Boundary relations, unitary colligations, and functional models

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Dedicated to Ilppo Simo Louhivaara on the occasion of his eightieth birthday

Abstract. Recently a new notion, the so-called *boundary relation*, has been introduced involving an analytic object, the so-called *Weyl family*. Weyl families and boundary relations establish a link between the class of *Nevanlinna families* and unitary relations acting from one Kreĭn space, a basic (state) space, to another Kreĭn space, a parameter space where the Nevanlinna family or Weyl family is acting. Nevanlinna families are a natural generalization of the class of operator-valued Nevanlinna functions and they are closely connected with the class of operator-valued Schur functions. This paper establishes the connection between boundary relations and their Weyl families on the one hand, and *unitary colligations* and their *transfer functions* on the other hand. From this connection there are various advances which will benefit the investigations on both sides, including operator theoretic as well as analytic aspects. As one of the main consequences a *functional model* for Nevanlinna families is obtained from a variant of the functional model of L. de Branges and J. Rovnyak for Schur functions. Here the model space is a reproducing kernel Hilbert space in which multiplication by the independent variable defines a closed simple symmetric operator. This operator gives rise to a boundary relation such that the given Nevanlinna family is realized as the corresponding Weyl family.

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1. Introduction

In this paper the interplay between the recent concepts of boundary relations and their Weyl families and the somewhat more classical notions of unitary colligations and their transfer functions is investigated. The concepts of boundary relations and corresponding Weyl families were introduced in [22], as generalizations of the notions of (ordinary and generalized) boundary triplets and their Weyl functions introduced and comprehensively studied by V.A. Derkach and M.M. Malamud in [25, 26]; see also [18, 31, 28, 40, 41]. Boundary triplets and Weyl functions play an important role in the theory of symmetric operators and the spectral analysis of their selfadjoint extensions; applications range from spectral theory of ordinary and partial differential operators to boundary value problems, perturbation theory, scattering theory and moment problems, see, e.g., [8, 9, 18, 19, 24, 25, 26, 28, 29].

The present paper is written in the language of linear relations, i.e., multivalued linear operators, since it is more general and often much more convenient to work with linear relations instead of linear operators. These objects are natural for many reasons in the present context, e.g., the Cayley transform of a unitary colligation is in general not a selfadjoint operator but a selfadjoint relation, the transfer function induces a relation-valued function, and the symmetric operators which are extended are often not densely defined, so that the selfadjoint extensions and the adjoint need not be operators.

For the convenience of the reader the definitions of boundary relations and Weyl families from [22, 23] are briefly recalled.

Definition 1.1. *Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be an auxiliary Hilbert space. A linear relation Γ from $\mathfrak{H}^2=\mathfrak{H}\times\mathfrak{H}$ to $\mathcal{H}^2=\mathcal{H}\times\mathcal{H}$ is said to be a* boundary relation *for S^* if:*

(i) $\mathcal{T}=\operatorname{dom}\Gamma$ *is dense in S^* and the identity*

$$(f',g)_{\mathfrak{H}}-(f,g')_{\mathfrak{H}}=(h',k)_{\mathcal{H}}-(h,k')_{\mathcal{H}}, \tag{1.1}$$

holds for every $\{\widehat f,\widehat h\}$, $\{\widehat g,\widehat k\}\in\Gamma$, where $\widehat f=\{f,f'\}$, $\widehat g=\{g,g'\}\in\mathfrak{H}^2$ and $\widehat h=\{h,h'\}$, $\widehat k=\{k,k'\}\in\mathcal{H}^2$;

(ii) *if $\{\widehat g,\widehat k\}\in\mathfrak{H}^2\times\mathcal{H}^2$ satisfies* (1.1) *for every $\{\widehat f,\widehat h\}\in\Gamma$, then $\{\widehat g,\widehat k\}\in\Gamma$.*

The Weyl family associated to a boundary relation Γ is the abstract analogue of the classical Titchmarsh-Weyl coefficient or m-function in Sturm-Liouville theory. Roughly speaking the values of the Weyl family are the images of the defect spaces of $\mathcal{T}$ under Γ.

Definition 1.2. *The* Weyl family $M(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, *of S corresponding to the boundary relation $\Gamma:\mathfrak{H}^2\to\mathcal{H}^2$ is defined by*

$$M(\lambda)=\bigl\{\widehat h\in\mathcal{H}^2\,:\,\{\widehat f_\lambda,\widehat h\}\in\Gamma \text{ for some } \widehat f_\lambda=\{f_\lambda,\lambda f_\lambda\}\in\mathfrak{H}^2\bigr\}. \tag{1.2}$$

If the values $M(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, are operators in $\mathcal{H}$, then one speaks of a Weyl function instead of a Weyl family. It follows directly from the definition that a Weyl family is a *Nevanlinna family*, that is, a holomorphic relation-valued function symmetric with respect to the real line and whose values on $\mathbb{C}_+$ ($\mathbb{C}_-$) are maximal dissipative (maximal accumulative, respectively) linear relations, cf., Definition 3.1. Conversely it was shown in [22] with the help of the Naimark dilation theorem that each Nevanlinna family can be realized as a Weyl family of some boundary relation in an abstract model space.

The present paper offers a new approach to boundary relations and their Weyl families. The underlying idea is that the Weyl family $M(\lambda)$ and a certain selfadjoint relation $\widetilde{A}$ in $\mathfrak{H}\times\mathcal{H}$ induced by the boundary relation $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ are connected via the Cayley transform with an operator-valued Schur function Θ in $\mathcal{H}$ and a unitary colligation U in $\mathfrak{H}\times\mathcal{H}$ with Θ as the corresponding transfer function. Several operator and function theoretic facts concerning this connection between Nevanlinna families and operator-valued Schur functions are established and expressed via explicit formulas. In particular, it is shown how these basic observations provide simple characterizations of various subclasses of the Nevanlinna families, e.g., operator-valued Nevanlinna functions with boundedly invertible imaginary part, and special types of boundary relations, e.g., ordinary boundary triplets. Furthermore, the realization of Schur functions in terms of unitary colligations gives rise to a simple alternative approach to the realization of Nevanlinna families as Weyl families of boundary relations. In particular, it is shown how a variant of the functional model of L. de Branges and J. Rovnyak (see [14, 15]) for Schur functions leads to a functional model for Nevanlinna families. The model space is a reproducing kernel Hilbert space in which multiplication by the independent variable is the closed simple symmetric operator which gives rise to the boundary relation. The operator of multiplication by the independent variable in the context of reproducing kernel Hilbert spaces of scalar entire functions was already considered by L. de Branges in [10, 11, 12, 13] for meromorphic scalar Nevanlinna functions.

The outline of the paper is as follows. In Section 2 some useful facts on linear operators and relations in Hilbert spaces are recalled. Section 3 deals with the notions of Schur functions, Nevanlinna families and Nevanlinna pairs. The connections between these objects are investigated and the corresponding reproducing kernel Hilbert spaces are introduced. The realization of a Schur function as the transfer function of a unitary colligation is explained in Section 4; the original de Branges-Rovnyak model in terms of a matrix kernel and vector functions on $\mathbb{D}$ is translated into a model with functions defined on $\mathbb{D}\cup\mathbb{D}^*$; cf., [3]. In Section 5 the connection between the Weyl family of a boundary relation and the transfer function of an associated unitary colligation is investigated by means of the Cayley transform. The realization of a given Nevanlinna family as the Weyl family of a boundary relation in a reproducing kernel Hilbert space can be found in Section 6. Moreover, it is shown that for the special classes of strict and uniformly strict Nevanlinna functions whose values are bounded linear operators the results in the present paper reduce to the ones obtained in [26, 33, 37, 38].

Recently the realization in terms of a functional model was also announced by V.A. Derkach in the International Conference on Modern Analysis and Applications in Odessa, Ukraine (2007), see [20], where different methods have been used. Furthermore it should be mentioned that several operator representations of so-called generalized Nevanlinna families and functions (generalized in the sense that they possess a finite number of negative squares) exist in the literature. A treatment along the lines of the present paper requires the introduction of boundary relations in the context of indefinite spaces, cf., [7]. This topic will be treated elsewhere.

2. Preliminaries

2.1. Linear relations in Hilbert spaces

Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces whose scalar products are denoted by $(\cdot,\cdot)$. A (closed) linear relation T from $\mathfrak{H}$ to $\mathfrak{K}$ is a (closed) linear subspace of the Cartesian product space $\mathfrak{H}\times\mathfrak{K}$. The elements of T are pairs denoted by $\widehat{f}=\{f,f'\}\in T$, $f\in\mathfrak{H}$, $f'\in\mathfrak{K}$. For a linear relation T from $\mathfrak{H}$ to $\mathfrak{K}$ the domain, kernel, range, and the multi-valued part are defined as

$$\begin{aligned}\operatorname{dom} T&=\{\,f\in\mathfrak{H}:\{f,f'\}\in T\,\}, &\quad \ker T&=\{\,f\in\mathfrak{H}:\{f,0\}\in T\,\},\\ \operatorname{ran} T&=\{\,f'\in\mathfrak{K}:\{f,f'\}\in T\,\}, &\quad \operatorname{mul} T&=\{\,f'\in\mathfrak{K}:\{0,f'\}\in T\,\},\end{aligned}$$

respectively. The inverse of T is a relation from $\mathfrak{K}$ to $\mathfrak{H}$ defined by

$$T^{-1}=\big\{\,\{f',f\}:\{f,f'\}\in T\,\big\}.$$

The sum T_1+T_2 and component-wise sum $T_1\mathbin{\widehat{+}}T_2$ of two linear relations T_1 and T_2 from $\mathfrak{H}$ to $\mathfrak{K}$ are defined by

$$\begin{aligned}T_1+T_2&=\big\{\,\{f,g+k\}:\{f,g\}\in T_1,\ \{f,k\}\in T_2\,\big\},\\ T_1\mathbin{\widehat{+}}T_2&=\big\{\,\{f+h,g+k\}:\{f,g\}\in T_1,\ \{h,k\}\in T_2\,\big\},\end{aligned}$$

respectively. Closed linear operators from $\mathfrak{H}$ to $\mathfrak{K}$ will be identified with closed linear relations via their graphs. The linear space of everywhere defined bounded linear operators from $\mathfrak{H}$ into $\mathfrak{K}$ will be denoted by $\mathbf{B}(\mathfrak{H},\mathfrak{K})$ and by $\mathbf{B}(\mathfrak{H})$ if $\mathfrak{H}=\mathfrak{K}$. The resolvent set $\rho(T)$ of a closed linear relation T in $\mathfrak{H}$ is the set of all $\lambda\in\mathbb{C}$ such that $(T-\lambda)^{-1}\in\mathbf{B}(\mathfrak{H})$. Often $T-\lambda$ is simply called boundedly invertible and it is then tacitly assumed that the inverse is defined on the whole space. For $\lambda\in\rho(T)$ one has the following identity

$$T=\big\{\,\{(T-\lambda)^{-1}h,(I+\lambda(T-\lambda)^{-1})h\}:\ h\in\mathfrak{H}\,\big\}. \tag{2.1}$$

The spectrum $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbb{C}$.

A linear relation T in $\mathfrak{H}$ is *accumulative* (*dissipative*) if $\operatorname{Im}(f',f)\le 0$ ($\operatorname{Im}(f',f)\ge 0$, respectively) holds for all $\{f,f'\}\in T$. The relation T is said to be *maximal accumulative* (*maximal dissipative*) if T is accumulative (dissipative) and there exists no proper accumulative (dissipative, respectively) extension of T in $\mathfrak{H}$. Note that T is maximal accumulative (maximal dissipative) if and only if T is accumulative (dissipative) and $\mathbb{C}_+ \subset \rho(T)$ ($\mathbb{C}_- \subset \rho(T)$, respectively).

