rho(H)$, and 0 is at most an isolated eigenvalue of H. Let $E_s(\cdot)$ be the spectral function of H_s in $\overline{\operatorname{dom}}\, H$. Define the spectral function $E(\cdot)$ for H in $\mathfrak{H}$ by $E(t) = E_s(t) \oplus 0_{\operatorname{mul} H}$, $t \in \mathbb{R}$, (cf. (2.3)) so that

$$(H - \lambda)^{-1} = \int_{\mathbb{R}} \frac{1}{s - \lambda}\, dE(s), \quad \lambda \in \rho(H). \tag{2.4}$$

For a measurable function $\varphi : \mathbb{R} \to \mathbb{C}$, define $\varphi(H) = \varphi(H_s) \mathbin{\widehat{\oplus}} (\{0\} \times \operatorname{mul} H)$.

A selfadjoint relation H in a Hilbert space $\mathfrak{H}$ is said to be *bounded from below* by $m \in \mathbb{R}$ if its operator part H_s is bounded from below by m:

$$(H_s h, h) \geq m(h, h) \quad \text{for all} \quad h \in \operatorname{dom} H = \operatorname{dom} H_s.$$

Any such number m is said to be *a lower bound*. The supremum of all lower bounds is called *the lower bound* of H. Any real number smaller than the lower bound belongs to $\rho(H)$. If the lower bound is nonnegative, then H is called *nonnegative*: $H \geq 0$. Note that if H has lower bound m, then $H - x$ has lower bound $m - x$ for any $x \in \mathbb{R}$. Therefore $H - x$ is nonnegative for all $x \leq m$. In particular, if $x < m$ then $(H - x)^{-1}$ is an everywhere defined positive bounded operator.

The *square root* $H^{1/2}$ of a nonnegative selfadjoint relation H is defined as

$$H^{1/2} = (H_s)^{1/2} \mathbin{\widehat{\oplus}} (\{0\} \times \operatorname{mul} H).$$

For a nonnegative selfadjoint relation H one has

$$\operatorname{dom} H \subset \operatorname{dom} H^{1/2}, \quad \overline{\operatorname{dom}}\, H = \overline{\operatorname{dom}}\, H^{1/2}, \quad \operatorname{mul} H = \operatorname{mul} H^{1/2}. \tag{2.5}$$

Clearly, if H_s is bounded, then $\operatorname{dom} H = \operatorname{dom} H^{1/2} = (\operatorname{mul} H)^{\perp}$.

2.3. **Ordering of selfadjoint relations.** Let H_1 and H_2 be selfadjoint relations in a Hilbert space $\mathfrak{H}$ with lower bounds m_1 and m_2, respectively. Then H_1 and H_2 are said to satisfy $H_1 \leq H_2$ if for a fixed $x < \min\{m_1, m_2\}$

$$0 \leq ((H_2 - x)^{-1} h, h) \leq ((H_1 - x)^{-1} h, h) \quad \text{for all} \quad h \in \mathfrak{H}, \tag{2.6}$$

see [5, 8, 11]. The next proposition gives a characterization for the ordering of selfadjoint relations, see [8, 11]. According to this proposition (2.6) holds automatically for all $x < \min\{m_1, m_2\}$ if it holds for some $x < \min\{m_1, m_2\}$.

Proposition 2.1. *Let H_1 and H_2 be selfadjoint relations in a Hilbert space $\mathfrak{H}$ with lower bounds m_1 and m_2, respectively. Then H_1 and H_2 satisfy $H_1 \leq H_2$ if and only if for any $x < \min\{m_1, m_2\}$*

$$\operatorname{dom} (H_2 - x)^{1/2} \subset \operatorname{dom} (H_1 - x)^{1/2} \tag{2.7}$$

and

$$\|(H_1 - x)_s^{1/2} h\| \leq \|(H_2 - x)_s^{1/2} h\| \ \text{for all}\ h \in \operatorname{dom} (H_2 - x)^{1/2}. \tag{2.8}$$

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If $\operatorname{dom} H_1$ and $\operatorname{dom} H_2$ are closed or, equivalently, if the operator parts $(H_1)_s$ and $(H_2)_s$ are bounded, then by Proposition 2.1 (cf. (2.5)) $H_1 \le H_2$ if and only if

$$\operatorname{dom} H_2 \subset \operatorname{dom} H_1 \quad \text{and} \quad ((H_1)_s h, h) \le ((H_2)_s h, h) \quad \text{for all} \quad h \in \operatorname{dom} H_2. \tag{2.9}$$

In particular, if $\operatorname{dom} H_1 = \operatorname{dom} H_2 = \mathfrak{H}$, i.e., if H_1 and H_2 are bounded selfadjoint operators, then the inequality $H_1 \le H_2$ has the usual meaning.