Let T be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. The *adjoint* T^* of T is a closed linear relation from $\mathfrak{K}$ to $\mathfrak{H}$ defined by

$$T^* := \{ \{g, g'\} : (f', g) = (f, g') \text{ for all } \{f, f'\} \in T \}.$$

This definition obviously extends the usual definition of the adjoint operator. If T is a closed linear relation in $\mathfrak{H}$ and $\lambda \in \mathbb{C}$, then $\operatorname{ran}(T - \lambda)$ is closed if and only if $\operatorname{ran}(T^* - \bar{\lambda})$ is closed. A linear relation T in $\mathfrak{H}$ is called *symmetric* (*selfadjoint*) if $T \subset T^*$ ($T = T^*$, respectively). It follows from the polarization identity that T is symmetric if and only if $(f', f) \in \mathbb{R}$ for all $\{f, f'\} \in T$.

For each linear relation T in the Hilbert space $\mathfrak{H}$ and each $\lambda \in \mathbb{C}$ the eigenspace $\mathfrak{N}_\lambda(T)$ is defined by $\mathfrak{N}_\lambda(T) = \ker(T - \lambda)$. Corresponding to this eigenspace is the following subset of T:

$$\widehat{\mathfrak{N}}_\lambda(T) = \{ \{f, f'\} \in T : f \in \mathfrak{N}_\lambda(T) \}.$$

Now let T be a closed symmetric relation in $\mathfrak{H}$. If A is an intermediate extension of T, i.e. $T \subset A \subset T^*$, with a nonempty resolvent set, then

$$T^* = A \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(T^*), \quad \text{direct sum},$$

holds for all $\lambda \in \rho(A)$. In particular, since for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) the relation $A = T \mathbin{\widehat{+}} \mathfrak{N}_{\bar{\lambda}}(T^*)$ is maximal accumulative (maximal dissipative, respectively), this leads to von Neumann's decomposition of a closed symmetric relation T:

$$T^* = T \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(T^*) \mathbin{\widehat{+}} \mathfrak{N}_{\bar{\lambda}}(T^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \text{direct sum}.$$

For a fixed $\mu \in \mathbb{C} \setminus \mathbb{R}$ the Cayley transform $C_\mu(T)$ of any linear relation T in $\mathfrak{H}$ is defined by

$$C_\mu(T) := \{ \{f' - \mu f, f' - \bar{\mu} f\} : \{f, f'\} \in T \}; \tag{2.2}$$

the corresponding inverse Cayley transform of a linear relation V in $\mathfrak{H}$ is given by

$$\{ \{h' - h, \mu h' - \bar{\mu} h\} : \{h, h'\} \in V \}. \tag{2.3}$$

Clearly the Cayley transform preserves closedness; and $C_\mu(T)$ maps $\operatorname{ran}(T - \mu)$ onto $\operatorname{ran}(T - \bar{\mu})$. Furthermore, the identity

$$\|f' - \bar{\mu} f\|^2 - \|f' - \mu f\|^2 = 4(\operatorname{Im} \mu) \operatorname{Im}(f', f), \quad f, f' \in \mathfrak{H}, \tag{2.4}$$

is straightforward to check. If $\mu \in \mathbb{C}_+$, the identity (2.4) shows that $C_\mu(T)$ is a expansive, contractive, isometric, or unitary operator if and only if T is a dissipative, accumulative, symmetric, or selfadjoint relation, respectively. Moreover, T is maximal dissipative or maximal accumulative if and only if $C_\mu(T)$ is a expansive or contractive operator, which is defined on $\mathfrak{H}$.

2.2. Some results for bounded linear operators

Let $\mathfrak{H}$ be a Hilbert space and let $T \in \mathbf{B}(\mathfrak{H})$. The following lemma lists some simple useful results.

Lemma 2.1. *Let $\mathfrak{H}$ be a Hilbert space, let $T \in \mathbf{B}(\mathfrak{H})$, and let $\lambda \in \mathbb{C}$. Then for $|\lambda| \geq \|T\|$:*

(i) $\ker(T-\lambda) = \ker(T^*-\bar\lambda) \subset \ker(T^*T-|\lambda|^2) \cap \ker(TT^*-|\lambda|^2)$*;*
(ii) $\overline{\operatorname{ran}}\,(T-\lambda) = \overline{\operatorname{ran}}\,(T^*-\bar\lambda)$ *(and* $\operatorname{ran}(T-\lambda)$ *is closed if and only if* $\operatorname{ran}(T^*-\bar\lambda)$ *is closed)*
(iii) *the following conditions are equivalent:* $\operatorname{ran}(T-\lambda) = \mathfrak{H}$, $\operatorname{ran}(T^*-\bar\lambda) = \mathfrak{H}$, $(T-\lambda)^{-1} \in \mathbf{B}(\mathfrak{H})$*, and* $(T^*-\bar\lambda)^{-1} \in \mathbf{B}(\mathfrak{H})$.

Moreover, for all $\lambda \neq 0$ one has:

(iv) $\ker(T^*T-|\lambda|^2) = \{0\}$ *if and only if* $\ker(TT^*-|\lambda|^2) = \{0\}$*;*
(v) $\operatorname{ran}(T^*T-|\lambda|^2)$ *is closed if and only if* $\operatorname{ran}(TT^*-|\lambda|^2)$ *is closed;*
(vi) $(T^*T-|\lambda|^2)^{-1} \in \mathbf{B}(\mathfrak{H})$ *if and only if* $(TT^*-|\lambda|^2)^{-1} \in \mathbf{B}(\mathfrak{H})$.

Proof. (i) Let $h \in \ker(T-\lambda)$. Since $\|T^*\| \leq |\bar\lambda|$, it follows that

$$0 \leq \big((T^*-\bar\lambda)h, (T^*-\bar\lambda)h\big) = (T^*h, T^*h) - (\bar\lambda h, \bar\lambda h) \leq 0,$$

and, hence, $h \in \ker(T^*-\bar\lambda)$. Thus $\ker(T-\lambda) \subset \ker(T^*-\bar\lambda)$. The reverse inclusion $\ker(T^*-\bar\lambda) \subset \ker(T-\lambda)$ holds by symmetry. The inclusion in (i) is obvious.

(ii) This follows by taking orthogonal complements in the identity in (i).

(iii) It suffices to note the following. If $\operatorname{ran}(T-\lambda) = \mathfrak{H}$, then $\ker(T^*-\bar\lambda) = \{0\}$, but then $\ker(T-\lambda) = \{0\}$ by (i). It follows from the closed graph theorem that $(T-\lambda)^{-1} \in \mathbf{B}(\mathfrak{H})$.

(iv), (v), & (vi) The following identities are easily verified:

$$\begin{aligned} A := |\lambda| \begin{pmatrix} T & |\lambda| \\ |\lambda| & T^* \end{pmatrix} &= \begin{pmatrix} |\lambda| & 0 \\ T^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & |\lambda|^2 - T^*T \end{pmatrix} \begin{pmatrix} T & |\lambda| \\ I & 0 \end{pmatrix}, \\ A^* = |\lambda| \begin{pmatrix} T^* & |\lambda| \\ |\lambda| & T \end{pmatrix} &= \begin{pmatrix} |\lambda| & 0 \\ T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & |\lambda|^2 - TT^* \end{pmatrix} \begin{pmatrix} T^* & |\lambda| \\ I & 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

For $\lambda \neq 0$ the first and the last factors of the products on the righthand side of (2.5) are bounded with bounded inverse. Therefore, $\ker A = \{0\}$, $\operatorname{ran} A$ is closed, or A is boundedly invertible if and only if $\ker(|\lambda|^2 - T^*T) = \{0\}$, $\operatorname{ran}(|\lambda|^2 - T^*T)$ is closed, or $|\lambda|^2 - T^*T$ is boundedly invertible, respectively. Analogously, $\ker A^* = \{0\}$, $\operatorname{ran} A^*$ is closed, or A^* is boundedly invertible if and only if $\ker(|\lambda|^2 - TT^*) = \{0\}$, $\operatorname{ran}(|\lambda|^2 - TT^*)$ is closed, or $|\lambda|^2 - TT^*$ is boundedly invertible, respectively. Finally, it remains to observe that $\ker A = \{0\}$ if and only if $\ker A^* = \{0\}$ (due to the special structure of A), $\operatorname{ran} A$ is closed if and only if $\operatorname{ran} A^*$ is closed, and A is boundedly invertible if and only A^* is boundedly invertible, respectively. □

Some more useful facts for an operator $T \in \mathbf{B}(\mathfrak{H})$ which will be used in the following are:

$$\ker TT^* = \ker T^*, \quad \ker T^*T = \ker T, \tag{2.6}$$

$$\operatorname{ran} TT^*,\ \operatorname{ran} T^*T,\ \operatorname{ran} T,\ \text{ and } \operatorname{ran} T^* \text{ are closed simultaneously,} \tag{2.7}$$

$$TT^* \text{ is boundedly invertible if and only if } \operatorname{ran} T = \mathfrak{H}, \tag{2.8}$$

$$T^*T \text{ is boundedly invertible if and only if } \operatorname{ran} T^* = \mathfrak{H}. \tag{2.9}$$

For the sake of completeness the following useful result from [26] is recalled.

Lemma 2.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, let $\mathfrak{M}$ be a closed linear subspace of $\mathfrak{X}$, and let $P \in \mathbf{B}(\mathfrak{X},\mathfrak{Y})$ be surjective. Then the image $P\mathfrak{M}$ is closed in $\mathfrak{Y}$ if and only if the sum $\mathfrak{M}+\mathfrak{N}$ is closed in $\mathfrak{X}$, where $\mathfrak{N} := \ker P$.*

3. Schur functions and Nevanlinna families

3.1. Schur functions, Nevanlinna families, and Nevanlinna pairs

Let $\mathcal{H}$ be a Hilbert space and let $\Theta(z)$ be an $\mathbf{B}(\mathcal{H})$-valued *Schur function*, i.e., $z \mapsto \Theta(z) \in \mathbf{B}(\mathcal{H})$ is a holomorphic function defined on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\|\Theta(z)\| \leq 1$, $z \in \mathbb{D}$. A Schur function $\Theta(z)$ will be extended to the exterior $\mathbb{D}^* = \{z \in \mathbb{C} \cup \{\infty\} : |z| > 1\}$ of the unit disc in the closed complex plane $\mathbb{C} \cup \{\infty\}$ by

$$\Theta(z) := \Theta(1/\bar{z})^*, \quad z \in \mathbb{D}^*, \tag{3.1}$$

cf., [27], [3, p. 112]. Here the following conventions $1/\infty = 0$ and $1/0 = \infty$ will be used, so that in particular $\Theta(\infty) = \Theta(0)^*$.

Let $\Theta(z)$ be an $\mathbf{B}(\mathcal{H})$-valued Schur function defined on $\mathbb{D} \cup \mathbb{D}^*$ as in (3.1). For a fixed $\mu \in \mathbb{C}_+$ define the function z by

$$z(\lambda) = \frac{\lambda - \mu}{\lambda - \bar{\mu}}, \tag{3.2}$$

so that z maps the upper halfplane $\mathbb{C}_+$ onto $\mathbb{D}$ and $\mathbb{C}_-$ onto $\mathbb{D}^*$. The argument λ in the mapping z will often be suppressed. It follows that the family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of linear relations defined by

$$M(\lambda) = \begin{cases} \{\, \{(I - \Theta(z))f, (\mu\Theta(z) - \bar{\mu})f\} : f \in \mathcal{H}\,\}, & \lambda \in \mathbb{C}_+, \\ \{\, \{(I - \Theta(z))f, (\bar{\mu}\Theta(z) - \mu)f\} : f \in \mathcal{H}\,\}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.3}$$

is a so-called Nevanlinna family in the sense of the next definition.

Definition 3.1. *A family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of linear relations in $\mathcal{H}$ is said to be a Nevanlinna family in $\mathcal{H}$ if*

(i) *$M(\lambda)$ is maximal dissipative (maximal accumulative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$);*
(ii) *$M(\bar{\lambda}) = M(\lambda)^*$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(iii) *for some, and hence for all, $\nu \in \mathbb{C}_+$ ($\nu \in \mathbb{C}_-$) the $\mathbf{B}(\mathcal{H})$-valued function $\lambda \mapsto (M(\lambda) + \nu)^{-1}$ is holomorphic on $\mathbb{C}_+$ ($\mathbb{C}_-$, respectively).*

If the Schur function $\Theta(z)$ and the Nevanlinna family $M(\lambda)$ are connected via (3.3), then $\Theta(z)$ can be recovered from $M(\lambda)$ as follows

$$\Theta(z) = \begin{cases} I - (\mu - \bar{\mu})(M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ I - (\bar{\mu} - \mu)(M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.4}$$

and $\Theta(z)$ satisfies (3.1). In fact, every Nevanlinna family $M(\lambda)$ defines via (3.4) a Schur function $\Theta(z)$, so that $M(\lambda)$ can be represented by (3.3).

Definition 3.2. *A pair $\{\Phi(\lambda), \Psi(\lambda)\}$ of $\mathbf{B}(\mathcal{H})$-valued functions is said to be a* Nevanlinna pair[1] *in $\mathcal{H}$ if $\Phi(\lambda)$ and $\Psi(\lambda)$ are holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and*

(i) $(\mathrm{Im}\,\lambda)\mathrm{Im}\,(\Psi(\lambda)\Phi(\lambda)^*) \geq 0$ *for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(ii) $\Psi(\lambda)\Phi(\bar{\lambda})^* = \Phi(\lambda)\Psi(\bar{\lambda})^*$ *for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(iii) $(\Psi(\lambda) + \nu\Phi(\lambda))^{-1} \in \mathbf{B}(\mathcal{H})$ *for some or, equivalently, for all $\lambda \in \mathbb{C}_\pm$, $\nu \in \mathbb{C}_\pm$.*

Two Nevanlinna pairs $\{\Phi(\lambda), \Psi(\lambda)\}$ and $\{\Phi'(\lambda), \Psi'(\lambda)\}$ are said to be equivalent *when*

$$\Phi'(\lambda) = \chi(\lambda)\Phi(\lambda) \quad \text{and} \quad \Psi'(\lambda) = \chi(\lambda)\Psi(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.5}$$

where $\chi(\lambda)$ is an $\mathbf{B}(\mathcal{H})$-valued holomorphic operator function on $\mathbb{C} \setminus \mathbb{R}$, such that $\chi(\lambda)^{-1} \in \mathbf{B}(\mathcal{H})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. A Nevanlinna pair $\{\Phi(\lambda), \Psi(\lambda)\}$ is said to be symmetric *if*

$$\Phi(\lambda) = \Phi(\bar{\lambda})^* \quad \text{and} \quad \Psi(\lambda) = \Psi(\bar{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.6}$$

Let $\{\Phi(\lambda), \Psi(\lambda)\}$ be a Nevanlinna pair in the Hilbert space $\mathcal{H}$ and define the family $M(\lambda)$ in $\mathcal{H}$ by

$$\begin{aligned} M(\lambda) &= \{ \{f, f'\} : \Phi(\lambda)f + \Psi(\lambda)f' = 0 \} \\ &= \{ \{\Psi(\bar{\lambda})^* g, -\Phi(\bar{\lambda})^* g\} : g \in \mathcal{H} \}. \end{aligned} \tag{3.7}$$

Then $M(\lambda)$ is a Nevanlinna family. If $\{\Phi'(\lambda), \Psi'(\lambda)\}$ is a second Nevanlinna pair in $\mathcal{H}$ such that (3.7) holds with $\Phi(\lambda)$ and $\Psi(\lambda)$ replaced by $\Phi'(\lambda)$ and $\Psi'(\lambda)$, respectively, then the Nevanlinna pairs $\{\Phi(\lambda), \Psi(\lambda)\}$ and $\{\Phi'(\lambda), \Psi'(\lambda)\}$ are equivalent. Equivalent Nevanlinna pairs determine via (3.7) the same Nevanlinna family.

Let $\mu \in \mathbb{C}_+$ and let $M(\lambda)$ be a Nevanlinna family in $\mathcal{H}$. Then the pair $\{A(\lambda), B(\lambda)\}$ defined by

$$A(\lambda) := \begin{cases} (M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ (M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.8}$$

$$B(\lambda) := \begin{cases} I - \mu(M(\lambda) + \mu)^{-1}, & \lambda \in \mathbb{C}_+, \\ I - \bar{\mu}(M(\lambda) + \bar{\mu})^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.9}$$

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[1]Note that Definition 3.2 differs slightly from the definition of Nevanlinna pairs used for instance in [21, 26]: $\{\Phi(\lambda), \Psi(\lambda)\}$ is a Nevanlinna pair as in Definition 3.2 if and only if $\{\widetilde{\Phi}(\lambda), \widetilde{\Psi}(\lambda)\}$, where $\widetilde{\Phi}(\lambda) := \Psi(\overline{\lambda})^*$ and $\widetilde{\Psi}(\lambda) := -\Phi(\overline{\lambda})^*$, is a Nevanlinna pair in the sense of [21, Definition 2.2].

is a Nevanlinna pair in $\mathcal{H}$, which is symmetric. It follows that

$$M(\lambda) = \{ \{A(\lambda)g, B(\lambda)g\} : g \in \mathcal{H} \}, \tag{3.10}$$

cf., (2.1). Thus, every Nevanlinna family $M(\lambda)$ can be represented in the forms (3.7), (3.10) where the Nevanlinna pair can be taken to be symmetric.

If the Nevanlinna family $M(\lambda)$ is represented in the form (3.10) by means of a symmetric Nevanlinna pair $\{A(\lambda), B(\lambda)\}$, then the transformation in (3.4) takes the form

$$\Theta(z) = \begin{cases} (B(\lambda) + \bar{\mu}A(\lambda))(B(\lambda) + \mu A(\lambda))^{-1}, & \lambda \in \mathbb{C}_+, \\ (B(\lambda) + \mu A(\lambda))(B(\lambda) + \bar{\mu}A(\lambda))^{-1}, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.11}$$

and (3.1) holds. Conversely, if $\Theta(z)$ is a Schur function extended to $\mathbb{D}^*$ via (3.1), then the formula (3.3) gives rise to a Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ in $\mathcal{H}$ of the form

$$A(\lambda) := I - \Theta(z), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad B(\lambda) := \begin{cases} \mu\Theta(z) - \bar{\mu}, & \lambda \in \mathbb{C}_+, \\ \bar{\mu}\Theta(z) - \mu, & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.12}$$

and the pair $\{A(\lambda), B(\lambda)\}$ satisfies the symmetry property (3.6).

In the special case that the Nevanlinna family $M(\lambda)$ is an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function it is clear that $\{I, M(\lambda)\}$ is a symmetric Nevanlinna pair.

3.2. Schur and Nevanlinna kernels and their associated reproducing kernel Hilbert spaces

Let $\mathsf{K}(w, z)$ be a kernel defined on $\mathcal{D} \times \mathcal{D}$, $\mathcal{D} \subset \mathbb{C}$, with values in a Hilbert space $\mathfrak{K}$. Recall the following notions, cf., [3, p. 6]. The kernel $\mathsf{K}(w, z)$ is said to be *hermitian* if

$$\mathsf{K}(w, z)^* = \mathsf{K}(z, w), \quad w, z \in \mathcal{D}.$$

If $\mathcal{D}$ is an open subset, the hermitian kernel $\mathsf{K}(w, z)$ is said to be *holomorphic* if it is holomorphic in z for each fixed w and holomorphic in $\bar{w}$ for each fixed z. The kernel $\mathsf{K}(w, z)$ is said to be *nonnegative*, if for any finite set of points $w_1, \dots, w_n$ in $\mathcal{D}$ and vectors $f_1, \dots f_n \in \mathfrak{K}$, the hermitian matrix

$$\big((\mathsf{K}(w_j, w_i) f_j, f_i) \big)_{i,j=1}^n$$

is nonnegative. Schur functions and Nevanlinna pairs generate in a natural way kernels which are hermitian, holomorphic, and nonnegative.

Let $\Theta(z)$ be an $\mathbf{B}(\mathcal{H})$-valued Schur function defined on the unit disc $\mathbb{D}$. The corresponding 2×2 operator matrix kernel $\mathsf{D}_\Theta(w, z)$ is defined by

$$\mathsf{D}_\Theta(w, z) = \begin{pmatrix} \frac{I - \Theta(z)\Theta(w)^*}{1 - z\bar{w}} & \frac{\Theta(z) - \Theta(\bar{w})}{z - \bar{w}} \\ \frac{\Theta(\bar{z})^* - \Theta(w)^*}{z - \bar{w}} & \frac{I - \Theta(\bar{z})^*\Theta(\bar{w})}{1 - z\bar{w}} \end{pmatrix}, \quad z, w \in \mathbb{D}. \tag{3.13}$$

The kernel $\mathsf{D}_\Theta(w, z)$ is hermitian, holomorphic, and moreover nonnegative, see [39]. Let $\mathfrak{D}(\Theta)$ be the reproducing kernel Hilbert space associated with the Schur kernel in (3.13). It consists of $(\mathcal{H} \oplus \mathcal{H})$-valued holomorphic vector functions on $\mathbb{D}$

obtained as the closed linear span of functions $z \mapsto \mathsf{D}_\Theta(w,z)F$, $w \in \mathbb{D}$, $F \in \mathcal{H}\oplus\mathcal{H}$, which is provided with the scalar product determined by

$$\big\langle \mathsf{D}_\Theta(w,\cdot)F, \mathsf{D}_\Theta(v,\cdot)G \big\rangle := \big(\mathsf{D}_\Theta(w,v)F, G\big), \quad F,G \in \mathcal{H}\oplus\mathcal{H},\ w,v \in \mathbb{D}. \tag{3.14}$$

The $\mathcal{H}\oplus\mathcal{H}$-valued functions $\Phi \in \mathfrak{D}(\Theta)$ satisfy the reproducing kernel property

$$\langle \Phi(\cdot), \mathsf{D}_\Theta(w,\cdot)F \rangle = (\Phi(w), F), \quad F \in \mathcal{H}\oplus\mathcal{H},\ w \in \mathbb{D}. \tag{3.15}$$

Now let $\Theta(z)$ be an $\mathbf{B}(\mathcal{H})$-valued Schur function extended to $\mathbb{D}\cup\mathbb{D}^*$ as in (3.1). The corresponding *Schur kernel* $\mathsf{S}_\Theta(w,z)$ on $(\mathbb{D}\cup\mathbb{D}^*)\times(\mathbb{D}\cup\mathbb{D}^*)$ is defined by

$$\mathsf{S}_\Theta(w,z) := \begin{cases} \frac{1-\Theta(z)\Theta(w)^*}{1-z\bar{w}}, & w \in \mathbb{D},\ z \in \mathbb{D}, \\ \frac{\Theta(z)-\Theta(w)^*}{1/z-\bar{w}}, & w \in \mathbb{D},\ z \in \mathbb{D}^*, \\ \frac{\Theta(z)-\Theta(w)^*}{z-1/\bar{w}}, & w \in \mathbb{D}^*,\ z \in \mathbb{D}, \\ \frac{1-\Theta(z)\Theta(w)^*}{1-1/(z\bar{w})}, & w \in \mathbb{D}^*,\ z \in \mathbb{D}^*. \end{cases} \tag{3.16}$$