The inclusion (2.7) combined with (2.2) and (2.5) yields the following implication

$$H_1 \le H_2 \quad \Rightarrow \quad \overline{\operatorname{dom}}\, H_2 \subset \overline{\operatorname{dom}}\, H_1 \quad \text{and} \quad \operatorname{mul} H_1 \subset \operatorname{mul} H_2. \tag{2.10}$$

Corollary 2.2. *Let H_1 and H_2 be selfadjoint relations in a Hilbert space $\mathfrak{H}$ with closed domains such that $H_1 \le H_2$. Then $-H_2 \le -H_1$ if and only if $\operatorname{dom} H_1 = \operatorname{dom} H_2$ or, equivalently, $\operatorname{mul} H_1 = \operatorname{mul} H_2$.*

Proof. By assumption the operator parts $(H_1)_s$ and $(H_2)_s$ are bounded, which guarantees that each of the relations $\pm H_1$ and $\pm H_2$ is bounded from below. Now, the implication $(\Rightarrow)$ is obtained by applying (2.10) to the inequalities $H_1 \le H_2$ and $-H_2 \le -H_1$. The implication $(\Leftarrow)$ follows directly from (2.9). □

Let H_j be a selfadjoint relation in a Hilbert space $\mathfrak{H}$ with lower bound m_j and let $E_j(\cdot)$ be its spectral function for $j = 1, 2$. Then for $x < m_j$,

$$x\|h\|^2 + \|(H_j - x)_s^{1/2} h\|^2 = \int_{\mathbb{R}} s\, d(E_j(s)h, h), \quad h \in \operatorname{dom}(H_j - x)^{1/2}.$$

Hence, the selfadjoint relations H_1 and H_2 satisfy $H_1 \le H_2$ if and only if the inclusion (2.7) and the following inequality are satisfied for any $x < \min\{m_1, m_2\}$:

$$\int_{\mathbb{R}} s\, d(E_1(s)h, h) \le \int_{\mathbb{R}} s\, d(E_2(s)h, h) \quad \text{for all} \quad h \in \operatorname{dom}(H_2 - x)^{1/2}. \tag{2.11}$$

The next lemma will be useful in the proofs of Proposition 2.6 and 2.7 below.

Lemma 2.3. *Let H_1 and H_2 be selfadjoint relations in a Hilbert space $\mathfrak{H}$ which are bounded from below and satisfy $H_1 \le H_2$. Let $E_1(\cdot)$ and $E_2(\cdot)$ denote the corresponding spectral measures. Then the following statements hold:*

(i) $\operatorname{ran} E_2((-\infty, 0]) \cap \operatorname{ran}(I - E_1((-\infty, 0))) \subset \ker H_1 \cap \ker H_2$;
(ii) $\operatorname{ran} E_2((-\infty, 0]) \cap \operatorname{ran}(I - E_1((-\infty, 0])) = \{0\}$;
(iii) $\operatorname{ran} E_2((-\infty, 0)) \cap \operatorname{ran}(I - E_1((-\infty, 0))) = \{0\}$.

Proof. Note first that since H_2 is semibounded,

$$\operatorname{ran} E_2((-\infty, 0]) \subset \operatorname{dom} H_2 \subset \operatorname{dom}(H_2 - x)^{1/2}, \quad x < \min\{m_1, m_2\}.$$

Hence (2.11) holds for $h \in \operatorname{ran} E_2((-\infty, 0])$.

(i) For $h \in \operatorname{ran} E_2((-\infty, 0]) \cap \operatorname{ran}(I - E_1((-\infty, 0)))$ the righthand side of (2.11) is nonpositive and the lefthand side is nonnegative. Hence,

$$\int_{\mathbb{R}} s\, d(E_1(s)h, h) = \int_{\mathbb{R}} s\, d(E_2(s)h, h) = 0$$

and this implies (i).

(ii) Let $h \in \operatorname{ran} E_2((-\infty, 0]) \cap \operatorname{ran}(I - E_1((-\infty, 0]))$. If $h \ne 0$, the righthand side of (2.11) is nonpositive and the lefthand side is positive. Hence $h = 0$ and (ii) holds.

(iii) Let $h \in \operatorname{ran} E_2((-\infty, 0)) \cap \operatorname{ran}(I - E_1((-\infty, 0)))$. If $h \ne 0$, the righthand side of (2.11) is negative and the lefthand side is nonnegative. Hence $h = 0$ and (iii) holds. □

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The following result is included as a preparation for Section 3.

Lemma 2.4. *Let H be a selfadjoint relation in a Hilbert space $\mathfrak{H}$ and let (α,β) be a spectral gap of H. Then the following statements hold:*

(i) *the relations $(H-\alpha)^{-1}$ and $(H-\beta)^{-1}$ are selfadjoint with $-(H-\alpha)^{-1}$ and $(H-\beta)^{-1}$ being bounded from below. They are limits of $(H-t)^{-1}$ as $t\downarrow\alpha$ and $t\uparrow\beta$ respectively:*

$$(H-t)^{-1}\to(H-\alpha)^{-1},\quad (H-t)^{-1}\to(H-\beta)^{-1},$$

where the convergence is in the strong resolvent sense. Moreover, the inequalities

$$-(H-t)^{-1}\le-(H-\alpha)^{-1}\quad\text{and}\quad(H-t)^{-1}\le(H-\beta)^{-1}$$

hold for $\alpha<t<\beta$;

(ii) *if K_α and K_β are selfadjoint relations in $\mathfrak{H}$ with $-K_\alpha$ and K_β being bounded from below, such that $-(H-t)^{-1}\le-K_\alpha$ or $(H-t)^{-1}\le K_\beta$, $\alpha<t<\beta$, then the limits $(H-\alpha)^{-1}$ and $(H-\beta)^{-1}$ satisfy*

$$-(H-\alpha)^{-1}\le-K_\alpha\quad\text{or}\quad(H-\beta)^{-1}\le K_\beta.$$

Proof. The statements are proved for the right endpoint β; a similar reasoning applies to the left endpoint α. Note first that if $E(\cdot)$ is the spectral function of H, then (2.4) shows that for all $t_1,t_2\in(\alpha,\beta)$ with $t_1\le t_2$ and all $h\in\mathfrak{H}$,

$$((H-t_2)^{-1}h,h)-((H-t_1)^{-1}h,h)=\int_{\mathbb{R}\setminus(\alpha,\beta)}\frac{t_2-t_1}{(s-t_1)(s-t_2)}\,d(E(s)h,h).$$

The support of the measure $d(E(\cdot)h,h)$ is contained in $\mathbb{R}\setminus(\alpha,\beta)$ and there the integrand is nonnegative. Hence, the operator function $(H-t)^{-1}$ is nondecreasing in $t\in(\alpha,\beta)$.