The kernel $\mathsf{S}_\Theta(\cdot,\cdot)$ is hermitian, holomorphic, and nonnegative, see [39]. Let $\mathfrak{S}(\Theta)$ be the reproducing kernel Hilbert space associated with the Schur kernel in (3.16). It consists of $\mathcal{H}$-valued holomorphic vector functions on $\mathbb{D}\cup\mathbb{D}^*$ obtained as the closed linear span of functions $z \mapsto \mathsf{S}_\Theta(w,z)f$, $w \in \mathbb{D}\cup\mathbb{D}^*$, $f \in \mathcal{H}$, which is provided with the scalar product determined by

$$\big\langle \mathsf{S}_\Theta(w,\cdot)f, \mathsf{S}_\Theta(v,\cdot)g \big\rangle := \big(\mathsf{S}_\Theta(w,v)f, g\big), \quad f,g \in \mathcal{H},\ w,v \in \mathbb{D}\cup\mathbb{D}^*. \tag{3.17}$$

The functions $\varphi \in \mathfrak{S}(\Theta)$ satisfy the reproducing kernel property

$$\langle \varphi(\cdot), \mathsf{S}_\Theta(w,\cdot)f \rangle = (\varphi(w), f), \quad h \in \mathcal{H},\ w \in \mathbb{D}\cup\mathbb{D}^*. \tag{3.18}$$

The elements of $\mathfrak{D}(\Theta)$ and $\mathfrak{S}(\Theta)$ can be identified: the function $F(z) \in \mathfrak{D}(\Theta)$ corresponds to the function $f(z) \in \mathfrak{S}(\Theta)$, as follows

$$F(z) = \begin{pmatrix} h(z) \\ k(z) \end{pmatrix}, \quad z \in \mathbb{D}, \quad \text{and} \quad f(z) = \begin{cases} h(z), & z \in \mathbb{D}, \\ k(1/\bar{z}), & z \in \mathbb{D}^*, \end{cases} \tag{3.19}$$

cf., [3, pp. 112–113].

Let $\{A(\lambda), B(\lambda)\}$ be a symmetric Nevanlinna pair in $\mathcal{H}$. The corresponding *Nevanlinna kernel* $\mathsf{N}_{A,B}(\xi,\lambda)$ on $(\mathbb{C}_+\cup\mathbb{C}_-)\times(\mathbb{C}_+\cup\mathbb{C}_-)$ is defined by

$$\mathsf{N}_{A,B}(\xi,\lambda) := \frac{B(\lambda)A(\xi)^* - A(\lambda)B(\xi)^*}{\lambda-\bar{\xi}}, \quad \lambda,\xi \in \mathbb{C}_+\cup\mathbb{C}_-, \quad \xi \neq \bar{\lambda}. \tag{3.20}$$

Let the Schur function $\Theta(z)$ be connected with the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ via (3.12). Then the corresponding Schur kernel $\mathsf{S}_\Theta(w,z)$ and the Nevanlinna kernel $\mathsf{N}_{A,B}(\xi,\lambda)$ are connected via

$$\mathsf{S}_\Theta(w,z) = r(\lambda)\mathsf{N}_{A,B}(\xi,\lambda)r(\xi)^*, \quad \lambda,\xi \in \mathbb{C}_+\cup\mathbb{C}_-, \quad \xi \neq \bar{\lambda} \tag{3.21}$$

where $z = (\lambda - \mu)/(\lambda - \bar{\mu})$ and $w = (\xi - \mu)/(\xi - \bar{\mu})$ and

$$r(\lambda) = \begin{cases} (\lambda - \bar{\mu})/(\mu - \bar{\mu}), & \lambda \in \mathbb{C}_+, \\ (\lambda - \mu)/(\bar{\mu} - \mu), & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.22}$$

Hence, the kernel $\mathsf{N}_{A,B}(\xi, \lambda)$ is hermitian, holomorphic, and nonnegative. The corresponding reproducing kernel Hilbert space will be denoted by $\mathfrak{H}(A, B)$. If the pair $\{I, M(\lambda)\}$ is an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function the notation $\mathfrak{H}(M)$ will be used instead of $\mathfrak{H}(I, M)$. The space $\mathfrak{H}(A, B)$ consists of $\mathcal{H}$-valued holomorphic vector functions on $\mathbb{C}_+ \cup \mathbb{C}_-$ obtained as the closed linear span of functions $\lambda \mapsto \mathsf{N}_{A,B}(\xi, \lambda) f$, $\xi \in \mathbb{C}_+ \cup \mathbb{C}_-$, $f \in \mathcal{H}$, which is provided with the scalar product determined by

$$\big\langle \mathsf{N}_{A,B}(\xi, \cdot) f, \mathsf{N}_{A,B}(\lambda, \cdot) g \big\rangle := \big(\mathsf{N}_{A,B}(\xi, \lambda) f, g\big), \quad f, g \in \mathcal{H},\ \xi, \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{3.23}$$

The functions $\varphi \in \mathfrak{H}(A, B)$ satisfy the reproducing kernel property

$$\langle \varphi(\cdot), \mathsf{N}_{A,B}(\xi, \cdot) f \rangle = (\varphi(\xi), f), \quad f \in \mathcal{H},\ \xi \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{3.24}$$

Assume that $\{A'(\lambda), B'(\lambda)\}$ is a symmetric Nevanlinna pair in $\mathcal{H}$ which is equivalent to $\{A(\lambda), B(\lambda)\}$ in the sense of (3.5). Then

$$\mathsf{N}_{A',B'}(\xi, \lambda) = \chi(\lambda) \mathsf{N}_{A,B}(\xi, \lambda) \chi(\xi)^*$$

and if $\langle \cdot, \cdot \rangle_{A',B'}$ and $\langle \cdot, \cdot \rangle_{A,B}$ denote the scalar products in $\mathfrak{H}(A', B')$ and $\mathfrak{H}(A, B)$, respectively, then

$$\langle \chi\varphi, \chi\psi \rangle_{A',B'} = \langle \varphi, \psi \rangle_{A,B} \tag{3.25}$$

holds for all functions $\varphi, \psi \in \mathfrak{H}(A, B)$. In particular a function φ belongs to $\mathfrak{H}(A, B)$ if and only if the function $\chi\varphi$ belongs to $\mathfrak{H}(A', B')$. Hence, multiplication by χ is a unitary mapping from $\mathfrak{H}(A, B)$ onto $\mathfrak{H}(A', B')$.

Multiplication by the function r in (3.22) is a unitary mapping from the Hilbert space $\mathfrak{H}(A, B)$ onto the Hilbert space $\mathfrak{S}(\Theta)$. That is, for $\varphi \in \mathfrak{H}(A, B)$ the function

$$\Phi(z) = r(\lambda)\varphi(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

belongs to $\mathfrak{S}(\Theta)$, $\|\Phi\|_{\mathfrak{S}(\Theta)} = \|\varphi\|_{\mathfrak{H}(A,B)}$, and every element $\Phi \in \mathfrak{S}(\Theta)$ can be written in this way with some $\varphi \in \mathfrak{H}(A, B)$.

3.3. Some connections between Nevanlinna families and Schur functions

Let $M(\lambda)$ be a Nevanlinna family in $\mathcal{H}$ and let $\Theta(z)$ be a Schur function such that (3.3), (3.4) hold. Then the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ given by (3.12) is symmetric. Various subspaces involving $M(\lambda)$ will be expressed in terms of $\{A(\lambda), B(\lambda)\}$ and $\Theta(z)$.

Lemma 3.3. *Let $M(\lambda)$ be a Nevanlinna family in the Hilbert space $\mathcal{H}$ and let $\Theta(z)$ be a Schur function such that* (3.3), (3.4) *hold, and let $\{A(\lambda), B(\lambda)\}$ be the Nevanlinna pair given by* (3.12)*. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

(i) $\operatorname{mul} M(\lambda) = \{ B(\lambda)g : A(\lambda)g = 0 \}$;
(ii) $\ker M(\lambda) = \{ A(\lambda)g : B(\lambda)g = 0 \}$;
(iii) $\{ \{B(\lambda)f - A(\lambda)f' : \{f, f'\} \in M(\lambda) \widehat{+} M(\lambda)^* \} = \operatorname{ran}(I - \Theta(z)\Theta(z)^*)$;
(iv) $M(\lambda) \cap M(\lambda)^* = \{ \{A(\bar\lambda)g, B(\bar\lambda)g\} : g \in \ker(I - \Theta(z)\Theta(z)^*) \}$;
(v) $\ker(M(\lambda) - M(\lambda)^*) = \{ A(\bar\lambda)g : g \in \ker(I - \Theta(z)\Theta(z)^*) \}$;
(vi) $\ker(M(\lambda)^{-1} - M(\lambda)^{-*}) = \{ B(\bar\lambda)g : g \in \ker(I - \Theta(z)\Theta(z)^*) \}$.

Proof. (i) & (ii) These statements follow directly from (3.10).

(iii) By (3.3) and (3.12) one has $M(\lambda) = \{\{A(\lambda)g, B(\lambda)g\} : g \in \mathcal{H}\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Thus by expressing the elements in the graph of $M(\lambda)$ as column vectors, one has

$$M(\lambda) \widehat{+} M(\lambda)^* = \operatorname{ran} \begin{pmatrix} A(\lambda) & A(\bar\lambda) \\ B(\lambda) & B(\bar\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.26}$$

since $M(\lambda)^* = M(\bar\lambda)$. On the other hand, using the symmetry of the Nevanlinna pair $\{A(\lambda), B(\lambda)\}$ and (3.12) leads to

$$\begin{aligned} \begin{pmatrix} B(\lambda) & -A(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & A(\bar\lambda) \\ B(\lambda) & B(\bar\lambda) \end{pmatrix} &= \begin{pmatrix} 0 & B(\lambda)A(\bar\lambda) - A(\lambda)B(\bar\lambda) \end{pmatrix} \\ &= \pm(\mu - \bar\mu) \begin{pmatrix} 0 & (I - \Theta(z)\Theta(z)^*) \end{pmatrix}, \end{aligned} \tag{3.27}$$

for $\lambda \in \mathbb{C}_\pm$, respectively. The stated identity is immediate from (3.26) and (3.27).

(iv) According to (iii) $g \in \ker(I - \Theta(z)\Theta(z)^*)$ if and only if for all $\{f, f'\} \in M(\lambda) \widehat{+} M(\lambda)^*$ one has

$$0 = (B(\lambda)f - A(\lambda)f', g) = (f, B(\bar\lambda)g) - (f', A(\bar\lambda)g).$$

This means that $\{A(\bar\lambda)g, B(\bar\lambda)g\} \in (M(\lambda) \widehat{+} M(\lambda)^*)^* = M(\lambda) \cap M(\lambda)^*$.

(v) & (vi) The assertions on $\ker(M(\lambda) - M(\lambda)^*)$ and $\ker(M(\lambda)^{-1} - M(\lambda)^{-*})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, are immediate from part (iii). □

Proposition 3.4. *Let $M(\lambda)$ be a Nevanlinna family in the Hilbert space $\mathcal{H}$ and let $\Theta(z)$ be a Schur function such that* (3.3), (3.4) *hold. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

(i) $\operatorname{mul} M(\lambda) = \ker(I - \Theta(z))$;
(ii) $\ker M(\lambda) = \begin{cases} \ker(\bar\mu/\mu - \Theta(z)), & \lambda \in \mathbb{C}_+, \\ \ker(\mu/\bar\mu - \Theta(z)), & \lambda \in \mathbb{C}_-; \end{cases}$
(iii) $M(\lambda) \cap M(\lambda)^* = \{0\}$ *if and only if* $\ker(I - \Theta(z)\Theta(z)^*) = \{0\}$;
(iv) $M(\lambda) \widehat{+} M(\lambda)^*$ *is closed if and only if* $\operatorname{ran}(I - \Theta(z)\Theta(z)^*)$ *is closed;*
(v) $M(\lambda) \widehat{+} M(\lambda)^* = \mathcal{H} \times \mathcal{H}$ *if and only if* $\operatorname{ran}(I - \Theta(z)\Theta(z)^*) = \mathcal{H}$.