(i) Fix some $c\in(\alpha,\beta)$ and let m_c be a lower bound for the bounded operator $(H-c)^{-1}$. As the function $t\mapsto(H-t)^{-1}$ is nondecreasing in (α,β) it follows that m_c is a lower bound for $(H-t)^{-1}$, $t\in(c,\beta)$. Hence by [5, Theorem 3.5] there exists a selfadjoint relation B in $\mathfrak{H}$, bounded from below by m_c, such that $(H-t)^{-1}\to B$ as $t\uparrow\beta$ in the strong resolvent sense, or, equivalently, in the graph sense; cf. [5, Proposition 2.3] and [17]. Moreover, $(H-t)^{-1}\le B$ holds for all $t\in(c,\beta)$.

Hence, to prove (i) it suffices to verify $B=(H-\beta)^{-1}$. For this let $\{\phi,\psi\}\in B$. Since B is the graph limit of $(H-t)^{-1}$ there exist $\{\phi_t,\psi_t\}\in(H-t)^{-1}$ with $\{\phi_t,\psi_t\}\to\{\phi,\psi\}$ as $t\uparrow\beta$. Since

$$\{\psi_t,\phi_t+(t-\beta)\psi_t\}\in H-\beta\quad\text{and}\quad\{\phi_t+(t-\beta)\psi_t,\psi_t\}\in(H-\beta)^{-1},$$

it follows that $\{\phi,\psi\}\in(H-\beta)^{-1}$, i.e., $B\subset(H-\beta)^{-1}$. Since both B and $(H-\beta)^{-1}$ are selfadjoint, the equality $B=(H-\beta)^{-1}$ follows.

(ii) Since $B=(H-\beta)^{-1}$ is bounded from below by m_c the relation $(H-\beta)^{-1}-m_c$ is nonnegative. Recall that $\operatorname{dom}((H-\beta)^{-1}-m_c)^{1/2}=\mathfrak{H}_0$, where

$$\mathfrak{H}_0=\{h\in\mathfrak{H}:\lim_{t\uparrow\beta}\|((H-t)^{-1}-m_c)^{1/2}h\|<\infty\};$$

cf. [5, Theorem 3.5]. Now let K_β be such that $(H-t)^{-1}\le K_\beta$, then by Proposition 2.1 for all $t\in(c,\beta)$

$$\operatorname{dom}(K_\beta-m_c)^{1/2}\subset\operatorname{dom}((H-t)^{-1}-m_c)^{1/2}$$

and

$$\|((H-t)^{-1}-m_c)^{1/2}h\| \leq \|(K_\beta - m_c)_s^{1/2}h\| \quad \text{for all} \quad h \in \mathrm{dom}\,(K_\beta - m_c)^{1/2}.$$

Since $(H-t)^{-1}$ is a nondecreasing operator function on (α, β), the preceding inequality implies that

$$\mathrm{dom}\,(K_\beta - m_c)^{1/2} \subset \mathfrak{H}_0 = \mathrm{dom}\,((H-\beta)^{-1}-m_c)^{1/2}.$$

Hence Proposition 2.1 yields $(H-\beta)^{-1} \leq K_\beta$. □

2.4. **Inertia of selfadjoint relations.** The notion of inertia of a selfadjoint relation in a Hilbert space is defined by means of its associated spectral measure. In what follows the Hilbert space is assumed to be separable.

Definition 2.5. Let H be a selfadjoint relation in a separable Hilbert space $\mathfrak{H}$ and let $E(\cdot)$ be the spectral measure of H. The inertia of H is defined as the ordered quadruplet $\mathsf{i}(H) = \{\mathsf{i}^+(H), \mathsf{i}^-(H), \mathsf{i}^0(H), \mathsf{i}^\infty(H)\}$, where

$$\mathsf{i}^+(H) = \dim \mathrm{ran}\, E((0,\infty)), \qquad \mathsf{i}^-(H) = \dim \mathrm{ran}\, E((-\infty,0)),$$
$$\mathsf{i}^0(H) = \dim \ker H, \qquad \mathsf{i}^\infty(H) = \dim \mathrm{mul}\, H.$$

In particular, for a selfadjoint relation H in $\mathbb{C}^n$, the quadruplet $\mathsf{i}(H)$ consists of the numbers of positive, negative, zero, and infinite eigenvalues of H; cf. [7]. Hence, if H is a selfadjoint matrix in $\mathbb{C}^n$, then $\mathsf{i}^\infty(H) = 0$ and the remaining numbers make up the usual inertia of H, see, e.g. [12, 14] or the introduction.

The inertia numbers of a selfadjoint relation H in a separable Hilbert space $\mathfrak{H}$ satisfy:

$$\mathsf{i}^+(H) + \mathsf{i}^-(H) + \mathsf{i}^0(H) + \mathsf{i}^\infty(H) = \dim \mathfrak{H}. \tag{2.12}$$

Furthermore, the following identities hold:

$$\begin{aligned} \mathsf{i}(H^{-1}) &= \{\mathsf{i}^+(H), \mathsf{i}^-(H), \mathsf{i}^\infty(H), \mathsf{i}^0(H)\}; \\ \mathsf{i}(-H^{-1}) &= \{\mathsf{i}^-(H), \mathsf{i}^+(H), \mathsf{i}^\infty(H), \mathsf{i}^0(H)\}. \end{aligned} \tag{2.13}$$

The next proposition shows that the ordering of two selfadjoint relations in a separable Hilbert space implies certain inequalities between their inertia numbers; cf. [7, Proposition 3.6] for a finite-dimensional variant of Proposition 2.6.

Proposition 2.6. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ which are bounded from below and satisfy $H_1 \leq H_2$. Then their inertia $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$, $j = 1, 2$, satisfy the following inequalities:*

(i) $\mathsf{i}_1^\infty \leq \mathsf{i}_2^\infty$ *and* $\mathsf{i}_1^- + \mathsf{i}_1^0 + \mathsf{i}_1^+ \geq \mathsf{i}_2^- + \mathsf{i}_2^0 + \mathsf{i}_2^+$;
(ii) $\mathsf{i}_1^- \geq \mathsf{i}_2^-$ *and* $\mathsf{i}_1^- + \mathsf{i}_1^0 \geq \mathsf{i}_2^- + \mathsf{i}_2^0$;
(iii) $\mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^+ + \mathsf{i}_2^\infty$ *and* $\mathsf{i}_1^0 + \mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^0 + \mathsf{i}_2^+ + \mathsf{i}_2^\infty$.

Proof. (i) This is a direct consequence of the implication in (2.10).

(ii) If $\mathsf{i}_1^- = \infty$, then automatically $\mathsf{i}_2^- \leq \mathsf{i}_1^-$. Hence, in order to show $\mathsf{i}_2^- \leq \mathsf{i}_1^-$, assume that $\mathsf{i}_1^- < \infty$ and let $\mathfrak{L}$ be a finite-dimensional subspace in $\mathrm{ran}\, E_2((-\infty,0))$. Since $E_1((-\infty,0))$ restricted to $\mathfrak{L}$ is injective by Lemma 2.3 (iii), one has

$$\dim \mathfrak{L} = \dim E_1((-\infty,0))\mathfrak{L} \leq \dim \mathrm{ran}\, E_1((-\infty,0)) = \mathsf{i}_1^-.$$

Thus any finite-dimensional subspace of $\mathrm{ran}\, E_2((-\infty,0))$ has dimension at most i_1^-, which implies that the space $\mathrm{ran}\, E_2((-\infty,0))$ itself has dimension at most i_1^-, i.e. $\mathsf{i}_2^- \leq \mathsf{i}_1^-$.

The inequality $\mathsf{i}_1^- + \mathsf{i}_1^0 \geq \mathsf{i}_2^- + \mathsf{i}_2^0$ can be shown in a similar way, when (ii) in Lemma 2.3 is used instead of (iii).

(iii) By Lemma 2.3 (ii) the identity

$$\operatorname{ran} E_2((-\infty,0]) \cap (\operatorname{ran} E_1((0,\infty)) \oplus \operatorname{mul} H_1) = \{0\} \tag{2.14}$$

holds. If $\mathsf{i}_2^+ + \mathsf{i}_2^\infty = \infty$, then automatically $\mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^+ + \mathsf{i}_2^\infty$. Hence, in order to show $\mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^+ + \mathsf{i}_2^\infty$, assume that $\mathsf{i}_2^+ + \mathsf{i}_2^\infty < \infty$ and let $\mathfrak{L}$ be a finite-dimensional subspace in $\operatorname{ran} E_1((0,\infty)) \oplus \operatorname{mul} H_1$. Since $I - E_2((-\infty,0])$ restricted to $\mathfrak{L}$ is injective by (2.14), one has

$$\dim \mathfrak{L} = \dim \big(I - E_2((-\infty,0])\big)\mathfrak{L} \leq \dim \operatorname{ran} \big(I - E_2((-\infty,0])\big) = \mathsf{i}_2^+ + \mathsf{i}_2^\infty.$$

Thus any finite-dimensional subspace of $\operatorname{ran} E_1((0,\infty)) \oplus \operatorname{mul} H_1$ has dimension at most $\mathsf{i}_2^+ + \mathsf{i}_2^\infty$, which implies that the space $\operatorname{ran} E_1((0,\infty)) \oplus \operatorname{mul} H_1$ itself has dimension at most $\mathsf{i}_2^+ + \mathsf{i}_2^\infty$, i.e., $\mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^+ + \mathsf{i}_2^\infty$.

The inequality $\mathsf{i}_1^0 + \mathsf{i}_1^+ + \mathsf{i}_1^\infty \leq \mathsf{i}_2^- + \mathsf{i}_2^+ + \mathsf{i}_2^\infty$ can be shown in a similar way, when (iii) in Lemma 2.3 is used instead of (ii). ☐

The case of equality in an inertia inequality of Proposition 2.6 has a specific geometric implication.