Proof. (i) & (ii) These statements follow from (i) & (ii) of Lemma 3.3 and (3.12).

(iii) Let $\{A(\lambda), B(\lambda)\}$ be the Nevanlinna pair given by (3.12). Then the statement is clear from Lemma 3.3 (iv), since $\ker A(\bar\lambda) \cap \ker B(\bar\lambda) = \{0\}$ by the property (iii) in Definition 3.2.

(iv) For a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$ consider in $\mathcal{H} \times \mathcal{H}$ the closed subspace $\mathfrak{M} = M(\bar\lambda)$ and let

$$P = \begin{pmatrix} B(\lambda) & -A(\lambda) \end{pmatrix} : \mathcal{H} \times \mathcal{H} \to \mathcal{H};$$

see Lemma 3.3 (iii). The property (iii) in Definition 3.2 implies that P is surjective. Moreover, the symmetry of $\{A(\lambda), B(\lambda)\}$ and (3.7) show that $\mathfrak{N} = \ker P = M(\lambda)$, while it follows from (3.27) that $P\mathfrak{M} = \operatorname{ran}(I - \Theta(z)\Theta(z)^*)$. Therefore, by Lemma 2.2 $\operatorname{ran}(I - \Theta(z)\Theta(z)^*)$ is closed in $\mathcal{H}$ if and only if $\mathfrak{M} \widehat{+} \mathfrak{N} = M(\lambda)^* \widehat{+} M(\lambda)$ is closed in $\mathcal{H} \times \mathcal{H}$.

(v) This statement is obtained by combining the assertions in (iii) and (iv). □

4. Unitary colligations and Schur functions

4.1. Unitary colligations and transfer functions

A *unitary colligation* consists of a pair of Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$ and a unitary operator $U \in \mathbf{B}(\mathfrak{H} \oplus \mathcal{H})$,

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathcal{H} \end{pmatrix}, \tag{4.1}$$

where $T \in \mathbf{B}(\mathfrak{H})$, $F \in \mathbf{B}(\mathcal{H}, \mathfrak{H})$, $G \in \mathbf{B}(\mathfrak{H}, \mathcal{H})$ and $H \in \mathbf{B}(\mathcal{H})$ are contractions; cf., [3, 16]. The unitarity of U is equivalent to the conditions

$$T^*T + G^*G = I, \quad F^*T + H^*G = 0, \quad F^*F + H^*H = I, \tag{4.2}$$

and

$$TT^* + FF^* = I, \quad GT^* + HF^* = 0, \quad GG^* + HH^* = I. \tag{4.3}$$

Note that in particular

$$\begin{aligned} \ker(I - T^*T) = \ker G, \quad \ker(I - TT^*) = \ker F^*, \\ \ker(I - H^*H) = \ker F, \quad \ker(I - HH^*) = \ker G^*. \end{aligned} \tag{4.4}$$

Some basic properties of the operators T, F, G, and H will be now collected.

Lemma 4.1. *Let $U \in \mathbf{B}(\mathfrak{H} \oplus \mathcal{H})$ be a unitary colligation as in* (4.1) *and let $\zeta \in \mathbb{T} = \{\, \xi \in \mathbb{C} : |\xi| = 1 \,\}$. Then:*

- (i) $\ker F^* = T(\ker G)$, $\ker G = T^*(\ker F^*)$ *and, moreover,* $(\zeta - T)(\ker G) = (\bar\zeta - T^*)(\ker F^*)$*;*
- (ii) $\ker G^* = H(\ker F)$, $\ker F = H^*(\ker G^*)$ *and, moreover,* $(\zeta - H)(\ker F) = (\bar\zeta - H^*)(\ker G^*)$*;*
- (iii) $\ker(\zeta - H) = \ker(\bar\zeta - H^*) \subset \ker F \cap \ker G^*$ *and* $\overline{\operatorname{ran}}\,(\zeta - H) = \overline{\operatorname{ran}}\,(\bar\zeta - H^*)$*;*
- (iv) $\ker F^* = \{0\}$ *if and only if* $\ker G = \{0\}$ *and* $\ker G^* = \{0\}$ *if and only if* $\ker F = \{0\}$*;*
- (v) *the ranges* $\operatorname{ran} G$, $\operatorname{ran} G^*$, $\operatorname{ran} F$, *and* $\operatorname{ran} F^*$ *are closed simultaneously;*
- (vi) $\operatorname{ran} G^* = \mathcal{H}$ *if and only if* $\operatorname{ran} F = \mathcal{H}$, *and* $\operatorname{ran} F^* = \mathcal{H}$ *if and only if* $\operatorname{ran} G = \mathcal{H}$.

Proof. (i) The second identity in (4.2) shows that $G\varphi = 0$ implies $F^*T\varphi = 0$, i.e., $T\varphi \in \ker F^*$. Conversely, if $F^*k = 0$, then the first two identities in (4.3) yield $k = TT^*k$ while $GT^*k = 0$, so that $k \in T(\ker G)$. This proves $\ker F^* = T(\ker G)$.

The proof of the second assertion is similar. As to the third identity: if $\varphi \in \ker G$, then $k = T\varphi \in \ker F^*$ and $(\zeta - T)\varphi = \zeta T^*T\varphi - T\varphi = \zeta(T^* - \bar{\zeta})k$, which implies that $(\zeta - T)(\ker G) \subset (\bar{\zeta} - T^*)(\ker F^*)$. The reverse inclusion follows in a similar manner.

(ii) The proof is completely analogous to that in part (i).

(iii) This is an immediate consequence of Lemma 2.1 and (4.4).

(iv) The statement follows from (4.4) and part (iv) of Lemma 2.1.

(v) In view of (4.2) $\operatorname{ran} G^*$ is closed if and only if $\operatorname{ran}(I - T^*T)$ is closed, or equivalently, if and only if $\operatorname{ran}(I - TT^*)$ is closed; see Lemma 2.1 (v). In view of (4.3) the last assertion is equivalent to $\operatorname{ran} F$ is closed. The remaining equivalences are due to $\operatorname{ran} G$ ($\operatorname{ran} F$) is closed if and only if $\operatorname{ran} G^*$ ($\operatorname{ran} F^*$, respectively) is closed.

(vi) This is obtained by combining the assertions in parts (iv) and (v). □

The *transfer function* $\Theta(z)$ of the unitary colligation (4.1) is defined by

$$\Theta(z) := H + zG(I - zT)^{-1}F, \quad z \in \mathbb{D}. \tag{4.5}$$

It is clear that $\Theta(z)$ is an $\mathbf{B}(\mathcal{H})$-valued holomorphic function on $\mathbb{D}$ and a straightforward computation using (4.2)-(4.3) shows

$$\frac{I - \Theta(z)\Theta(w)^*}{1 - z\bar{w}} = G(I - zT)^{-1}(I - \bar{w}T^*)^{-1}G^*, \qquad z, w \in \mathbb{D}. \tag{4.6}$$

Thus $\|\Theta(z)\| = \|\Theta(z)^*\| \leq 1$, $z \in \mathbb{D}$, and therefore $\Theta(z)$ is a Schur function. If $\Theta(z)$ is extended to $\mathbb{D}^*$ by (3.1), then

$$\Theta(z) = H^* + \tfrac{1}{z}F^*\left(I - \tfrac{1}{z}T^*\right)^{-1}G^*, \quad z \in \mathbb{D}^*, \tag{4.7}$$

and

$$\frac{I - \Theta(z)\Theta(w)^*}{1 - 1/(z\bar{w})} = F^*\big(I - \tfrac{1}{z}T^*\big)^{-1}\big(I - \tfrac{1}{\bar{w}}T\big)^{-1}F, \quad z, w \in \mathbb{D}^*. \tag{4.8}$$

In fact, it follows from (4.2) and (4.3) that the nonnegative kernel (3.16) is given by

$$\mathsf{S}_\Theta(w,z) = \begin{cases} G(I - zT)^{-1}(I - \bar{w}T^*)^{-1}G^*, & w \in \mathbb{D},\, z \in \mathbb{D}, \\ F^*(I - z^{-1}T^*)^{-1}(I - \bar{w}T^*)^{-1}G^*, & w \in \mathbb{D},\, z \in \mathbb{D}^*, \\ G(I - zT)^{-1}(I - \bar{w}^{-1}T)^{-1}F, & w \in \mathbb{D}^*,\, z \in \mathbb{D}, \\ F^*(I - z^{-1}T^*)^{-1}(I - \bar{w}^{-1}T)^{-1}F, & w \in \mathbb{D}^*,\, z \in \mathbb{D}^*. \end{cases}$$

The unitary colligation U in (4.1) is said to be *closely connected* if

$$\mathfrak{H} = \overline{\operatorname{span}}\left\{(I - wT)^{-1}Ff, (I - \lambda T^*)^{-1}G^*g : \, w, \lambda \in \mathbb{D},\, f, g \in \mathcal{H}\right\}. \tag{4.9}$$

The symmetry property (3.1) gives the following analogues for (4.6) and (4.8):

$$\begin{aligned} \frac{I-\Theta(z)^*\Theta(w)}{1-\bar{z}w} &= F^*(I-\bar{z}T^*)^{-1}(I-wT)^{-1}F, \quad z,w\in\mathbb{D}, \\ \frac{I-\Theta(z)^*\Theta(w)}{1-1/(\bar{z}w)} &= G\big(I-\tfrac{1}{\bar{z}}T\big)^{-1}\big(I-\tfrac{1}{w}T^*\big)^{-1}G^*, \quad z,w\in\mathbb{D}^*. \end{aligned} \tag{4.10}$$

Observe also that the transfer function $\Theta(z)$ is connected with the coresolvent of U and U^* via

$$\begin{aligned} P_{\mathcal{H}}(I-zU)^{-1}\restriction_{\mathcal{H}} &= (I-z\Theta(z))^{-1}, \qquad z\in\mathbb{D}, \\ P_{\mathcal{H}}\big(I-\tfrac{1}{z}U^*\big)^{-1}\restriction_{\mathcal{H}} &= \big(I-\tfrac{1}{z}\Theta(z)\big)^{-1}, \qquad z\in\mathbb{D}^*. \end{aligned} \tag{4.11}$$

The unitary colligation U in (4.1) induces for each $\zeta\in\mathbb{T}$ the operators $V(\zeta)$ and $V_*(\zeta)$ in the Hilbert space $\mathfrak{H}$ defined by

$$V(\zeta)=\big\{\,\{h,Th+Ff\}:\, h\in\mathfrak{H},\, f\in\mathcal{H},\, Gh+Hf=\zeta f\,\big\}, \tag{4.12}$$

and

$$V_*(\zeta)=\big\{\,\{h,T^*h+G^*f\}:\, h\in\mathfrak{H},\, f\in\mathcal{H},\, F^*h+H^*f=\bar{\zeta}f\,\big\}, \tag{4.13}$$

respectively. Observe that $V(\zeta)$ is a restriction of U and $V_*(\zeta)$ is a restriction of U^*. Lemma 4.1 (iii) shows that $\operatorname{mul} V(\zeta)=\operatorname{mul} V_*(\zeta)=\{0\}$. The operators $V(\zeta)$ and $V_*(\zeta)$ will play an important role in some considerations that follow.