Proposition 2.7. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ which are bounded from below. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1,2$, and assume that $H_1 \leq H_2$. Then the following statements hold:*

(i) *if $\mathsf{i}_1^\infty = \mathsf{i}_2^\infty < \infty$, then $\operatorname{mul} H_1 = \operatorname{mul} H_2$;*
(ii) *if $\mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0 < \infty$, then $\ker H_1 \subset \ker H_2$;*
(iii) *if $\mathsf{i}_1^- = \mathsf{i}_2^- < \infty$, then $\ker H_2 \subset \ker H_1$.*

In particular, if $\mathsf{i}_1^- = \mathsf{i}_2^- < \infty$ and $\mathsf{i}_1^0 = \mathsf{i}_2^0 < \infty$, then $\ker H_1 = \ker H_2$.

Proof. (i) This is a direct consequence of (2.10).

(ii) & (iii) Define the subspace $\mathfrak{L}_0 = \operatorname{ran} E_2((-\infty,0]) \cap \operatorname{ran}(I - E_1((-\infty,0)))$. According to Lemma 2.3 (i) $\mathfrak{L}_0 \subset \ker H_1 \cap \ker H_2$. Furthermore, note that $\mathfrak{L}_0$ can be rewritten as

$$\mathfrak{L}_0 = \operatorname{ran} E_2((-\infty,0]) \cap \big(\operatorname{ran} E_1((-\infty,0))\big)^\perp.$$

Since $\dim \operatorname{ran} E_2((-\infty,0]) = \mathsf{i}_2^- + \mathsf{i}_2^0$ and $\dim \operatorname{ran} E_1((-\infty,0)) = \mathsf{i}_1^- < \infty$,

$$\dim \mathfrak{L}_0 \geq \mathsf{i}_2^- + \mathsf{i}_2^0 - \mathsf{i}_1^-. \tag{2.15}$$

In case (ii), the assumption together with (2.15) implies that $\dim \mathfrak{L}_0 \geq \mathsf{i}_1^0 = \dim \ker H_1$. Combining this observation with the inclusion $\mathfrak{L}_0 \subset \ker H_1 \cap \ker H_2 \subset \ker H_1$ yields that $\ker H_1 \cap \ker H_2 = \ker H_1$ and, hence, that $\ker H_1 \subset \ker H_2$.

In case (iii), the assumption together with (2.15) implies that $\dim \mathfrak{L}_0 \geq \mathsf{i}_2^0 = \dim \ker H_2$. Combining this observation with the inclusion $\mathfrak{L}_0 \subset \ker H_1 \cap \ker H_2 \subset \ker H_2$ yields that $\ker H_1 \cap \ker H_2 = \ker H_2$ and, hence, that $\ker H_2 \subset \ker H_1$. ☐

3. Antitonicity for selfadjoint relations

The infinite-dimensional variants of the antitonicity theorems from the introduction are here proved by means of perturbation arguments, the spectral mapping result (2.1), and limit properties of monotone operator functions. Furthermore, various consequences and special cases of these results are also discussed.

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3.1. **An antitonicity theorem for bounded and boundedly invertible operators.** The following theorem is the infinite-dimensional variant of Theorem 1.1 from the introduction; it was proved independently in [18, 9]; cf. [10]. A simple proof is included here; it relies on the main arguments used in [9, 10].

Theorem 3.1. *Let H_1 and H_2 be bounded and boundedly invertible selfadjoint operators in a separable Hilbert space $\mathfrak{H}$. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\min\{\mathsf{i}_2^+, \mathsf{i}_1^-\} < \infty$ and that $H_1 \leq H_2$. Then*

$$H_2^{-1} \leq H_1^{-1} \quad \textit{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

Proof. Observe that for the bounded and boundedly invertible selfadjoint operators H_1 and H_2 one has $\mathsf{i}_j^0 = 0 = \mathsf{i}_j^\infty$, $j = 1, 2$. Hence $\mathsf{i}(H_1) = \mathsf{i}(H_2)$ is equivalent to $\mathsf{i}_1^- = \mathsf{i}_2^-$ and $\mathsf{i}_1^+ = \mathsf{i}_2^+$. Furthermore, observe that $\mathsf{i}_1^- < \infty$ implies that $\mathsf{i}_2^- < \infty$ and that $\mathsf{i}_2^+ < \infty$ implies that $\mathsf{i}_1^+ < \infty$; cf. Proposition 2.6.

($\Rightarrow$) In view of (2.13) the equalities $\mathsf{i}_1^- = \mathsf{i}_2^-$ and $\mathsf{i}_1^+ = \mathsf{i}_2^+$ follow by applying Proposition 2.6 to $H_1 \leq H_2$ and $H_2^{-1} \leq H_1^{-1}$.

($\Leftarrow$) Assume that $\mathsf{i}(H_1) = \mathsf{i}(H_2)$, so that $\mathsf{i}_1^- = \mathsf{i}_2^-$ and $\mathsf{i}_1^+ = \mathsf{i}_2^+$. The asserted implication will be shown in two steps.