Lemma 4.2. *Let $\zeta\in\mathbb{T}$. The operators $V(\zeta)$ and $V_*(\zeta)$ are isometric in $\mathfrak{H}$. If* $\operatorname{ran}(\zeta-H)$ *is closed or, equivalently,* $\operatorname{ran}(\bar{\zeta}-H^*)$ *is closed, then $V(\zeta)$ and $V_*(\zeta)$ are unitary, and $V_*(\zeta)=V(\zeta)^*$. In particular, if* $\operatorname{ran}(\zeta-H)=\mathcal{H}$ *or, equivalently,* $\operatorname{ran}(\bar{\zeta}-H^*)=\mathcal{H}$*, then*

$$V(\zeta)=T+F(\zeta-H)^{-1}G, \quad V_*(\zeta)=T^*+G^*(\bar{\zeta}-H^*)^{-1}F^*. \tag{4.14}$$

Proof. It is a consequence of the operator in (4.1) being unitary, that the restricted operators $V(\zeta)$ and $V_*(\zeta)$ in (4.12) and (4.13) are isometric. If, in particular, $\operatorname{ran}(\zeta-H)=\mathcal{H}$ or, equivalently, $\operatorname{ran}(\bar{\zeta}-H^*)=\mathcal{H}$, it follows from Lemma 4.1 (iii) and the definition that (4.14) holds. Hence $V_*(\zeta)=V(\zeta)^*$, which also shows that each of these operators is unitary in this case.

Now consider the general case, where $\operatorname{ran}(\zeta-H)$ or, equivalently by Lemma 4.1 (iii), $\operatorname{ran}(\bar{\zeta}-H^*)$ is only assumed to be closed. It is clear that

$$\mathcal{L}:=\ker(\zeta-H)=\ker(\bar{\zeta}-H^*),$$

cf., Lemma 4.1 (iii), is an invariant subspace for H and H^*. Furthermore, Lemma 4.1 (iii) implies $\mathcal{L}\subset\ker F$ and $\operatorname{ran} G\subset\operatorname{ran}(\zeta-H)=\mathcal{H}\ominus\mathcal{L}=\mathcal{L}^\perp$. Therefore, if $P_{\mathcal{L}}$, $P_{\mathcal{L}^\perp}$ denote the orthogonal projections in $\mathcal{H}$ onto $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively, then the unitary colligation $U\in\mathbf{B}(\mathfrak{H}\oplus\mathcal{H})$ in (4.1) can be written as

$$U=\begin{pmatrix} T & F\restriction_{\mathcal{L}^\perp} & 0 \\ P_{\mathcal{L}^\perp}G & P_{\mathcal{L}^\perp}H\restriction_{\mathcal{L}^\perp} & 0 \\ 0 & 0 & P_{\mathcal{L}}H\restriction_{\mathcal{L}} \end{pmatrix} : \begin{pmatrix}\mathfrak{H}\\ \mathcal{L}^\perp \\ \mathcal{L}\end{pmatrix} \to \begin{pmatrix}\mathfrak{H}\\ \mathcal{L}^\perp \\ \mathcal{L}\end{pmatrix}.$$

Clearly $P_{\mathcal{L}}H\upharpoonright_{\mathcal{L}}$ is unitary in $\mathcal{L}$ and

$$U' = \begin{pmatrix} T & F\upharpoonright_{\mathcal{L}^\perp} \\ P_{\mathcal{L}^\perp}G & P_{\mathcal{L}^\perp}H\upharpoonright_{\mathcal{L}^\perp} \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathcal{L}^\perp \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathcal{L}^\perp \end{pmatrix}$$

is a unitary colligation in $\mathfrak{H} \oplus \mathcal{L}^\perp$. Let $V'(\zeta)$ and $V_*'(\zeta)$ be the operators induced by U'; compare (4.12) and (4.13). By the first part of the proof $V'(\zeta)$ and $V_*'(\zeta)$ are unitary in $\mathfrak{H}$ and $V_*'(\zeta) = V'(\zeta)^*$ holds. Lemma 4.1 (iii) implies $V'(\zeta) = V(\zeta)$ and $V_*'(\zeta) = V_*(\zeta)$ and hence also the operators $V(\zeta)$ and $V_*(\zeta)$ are unitary in $\mathfrak{H}$ and satisfy $V_*(\zeta) = V(\zeta)^*$. □

Observe, that $V(z) = T + zF(I - zH)^{-1}G$, $z \in \mathbb{D}$, can be interpreted as the transfer function associated to the unitary colligation (4.1), after interchanging the roles of the underlying Hilbert spaces $\mathfrak{H}$ and $\mathcal{H}$; cf., (4.5). Likewise the function $V_*(z) = T^* + G^*(z - H^*)^{-1}F^*$, $z \in \mathbb{D}^*$, can be seen as an extension of $V(z)$ to the exterior $\mathbb{D}^*$; cf., (4.7). If $\zeta \in \mathbb{T}$ and $\operatorname{ran}(\zeta - H)$ (or $\operatorname{ran}(\bar\zeta - H^*)$) is closed, then $V(\zeta)$ and $V_*(\zeta)$ still admit representations analogous to those in (4.14):

$$V(\zeta) = T + F(\zeta - H)^{(-1)}G, \quad V_*(\zeta) = T^* + G^*(\bar\zeta - H^*)^{(-1)}F^*,$$

where the inverse stands for the (Moore-Penrose type) generalized inverse:

$$B^{(-1)} = \{\, \{g, f\} : \{f, g\} \in B,\ f \perp \ker B \,\}.$$

Proposition 4.3. *Let $\Theta(z)$ be the transfer function of the unitary colligation in* (4.1) *and let $\zeta \in \mathbb{T}$. Then:*

(i) $\ker(\zeta - \Theta(z)) = \ker(\bar\zeta - \Theta(z)^*) = \begin{cases} \ker(\zeta - H) = \ker(\bar\zeta - H^*), & z \in \mathbb{D}, \\ \ker(\bar\zeta - H) = \ker(\zeta - H^*), & z \in \mathbb{D}^*; \end{cases}$

(ii) $\operatorname{ran}(\zeta - \Theta(z))$ *is closed if and only if* $\begin{cases} \operatorname{ran}(\zeta - H) \text{ is closed}, & z \in \mathbb{D}, \\ \operatorname{ran}(\bar\zeta - H) \text{ is closed}, & z \in \mathbb{D}^*; \end{cases}$

(iii) $\operatorname{ran}(\zeta - \Theta(z)) = \mathcal{H}$ *if and only if* $\begin{cases} \operatorname{ran}(\zeta - H) = \mathcal{H}, & z \in \mathbb{D}, \\ \operatorname{ran}(\bar\zeta - H) = \mathcal{H}, & z \in \mathbb{D}^*; \end{cases}$

(iv) $\ker(I - \Theta(z)\Theta(z)^*) = \begin{cases} \ker G^*, & z \in \mathbb{D}, \\ \ker F, & z \in \mathbb{D}^*; \end{cases}$

(v) $\ker(I - \Theta(z)^*\Theta(z)) = \begin{cases} \ker F, & z \in \mathbb{D}, \\ \ker G^*, & z \in \mathbb{D}^*; \end{cases}$

(vi) $\operatorname{ran}(I - \Theta(z)\Theta(z)^*)^{\frac{1}{2}} = \begin{cases} \operatorname{ran} G, & z \in \mathbb{D}, \\ \operatorname{ran} F^*, & z \in \mathbb{D}^*; \end{cases}$

(vii) $\operatorname{ran}(I - \Theta(z)^*\Theta(z))^{\frac{1}{2}} = \begin{cases} \operatorname{ran} F^*, & z \in \mathbb{D}, \\ \operatorname{ran} G, & z \in \mathbb{D}^*; \end{cases}$

(viii) *the ranges* $\operatorname{ran}(I - \Theta(z)\Theta(z)^*)$, $\operatorname{ran}(I - \Theta(w)^*\Theta(w))$, $z, w \in \mathbb{D} \cup \mathbb{D}^*$, $\operatorname{ran} G$, *and* $\operatorname{ran} F^*$ *are closed simultaneously;*

(ix) *the equalities* $\operatorname{ran}(I-\Theta(z)\Theta(z)^*)=\mathcal{H}$, $\operatorname{ran}(I-\Theta(w)^*\Theta(w))=\mathcal{H}$, $z,w\in\mathbb{D}\cup\mathbb{D}^*$, $\operatorname{ran}G=\mathcal{H}$, *and* $\operatorname{ran}F^*=\mathcal{H}$ *hold simultaneously.*

Proof. (i) The equalities $\ker(\zeta-\Theta(z))=\ker(\bar\zeta-\Theta(z)^*)$, $z\in\mathbb{D}\cup\mathbb{D}^*$, and $\ker(\zeta-H)=\ker(\bar\zeta-H^*)$, $\ker(\bar\zeta-H)=\ker(\zeta-H^*)$ follow from Lemma 2.1, cf., Lemma 4.1 (iii).

Let $z\in\mathbb{D}$. Then $\ker(\zeta-H)\subset\ker F$ implies $\ker(\zeta-H)\subset\ker(\zeta-\Theta(z))$, cf., (4.5). Conversely, let $h\in\ker(\zeta-\Theta(z))=\ker(\bar\zeta-\Theta(z)^*)$ for some $z\in\mathbb{D}$. Then (4.6) implies $(I-\bar zT^*)^{-1}G^*h=0$, i.e., $G^*h=0$ and by taking the adjoint in (4.5) $h\in\ker(\bar\zeta-H^*)$ follows.

For $z\in\mathbb{D}^*$ the assertion follows in a similar way from (4.7) and (4.8).

(ii) Due to (i) $\mathcal{L}=\ker(\zeta-H)$ ($\mathcal{M}=\ker(\bar\zeta-H)$) is a reducing subspace for H and $\Theta(z)$ for $z\in\mathbb{D}$ ($z\in\mathbb{D}^*$, respectively). Moreover, by Lemma 4.1 (iii) $\mathcal{L},\mathcal{M}\subset\ker F\cap\ker G^*$, in particular, $\operatorname{ran}G\subset\mathcal{H}\ominus\mathcal{L},\mathcal{H}\ominus\mathcal{M}$. Hence, by decomposing the unitary colligation U in (4.1) and its transfer function in (4.5) and (4.7) according to $\mathcal{H}=(\mathcal{H}\ominus\mathcal{L})\oplus\mathcal{L}$ ($\mathcal{H}=(\mathcal{H}\ominus\mathcal{M})\oplus\mathcal{M}$, respectively) as in the proof of Lemma 4.2, the assertion in (ii) reduces to the statement in part (iii), which is proved in the next item.

(iii) For $z=0$ the assertion is trivial, since $\Theta(0)=H$. Fix $\zeta\in\mathbb{T}$ and $z\in\mathbb{D}\setminus\{0\}$, and consider

$$A:=\begin{pmatrix}\frac{1}{z}-T & -F\\ -G & \zeta-H\end{pmatrix}=L\begin{pmatrix}\frac{1}{z}-T & 0\\ 0 & \zeta-\Theta(z)\end{pmatrix}R, \tag{4.15}$$

where

$$L=\begin{pmatrix}I & 0\\ -G(\frac{1}{z}-T)^{-1} & I\end{pmatrix}\quad\text{and}\quad R=\begin{pmatrix}I & -(\frac{1}{z}-T)^{-1}F\\ 0 & I\end{pmatrix}$$

are bounded and boundedly invertible operators. Hence A is boundedly invertible if and only if $\zeta-\Theta(z)$ is boundedly invertible. Now, if $\operatorname{ran}(\zeta-\Theta(z))=\mathcal{H}$, then by (i) $\ker(\zeta-\Theta(z))=\{0\}$ and hence A is boundedly invertible. In particular, $(-G\ \ \zeta-H):\mathfrak{H}\oplus\mathcal{H}\to\mathcal{H}$ is surjective and since

$$0\le GG^*+(\zeta-H)(\zeta-H)^*=2(I-\operatorname{Re}\bar\zeta H)$$

also $I-\operatorname{Re}\bar\zeta H$ is boundedly invertible; cf., (2.8). Consequently $I-\bar\zeta H$ is boundedly invertible, since its real part $I-\operatorname{Re}\bar\zeta H$ is a boundedly invertible positive operator. Thus, $\operatorname{ran}(\zeta-H)=\mathcal{H}$.