First consider the case that $\mathsf{i}_1^- < \infty$. Then $\mathsf{i}_2^- = \mathsf{i}_1^- < \infty$. Now define the operator J as $I_{\mathsf{i}_1^+} \oplus -I_{\mathsf{i}_1^-}$. Then a result of G. Köthe, cf. [13, Satz 1.2], shows the existence of bounded and boundedly invertible operators V_1 and V_2 such that

$$H_1 = V_1^* J V_1 \quad \text{and} \quad H_2 = V_2^* J V_2.$$

By means of the above notation the inequality $H_1 \leq H_2$ can be written as

$$0 \leq J - U^* J U, \qquad U = V_1 V_2^{-1}. \tag{3.1}$$

A simple calculation shows that

$$\begin{pmatrix} I & 0 \\ JU^* & I \end{pmatrix}^* \begin{pmatrix} J - UJU^* & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} I & 0 \\ JU^* & I \end{pmatrix} = \begin{pmatrix} I & JU \\ 0 & I \end{pmatrix}^* \begin{pmatrix} J & 0 \\ 0 & J - U^*JU \end{pmatrix} \begin{pmatrix} I & JU \\ 0 & I \end{pmatrix}.$$

Since congruence does not change the inertia of bounded operators, the inertia of the diagonal matrices in the above equation coincide, i.e.,

$$\mathsf{i}^-(J - UJU^*) + \mathsf{i}^-(J) = \mathsf{i}^-(J) + \mathsf{i}^-(J - U^*JU).$$

As $\mathsf{i}^-(J) = \mathsf{i}_1^- < \infty$ and $\mathsf{i}^-(J - U^*JU) = 0$ by (3.1) it follows that $\mathsf{i}^-(J - UJU^*) = 0$ and hence $J - UJU^*$ is a nonnegative operator. Using the definition of U, this yields

$$H_2^{-1} = (V_2^* J V_2)^{-1} = V_2^{-1} J V_2^{-*} \leq V_1^{-1} J V_1^{-*} = (V_1^* J V_1)^{-1} = H_1^{-1},$$

which completes the proof in the case $\mathsf{i}_1^- < \infty$.

Next consider the case $\mathsf{i}_2^+ < \infty$. Then it follows that $\mathsf{i}_1^+ = \mathsf{i}_2^+ < \infty$. By (2.13) this implies that $\mathsf{i}^-(-H_1) = \mathsf{i}^-(-H_2) < \infty$. Since $H_1 \leq H_2$ is equivalent to $-H_2 \leq -H_1$, the previous step shows that $-H_1^{-1} \leq -H_2^{-1}$, which is equivalent to $H_2^{-1} \leq H_1^{-1}$; see Corollary 2.2. This completes the proof of Theorem 3.1. □

3.2. **First main antitonicity theorem.** The following theorem is the infinite-dimensional version of Theorem 1.3 from the introduction. Recall that for selfadjoint relations H_1 and H_2 with closed ranges the operator parts of H_1^{-1} and H_2^{-1} are bounded; in particular, the relations $-H_1^{-1}$ and $-H_2^{-1}$ are bounded from below.

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Theorem 3.2. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ which are bounded from below and have closed ranges. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- + \mathsf{i}_1^0 < \infty$ and that $H_1 \le H_2$. Then*

$$-H_1^{-1} \le -H_2^{-1} \quad \textit{if and only if} \quad \mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0.$$

Proof. ($\Rightarrow$) Apply Proposition 2.6 and (2.13) to the inequalities $H_1 \le H_2$ and $-H_1^{-1} \le -H_2^{-1}$. Then the inertia equality $\mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0$ follows.

($\Leftarrow$) Let $H_1 \le H_2$ and assume that $\mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0 < \infty$ holds. Since the ranges of H_1 and H_2 are closed, there exists a constant $\delta > 0$, such that $(-\delta, \delta)\backslash\{0\} \subset \rho(H_j)$; i.e. H_j has a spectral gap around 0 and the point 0 is possibly an isolated eigenvalue of finite multiplicity, $j = 1, 2$. Define $\mu^+ := \min\{1, \delta\}$, then Proposition 2.1 implies that the inequality

$$H_1 - \epsilon_1 \le H_2 - \epsilon_2, \qquad 0 < \epsilon_2 \le \epsilon_1 < \mu^+, \tag{3.2}$$

holds. Clearly, $H_j(\epsilon_j) := H_j - \epsilon_j$ is boundedly invertible and its inertia is

$$\mathsf{i}\big(H_j(\epsilon_j)\big) = \big\{\mathsf{i}_j^+(\epsilon_j), \mathsf{i}_j^-(\epsilon_j), \mathsf{i}_j^0(\epsilon_j), \mathsf{i}_j^\infty(\epsilon_j)\big\} = \big\{\mathsf{i}_j^+, \mathsf{i}_j^- + \mathsf{i}_j^0, 0, \mathsf{i}_j^\infty\big\}, \quad j = 1, 2. \tag{3.3}$$

Let m_j be a lower bound for H_j, $j = 1, 2$. Then $m_j - 1 < m_j - \epsilon_j$ is a lower bound for $H_j(\epsilon_j)$, $j = 1, 2$. Hence $H_1(\epsilon_1) \le H_2(\epsilon_2)$ in (3.2) implies that

$$0 \le (H_2(\epsilon_2) - x)^{-1} \le (H_1(\epsilon_1) - x)^{-1}, \quad x < \min\{0, m_1 - 1, m_2 - 1\};$$

cf. (2.6). Using (2.1), this yields the inequality

$$\big(-H_2(\epsilon_2)^{-1} + 1/x\big)^{-1} \le \big(-H_1(\epsilon_1)^{-1} + 1/x\big)^{-1}. \tag{3.4}$$

By (2.13) and (3.3) the inertia numbers of $-H_j(\epsilon_j)^{-1}$, $j = 1, 2$, are given by

$$\mathsf{i}\big(-H_j(\epsilon_j)^{-1}\big) = \big\{\mathsf{i}_j^-(\epsilon_j), \mathsf{i}_j^+(\epsilon_j), \mathsf{i}_j^\infty(\epsilon_j), 0\big\}, \quad j = 1, 2. \tag{3.5}$$

Since $(0, -1/(m_j - 1)) \subset \rho(-H_j(\epsilon_j)^{-1})$ the operator $-H_j(\epsilon_j)^{-1} + 1/x$ is bounded and boundedly invertible for all $x < \min\{0, m_1 - 1, m_2 - 1\}$, $j = 1, 2$. Hence (3.5) and (3.3) imply that for $j = 1, 2$:

$$\mathsf{i}\big(-H_j(\epsilon_j)^{-1} + 1/x\big) = \big\{\mathsf{i}_j^-(\epsilon_j), \mathsf{i}_j^+(\epsilon_j) + \mathsf{i}_j^\infty(\epsilon_j), 0, 0\big\} = \big\{\mathsf{i}_j^- + \mathsf{i}_j^0, \mathsf{i}_j^+ + \mathsf{i}_j^\infty, 0, 0\big\}.$$

Since by assumption $\mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0 < \infty$, Theorem 3.1 applied to (3.4) yields

$$-H_1(\epsilon_1)^{-1} + 1/x \le -H_2(\epsilon_2)^{-1} + 1/x, \quad 0 < \epsilon_2 \le \epsilon_1 < \mu^+$$

or, equivalently,

$$-(H_1 - \epsilon_1)^{-1} \le -(H_2 - \epsilon_2)^{-1}, \quad 0 < \epsilon_2 \le \epsilon_1 < \mu^+. \tag{3.6}$$

Now letting subsequently $\epsilon_2 \downarrow 0$ and $\epsilon_1 \downarrow 0$ in (3.6) in the strong resolvent sense and using Lemma 2.4 in each step, the inequality $-H_1^{-1} \le -H_2^{-1}$ is obtained. □

It is emphasized that the equivalence in Theorem 3.2 is not true without the minus signs; see Corollary 2.2.

Corollary 3.3. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ with closed domains and closed ranges. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- + \mathsf{i}_1^0 < \infty$, $\mathsf{i}_2^\infty < \infty$, and that $H_1 \le H_2$. Then the following statements are equivalent:*

(i) $\mathsf{i}(H_1) = \mathsf{i}(H_2)$;

(ii) (a) $-H_1^{-1} \le -H_2^{-1}$;

(b) $\operatorname{mul} H_1 = \operatorname{mul} H_2$;
(c) $\ker H_1 = \ker H_2$;
(iii) $-H_2 \le -H_1$, $-H_1^{-1} \le -H_2^{-1}$, *and* $H_2^{-1} \le H_1^{-1}$.

Proof. (i) ⇒ (ii) This follows from Theorem 3.2 and Proposition 2.7.

(ii) ⇒ (iii) Apply Corollary 2.2 to the inequalities $H_1 \le H_2$ and $-H_1^{-1} \le -H_2^{-1}$. Then the desired inequalities follow.

(iii) ⇒ (i) If the stated inequalities hold, then by Corollary 2.2 $\operatorname{mul} H_1 = \operatorname{mul} H_2$ and $\ker H_1 = \ker H_2$, i.e. $\mathsf{i}_1^\infty = \mathsf{i}_2^\infty$ and $\mathsf{i}_1^0 = \mathsf{i}_2^0$. Furthermore, the inequality $-H_1^{-1} \le -H_2^{-1}$ implies that $\mathsf{i}_1^- + \mathsf{i}_1^0 = \mathsf{i}_2^- + \mathsf{i}_2^0$. Since i_j^-, i_j^0, and i_j^∞ are finite for $j = 1, 2$, (2.12) shows that (i) holds. □

3.3. **Second main antitonicity theorem.** The following theorem is the infinite-dimensional version of Theorem 1.2 from the introduction. It is emphasized that in contrast to Theorem 3.2 there is no closed range assumption on the relations. However, the conditions $H_1 \le H_2$ and $\mathsf{i}_1^- < \infty$ imply $\mathsf{i}_2^- < \infty$; hence H_1^{-1} and H_2^{-1} are both semibounded from below.

Theorem 3.4. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ which are bounded from below. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- < \infty$ and that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1} \quad \text{if and only if} \quad \mathsf{i}_1^- = \mathsf{i}_2^- .$$

Proof. (⇒) Apply Proposition 2.6 and (2.13) to $H_1 \le H_2$ and $H_2^{-1} \le H_1^{-1}$. Then the inertia equality $\mathsf{i}_1^- = \mathsf{i}_2^-$ follows.

(⇐) Let $H_1 \le H_2$ and assume that $\mathsf{i}_1^- = \mathsf{i}_2^- < \infty$ holds. Then the negative spectrum of H_j consists of $0 \le \mathsf{i}_j^- < \infty$ eigenvalues (counting multiplicities), $j = 1, 2$. Let μ_j^- be the largest negative eigenvalue of H_j if $0 < \mathsf{i}_j^-$ and define

$$\mu^- := \begin{cases} \min\{1, -\mu_1^-, -\mu_2^-\}, & \mathsf{i}_1^- = \mathsf{i}_2^- > 0, \\ 1, & \mathsf{i}_1^- = \mathsf{i}_2^- = 0. \end{cases}$$