Conversely, if $\operatorname{ran}(\zeta-H)=\mathcal{H}$, then by Lemma 4.2 the operator $V(\zeta)$ in (4.12) is unitary and satisfies (4.14). Thus $\frac{1}{z}-V(\zeta)$ is boundedly invertible and therefore also A in (4.15) is boundedly invertible, so that $\operatorname{ran}(\zeta-\Theta(z))=\mathcal{H}$.

For $z\in\mathbb{D}^*$ it follows from (i) and (3.1) that $\operatorname{ran}(\zeta-\Theta(z))=\mathcal{H}$ holds if and only if

$$\mathcal{H}=\operatorname{ran}(\bar\zeta-\Theta(z)^*)=\operatorname{ran}\left(\bar\zeta-\Theta(\tfrac{1}{\bar z})\right)$$

holds. Hence $\operatorname{ran}(\zeta-\Theta(z))=\mathcal{H}$, $z\in\mathbb{D}^*$, if and only if $\operatorname{ran}(\bar\zeta-H)=\mathcal{H}$.

(iv) & (vi) It follows from (4.6) that

$$I-\Theta(z)\Theta(z)^* = (1-z\bar z)G(I-zT)^{-1}(I-\bar zT^*)^{-1}G^*, \quad z\in\mathbb{D}. \tag{4.16}$$

Since T is contractive (4.16) shows that

$$\operatorname{ran}(I-\Theta(z)\Theta(z)^*)^{\frac12} = \operatorname{ran} G(I-zT)^{-1} = \operatorname{ran} G, \quad z\in\mathbb{D}.$$

Similarly (4.8) implies that

$$I-\Theta(z)\Theta(z)^* = (1-1/(z\bar z))F^*\big(I-\tfrac{1}{z}T^*\big)^{-1}\big(I-\tfrac{1}{\bar z}T\big)^{-1}F, \quad z\in\mathbb{D}^*. \tag{4.17}$$

Hence, $\operatorname{ran}(I-\Theta(z)\Theta(z)^*)^{\frac12} = \operatorname{ran} F^*$ for $z\in\mathbb{D}^*$. To get the formula in (iv) for $\ker(I-\Theta(z)\Theta(z)^*)$, $z\in\mathbb{D}\cup\mathbb{D}^*$, take orthogonal complements in (vi).

(v) & (vii) Apply (3.1) and parts (iv) and (vi), respectively; see also (4.10).

(viii) Clearly, $\operatorname{ran}(I-\Theta(z)\Theta(z)^*)$ is closed if and only if $\operatorname{ran}(I-\Theta(z)\Theta(z)^*)^{\frac12}$ is closed; see (2.7). Therefore, the assertion is obtained from Lemma 4.1 (v) and the formulas in parts (vi) and (vii).

(ix) For this apply Lemma 4.1 (vi) and the formulas in parts (vi) and (vii). □

4.2. A functional model for Schur functions

Each $\mathbf{B}(\mathcal{H})$-valued Schur function $\Theta(z)$ can be realized as the transfer function of a unitary colligation U of the form (4.1). The de Branges-Rovnyak model provides a unitary colligation via the reproducing kernel space $\mathfrak{D}(\Theta)$. However, for the present purposes it is more convenient to use the reproducing kernel space $\mathfrak{S}(\Theta)$ for the extended Schur function $\Theta(z)$. The following theorem can be obtained via an identification with the de Branges-Rovnyak model as in (3.19); whereas the de Branges-Rovnyak model is in terms of functions on $\mathbb{D}$ the present model concerns functions defined on $\mathbb{D}\cup\mathbb{D}^*$. For the convenience of the reader a direct proof will be given.

Theorem 4.4. *Let $\Theta(z)$ be an $\mathbf{B}(\mathcal{H})$-valued Schur function and let $\mathfrak{S}(\Theta)$ be the corresponding reproducing kernel Hilbert space. Then*

$$U=\begin{pmatrix} T & F\\ G & H\end{pmatrix} : \begin{pmatrix}\mathfrak{S}(\Theta)\\ \mathcal{H}\end{pmatrix} \to \begin{pmatrix}\mathfrak{S}(\Theta)\\ \mathcal{H}\end{pmatrix}, \tag{4.18}$$

is a closely connected unitary colligation, such that the operators T, F, G, and H have the representation

$$\begin{aligned}
(Th)(z) &:= \begin{cases} \frac{1}{z}(h(z)-h(0)), & z\in\mathbb{D},\\ \frac{1}{z}h(z)-\Theta(z)h(0), & z\in\mathbb{D}^*,\end{cases}\\
(Ff)(z) &:= \begin{cases} \frac{1}{z}(\Theta(z)-\Theta(0))f, & z\in\mathbb{D},\\ (I-\Theta(z)\Theta(0))f, & z\in\mathbb{D}^*,\end{cases}\\
Gh &:= h(0),\\
Hf &:= \Theta(0)f,
\end{aligned} \tag{4.19}$$

for elements $h\in\mathfrak{S}(\Theta)$ and $f\in\mathcal{H}$. The transfer function of U coincides with the function $\Theta(z)$ on $\mathbb{D}$.

The formulas in (4.19) for $z=0$ can be interpreted via limit values due to holomorphy of $h(z)$ and $\Theta(z)$ at $z=0$. Observe that for $w\in\mathbb{D}$ and $h\in\mathfrak{S}(\Theta)$ the definition of the operator T in Theorem 4.4 implies

$$\big((I-wT)^{-1}h\big)(z)=\begin{cases}\frac{zh(z)-wh(w)}{z-w}, & z\in\mathbb{D},\\ \frac{h(z)-\Theta(z)wh(w)}{1-w/z}, & z\in\mathbb{D}^*,\end{cases}\tag{4.20}$$

and

$$\big((I-wT)^{-1}Th\big)(z)=\begin{cases}\frac{h(z)-h(w)}{z-w}, & z\in\mathbb{D},\\ \frac{h(z)/z-\Theta(z)h(w)}{1-w/z}, & z\in\mathbb{D}^*.\end{cases}\tag{4.21}$$

In particular, with $f\in\mathcal{H}$ (4.20) gives rise to the useful formula

$$\big((I-wT)^{-1}Ff\big)(z)=\begin{cases}\frac{\Theta(z)-\Theta(w)}{z-w}f, & z\in\mathbb{D},\\ \frac{I-\Theta(z)\Theta(w)}{1-w/z}f, & z\in\mathbb{D}^*.\end{cases}\tag{4.22}$$

Proof. The proof of Theorem 4.4 consists of four steps. First it will be verified that U in (4.18) is well defined and the adjoints of the operators T, F, G, and H in (4.19) will be calculated. In the second step it will be proved that U is a unitary operator in the Hilbert space $\mathfrak{S}(\Theta)\oplus\mathcal{H}$ and in the third step the closely connectedness of U will be shown. In the last step it will be checked that $\Theta(z)$ is the transfer function of U.

Step 1. By the definition of the Hilbert space $\mathfrak{S}(\Theta)$ the functions of the form

$$z\mapsto h(z):=\mathsf{S}_\Theta(w,z)f+\mathsf{S}_\Theta(\tfrac{1}{w},z)\tilde f,\qquad z\in\mathbb{D}\cup\mathbb{D}^*,\tag{4.23}$$

where $w\in\mathbb{D}\setminus\{0\}$, $f,\tilde f\in\mathcal{H}$, span a dense subset in $\mathfrak{S}(\Theta)$; see (3.16). A straightforward calculation using (3.16) shows that

$$(Th)(z)=\bar w\mathsf{S}_\Theta(w,z)f+\tfrac{1}{\bar w}\mathsf{S}_\Theta(\tfrac{1}{w},z)\tilde f-\mathsf{S}_\Theta(\infty,z)\big(\Theta(w)^*f+\tfrac{1}{\bar w}\tilde f\big)\tag{4.24}$$

holds for all $z\in\mathbb{D}\cup\mathbb{D}^*$ and hence T is well defined on the dense subspace spanned by linear combinations of functions of the form (4.23). Moreover, for h as in (4.23) one verifies the relation

$$\langle Th,Th\rangle=\langle h,h\rangle-(h(0),h(0))\le\langle h,h\rangle\tag{4.25}$$

with the help of (4.24) and the reproducing kernel property. This implies that T in (4.19) is a well defined contraction on $\mathfrak{S}(\Theta)$. Moreover $(Ff)(z)=\mathsf{S}_\Theta(\infty,z)f$ implies $F\in\mathbf{B}(\mathcal{H},\mathfrak{S}(\Theta))$, $G\in\mathbf{B}(\mathfrak{S}(\Theta),\mathcal{H})$ follows also from (4.25) and $H\in\mathbf{B}(\mathcal{H})$ is clear. Therefore U in (4.18) is a well defined operator in $\mathbf{B}(\mathfrak{S}(\Theta)\oplus\mathcal{H})$.

Next the adjoint of the operators T, F, G, and H in (4.19) will be calculated. First of all, by definition $\Theta(\infty)=\Theta(0)^*$, so that $H^*=\Theta(\infty)$. In order to determine the adjoint $G^*\in\mathbf{B}(\mathcal{H},\mathfrak{S}(\Theta))$ of G let $g\in\mathcal{H}$ and let h be as in (4.23). Then

$$\begin{aligned}(Gh,g)=(h(0),g)&=\big(\mathsf{S}_\Theta(w,0)f,g\big)+\big(\mathsf{S}_\Theta(\tfrac{1}{w},0)\tilde f,g\big)\\&=\big(f,(I-\Theta(w)\Theta(0)^*)g\big)+\big(\tilde f,\tfrac{1}{w}\big(\Theta(\tfrac{1}{w})-\Theta(0)^*\big)g\big)\end{aligned}$$

and on the other hand

$$(Gh, g) = \langle h, G^*g\rangle = (f, (G^*g)(w)) + \big(\tilde{f}, (G^*g)(\tfrac{1}{w})\big)$$

holds for arbitrary $f, \tilde{f} \in \mathcal{H}$. Therefore G^* is given by

$$(G^*g)(z) = \begin{cases} (I - \Theta(z)\Theta(\infty))g, & z \in \mathbb{D}, \\ z(\Theta(z) - \Theta(\infty))g, & z \in \mathbb{D}^*, \end{cases} \qquad g \in \mathcal{H}. \tag{4.26}$$