Then

$$H_1 + \epsilon_1 \le H_2 + \epsilon_2, \quad 0 < \epsilon_1 \le \epsilon_2 < \mu^-, \tag{3.7}$$

where $H_j + \epsilon_j$ is boundedly invertible and $\mathsf{i}(H_j + \epsilon_j) = \{\mathsf{i}_j^+ + \mathsf{i}_j^0, \mathsf{i}_j^-, 0, \mathsf{i}_j^\infty\}$, $j = 1, 2$. Since by assumption $\mathsf{i}_1^- = \mathsf{i}_2^- < \infty$, Theorem 3.2 can be applied to (3.7) yielding

$$-(H_1 + \epsilon_1)^{-1} \le -(H_2 + \epsilon_2)^{-1}, \quad 0 < \epsilon_1 \le \epsilon_2 < \mu^-.$$

Because $(H_j + \epsilon_j)^{-1}$, $j = 1, 2$, is a bounded operator, this inequality can be rewritten as

$$(H_2 + \epsilon_2)^{-1} \le (H_1 + \epsilon_1)^{-1}, \quad 0 < \epsilon_1 \le \epsilon_2 < \mu^-. \tag{3.8}$$

Now letting subsequently $\epsilon_1 \downarrow 0$ and $\epsilon_2 \downarrow 0$ in (3.8) in the strong resolvent sense and using Lemma 2.4 in each step (which is possible since $(-\mu^-, 0) \subset \rho(H_j)$, $j = 1, 2$), the inequality $H_2^{-1} \le H_1^{-1}$ is obtained. □

Theorem 3.4 with $\mathsf{i}_1^- = 0$ implies the following well-known result for nonnegative selfadjoint operators and relations; cf. [2, 8].

Corollary 3.5. *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$. Then*

$$0 \le H_1 \le H_2 \quad \text{if and only if} \quad 0 \le H_2^{-1} \le H_1^{-1}.$$

The following corollary for (not necessarily bounded) selfadjoint operators extends Theorem 3.1; cf. [18, Theorems 1, 2], [9, Theorem 2], and [10, Theorem 1.4].

Corollary 3.6. *Let H_1 and H_2 be injective selfadjoint operators in a separable Hilbert space $\mathfrak{H}$ and let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$. Then the following statements hold:*

(i) *if H_1 and H_2 are bounded from below, $\mathsf{i}_1^- < \infty$, and $H_1 \le H_2$, then*

$$H_2^{-1} \le H_1^{-1} \quad \text{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2);$$

(ii) *if $-H_1$ and $-H_2$ are bounded from below, $\mathsf{i}_2^+ < \infty$, and $-H_2 \le -H_1$, then*

$$-H_1^{-1} \le -H_2^{-1} \quad \text{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

Combining Theorem 3.2, Theorem 3.4, and Proposition 2.7 yields the following result.

Corollary 3.7. *Let H_1 and H_2 be selfadjoint operators in a separable Hilbert space $\mathfrak{H}$ which are bounded from below and have closed ranges. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- + \mathsf{i}_1^0 < \infty$ and that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1}, \quad -H_1^{-1} \le -H_2^{-1} \quad \text{if and only if} \quad \mathsf{i}_1^- = \mathsf{i}_2^-, \quad \mathsf{i}_1^0 = \mathsf{i}_2^0,$$

in which case $\ker H_1 = \ker H_2$.

3.4. **An antitonicity theorem for Moore-Penrose inverses.** The Moore-Penrose inverse H^+ of a selfadjoint operator H in a Hilbert space is defined as

$$H^+ := PH^{-1}P,$$

where H^{-1} is the inverse of H (in the sense of relations) and P denotes the orthogonal projection onto $\overline{\operatorname{ran}}\, H$ in $\mathfrak{H}$. It follows that

$$H^+ = (H^{-1})_s \,\widehat{\oplus}\, (\ker H \times \{0\}) \tag{3.9}$$

holds. Note that the assumption $\operatorname{mul} H = \{0\}$ implies $\ker\, (H^{-1})_s = \ker\, H^{-1} = \{0\}$ and hence $\ker\, H^+ = \ker\, H$ and $\mathsf{i}(H^+) = \mathsf{i}(H)$ hold.

The following theorem is the infinite-dimensional version of Corollary 1.4 from the introduction.

Theorem 3.8. *Let H_1 and H_2 be selfadjoint operators in a separable Hilbert space $\mathfrak{H}$ which are bounded from below. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- + \mathsf{i}_1^0 < \infty$ and that $H_1 \le H_2$. Then*

$$H_2^+ \le H_1^+ \quad \text{if and only if} \quad \ker\, H_1 = \ker\, H_2 \text{ and } \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

Proof. ($\Rightarrow$) Since $\ker\, H_j^+ = \ker\, H_j$, $j = 1, 2$, it follows from Proposition 2.6 and (2.13) that $\mathsf{i}_1^- = \mathsf{i}_2^- < \infty$ and $\mathsf{i}_1^0 = \mathsf{i}_2^0 < \infty$ hold. Since $\mathsf{i}_1^\infty = \mathsf{i}_2^\infty = 0$ by assumption, (2.12) implies that $\mathsf{i}_1^+ = \mathsf{i}_2^+$ and, therefore, $\mathsf{i}(H_1) = \mathsf{i}(H_2)$. The assertion $\ker\, H_1 = \ker\, H_2$ follows from Proposition 2.7.

($\Leftarrow$) The assumption $\mathsf{i}(H_1) = \mathsf{i}(H_2)$ together with Theorem 3.4 implies the inequalities $H_2^{-1} \le H_1^{-1}$ and $(H_2^{-1})_s \le (H_1^{-1})_s$. Therefore, as $\ker\, H_1 = \ker\, H_2$ it follows from (3.9) and Proposition 2.1 that $H_2^+ \le H_1^+$ holds. $\square$

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