Similarly, for a function h as in (4.23) and $g \in \mathcal{H}$ one verifies

$$\begin{aligned} \langle Fg, h\rangle &= \big(\tfrac{1}{w}(\Theta(w) - \Theta(0))g, f\big) + \big((I - \Theta(\tfrac{1}{w})\Theta(0)\big)g, \tilde{f}\big) \\ &= \big(g, \mathsf{S}_\Theta(w, \infty)f + \mathsf{S}_\Theta(\tfrac{1}{w}, \infty)\tilde{f}\big) = (g, h(\infty)) \end{aligned}$$

and therefore

$$F^*h = h(\infty), \quad h \in \mathfrak{S}(\Theta). \tag{4.27}$$

It remains to calculate $T^* \in \mathbf{B}(\mathfrak{S}(\Theta))$. For this, let again h be as in (4.23) and let

$$k(z) = \mathsf{S}_\Theta(\lambda, z)g + \mathsf{S}_\Theta(\tfrac{1}{\lambda}, z)\tilde{g}, \qquad z \in \mathbb{D} \cup \mathbb{D}^*, \tag{4.28}$$

where $\lambda \in \mathbb{D} \setminus \{0\}$ and $g, \tilde{g} \in \mathcal{H}$. By inserting h from (4.23) into

$$\langle Th, k\rangle = \big(\tfrac{1}{\lambda}(h(\lambda) - h(0)), g\big) + \big(\lambda h(\tfrac{1}{\lambda}) - \Theta(\tfrac{1}{\lambda})h(0), \tilde{g}\big)$$

and making use of the identities

$$\begin{aligned} &w\mathsf{S}_\Theta(\lambda, w) - \Theta(w)\mathsf{S}_\Theta(\lambda, \infty) = \big(\tfrac{1}{\lambda}(\mathsf{S}_\Theta(w, \lambda) - \mathsf{S}_\Theta(w, 0))\big)^*, \\ &\tfrac{1}{w}\big(\mathsf{S}_\Theta(\lambda, \tfrac{1}{w}) - \mathsf{S}_\Theta(\lambda, \infty)\big) = \big(\tfrac{1}{\lambda}\big(\mathsf{S}_\Theta(\tfrac{1}{w}, \lambda) - \mathsf{S}_\Theta(\tfrac{1}{w}, 0)\big)\big)^*, \\ &w\mathsf{S}_\Theta(\tfrac{1}{\lambda}, w) - \Theta(w)\mathsf{S}_\Theta(\tfrac{1}{\lambda}, \infty) = \big(\lambda\mathsf{S}_\Theta(w, \tfrac{1}{\lambda}) - \Theta(\tfrac{1}{\lambda})\mathsf{S}_\Theta(w, 0)\big)^*, \\ &\tfrac{1}{w}\big(\mathsf{S}_\Theta(\tfrac{1}{\lambda}, \tfrac{1}{w}) - \mathsf{S}_\Theta(\tfrac{1}{\lambda}, \infty)\big) = \big(\lambda\mathsf{S}_\Theta(\tfrac{1}{w}, \tfrac{1}{\lambda}) - \Theta(\tfrac{1}{\lambda})\mathsf{S}_\Theta(\tfrac{1}{w}, 0)\big)^*, \end{aligned}$$

it follows that

$$\langle Th, k\rangle = \big(f, wk(w) - \Theta(w)k(\infty)\big) + \big(\tilde{f}, \tfrac{1}{w}\big(k(\tfrac{1}{w}) - k(\infty)\big)\big).$$

On the other hand

$$\langle Th, k\rangle = \langle h, T^*k\rangle = (f, (T^*k)(w)) + \big(\tilde{f}, (T^*k)(\tfrac{1}{w})\big)$$

and therefore T^* is given by

$$(T^*k)(z) = \begin{cases} zk(z) - \Theta(z)k(\infty), & z \in \mathbb{D}, \\ z(k(z) - k(\infty)), & z \in \mathbb{D}^*, \end{cases} \qquad k \in \mathfrak{S}(\Theta). \tag{4.29}$$

Step 2. In order to check that U is unitary it is sufficient to verify the identities (4.2)-(4.3). For $h \in \mathfrak{S}(\Theta)$ it follows from (4.19), (4.29), (4.27) and (4.26) that

$$(TT^*h)(z) = \begin{cases} \frac{1}{z}(zh(z) - \Theta(z)h(\infty) + \Theta(0)h(\infty)), & z \in \mathbb{D}, \\ h(z) - h(\infty) + \Theta(z)\Theta(0)h(\infty), & z \in \mathbb{D}^*, \end{cases}$$

$$(FF^*h)(z) = \begin{cases} \frac{1}{z}(\Theta(z) - \Theta(0))h(\infty), & z \in \mathbb{D}, \\ (I - \Theta(z)\Theta(0))h(\infty), & z \in \mathbb{D}^*, \end{cases}$$

and

$$(T^*Th)(z) = \begin{cases} h(z) - h(0) + \Theta(z)\Theta(\infty)h(0), & z \in \mathbb{D}, \\ z(\frac{1}{z}h(z) - \Theta(z)h(0) + \Theta(\infty)h(0)), & z \in \mathbb{D}^*, \end{cases}$$

$$(G^*Gh)(z) = \begin{cases} (I - \Theta(z)\Theta(\infty))h(0), & z \in \mathbb{D}, \\ z(\Theta(z) - \Theta(\infty))h(0), & z \in \mathbb{D}^*, \end{cases}$$

hold, and this immediately implies $TT^* + FF^* = T^*T + G^*G = I$. Furthermore,

$$HF^*h = -GT^*h = \Theta(0)h(\infty), \quad H^*Gh = -F^*Th = \Theta(\infty)h(0),$$

and

$$\begin{aligned} GG^*f + HH^*f &= (I - \Theta(0)\Theta(\infty))f + \Theta(0)\Theta(\infty)f = f, \\ F^*Ff + H^*Hf &= (I - \Theta(\infty)\Theta(0))f + \Theta(\infty)\Theta(0)f = f, \end{aligned}$$

for all $h \in \mathfrak{S}(\Theta)$ and $f \in \mathcal{H}$. Therefore the identities (4.2)-(4.3) hold and U is a unitary operator in $\mathfrak{S}(\Theta) \oplus \mathcal{H}$.

Step 3. In this step it will be shown that U is closely connected. For $\lambda \in \mathbb{D}$, $k \in \mathfrak{S}(\Theta)$, and $g \in \mathcal{H}$ it is not difficult to check the formulas

$$\big((I - \lambda T^*)^{-1}k\big)(z) = \begin{cases} \frac{k(z) - \lambda\Theta(z)k(1/\lambda)}{1 - z\lambda}, & z \in \mathbb{D}, \\ \frac{k(z)/z - \lambda k(1/\lambda)}{\lambda - 1/w}, & z \in \mathbb{D}^*, \end{cases}$$

and

$$\big((I - \lambda T^*)^{-1}G^*g\big)(z) = \mathsf{S}_\Theta(\bar\lambda, z)g = \begin{cases} \frac{I - \Theta(z)\Theta(1/\lambda)}{1 - z\lambda}g, & z \in \mathbb{D}, \\ \frac{\Theta(z) - \Theta(1/\lambda)}{1/z - \lambda}g, & z \in \mathbb{D}^*. \end{cases} \tag{4.30}$$

Then (4.22) and (4.30) imply

$$\begin{aligned} &\overline{\text{span}}\left\{(I - wT)^{-1}Ff, (I - \lambda T^*)^{-1}G^*g : w, \lambda \in \mathbb{D},\ f, g \in \mathcal{H}\right\} \\ &\quad = \overline{\text{span}}\left\{z \mapsto \mathsf{S}_\Theta(\tfrac{1}{\bar w}, z)f, z \mapsto \mathsf{S}_\Theta(\bar\lambda, z)g : w, \lambda \in \mathbb{D},\ f, g \in \mathcal{H}\right\} = \mathfrak{S}(\Theta). \end{aligned}$$

Step 4. It remains to check that the transfer function of U is given by $\Theta(z)$. Let $w \in \mathbb{D}$. Then for $z \in \mathbb{D}$ and $f \in \mathcal{H}$ according to (4.22)

$$\big((I - wT)^{-1}Ff\big)(z) = \frac{\Theta(z) - \Theta(w)}{z - w}\, f$$

holds and therefore

$$Hf + wG(I - wT)^{-1}Ff = \Theta(0)f + w\frac{\Theta(0) - \Theta(w)}{-w} f = \Theta(w)f, \quad w \in \mathbb{D}.$$

This completes the proof of Theorem 4.4. □

4.3. Some consequences for Nevanlinna families

Let $M(\lambda)$ be a Nevanlinna family in $\mathcal{H}$ and let $\Theta(z)$ be a Schur function such that (3.3), (3.4) hold. From the fact that $\Theta(z)$ can be realized as the transfer function of a unitary colligation as in Theorem 4.4 and from Proposition 4.3 it follows that the expressions in Lemma 3.3 have certain natural invariance properties with respect to λ. In particular, the following statements hold.

Proposition 4.5. *Let $M(\lambda)$ be a Nevanlinna family in $\mathcal{H}$ and let $\Theta(z)$ be a Schur function such that* (3.3), (3.4) *hold. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

(i) $\operatorname{mul} M(\lambda) = \ker(I - H)$;
(ii) $\ker M(\lambda) = \ker(\bar{\mu}/\mu - H)$;
(iii) $\operatorname{dom} M(\lambda)$ *is closed if and only if* $\operatorname{ran}(I - H)$ *is closed;*
(iv) $\operatorname{ran} M(\lambda)$ *is closed if and only if* $\operatorname{ran}(\bar{\mu}/\mu - H))$ *is closed;*
(v) $M(\lambda) \cap M(\lambda)^* = \{0\}$ *if and only if* $\ker G^* = \{0\}$ *or, equivalently,* $\ker F = \{0\}$;
(vi) $M(\lambda) \,\widehat{+}\, M(\lambda)^*$ *is closed if and only if* $\operatorname{ran} G$ *or, equivalently,* $\operatorname{ran} F^*$ *is closed;*
(vii) $M(\lambda) \,\widehat{+}\, M(\lambda)^* = \mathcal{H}^2$ *if and only if* $\operatorname{ran} G = \mathcal{H}$ *or, equivalently,* $\operatorname{ran} F^* = \mathcal{H}$;
(viii) $\ker(M(\lambda) - M(\lambda)^*) = (I - H^*)(\ker G^*) = (I - H)(\ker F)$;
(ix) $\ker(M(\lambda)^{-1} - M(\lambda)^{-*}) = (\mu/\bar{\mu} - H^*)(\ker G^*) = (\bar{\mu}/\mu - H)(\ker F)$.

Proof. The statements (i)–(vii) are immediate from Propositions 3.4, 4.3 by using (3.3) and Theorem 4.4.

(viii) It follows from Lemma 3.3 (v), Proposition 4.3 (iv), (3.12), and (4.5) that for $z \in \mathbb{D}$,

$$\ker\big(M(\lambda) - M(\lambda)^*\big) = (I - \Theta(z)^*) \ker G^* = (I - H^*) \ker G^*.$$

Similarly for $z \in \mathbb{D}^*$ one obtains $\ker(M(\lambda) - M(\lambda)^*) = (I - H) \ker F$. Finally, according to Lemma 4.1 (ii) these two subspaces coincide.

(xi) The proof is completely analogous with the previous item. □

This proposition shows that the various expressions involving $M(\lambda)$ are independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Such invariance properties were obtained in a different manner in [22]. Since $\operatorname{mul} M(\lambda)$ is independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the Nevanlinna family $M(\lambda)$ can be decomposed into the direct orthogonal sum of a Nevanlinna family $M_0(\lambda)$ of densely defined operators in $\mathcal{H}_0 := \mathcal{H} \ominus \operatorname{mul} M(\lambda)$ and the relation $M_\infty = \{0\} \times \operatorname{mul} M(\lambda)$,

$$M(\lambda) = M_0(\lambda) \oplus M_\infty, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.31}$$

In particular, $M(\lambda)$ is an operator function if and only if $\operatorname{dom} M(\lambda)$ is dense for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In this case $M(\lambda)$ will be called a *Nevanlinna function*. The following definition can be found in [22].

Definition 4.6. *A Nevanlinna family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in $\mathcal{H}$ is said to be strict if $M(\lambda) \cap M(\lambda)^* = \{0\}$ for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. A Nevanlinna family $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in $\mathcal{H}$ is said to be uniformly strict if $M(\lambda) \,\widehat{+}\, M(\lambda)^* = \mathcal{H}^2$ for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Since $M_\infty \subset M(\lambda) \cap M(\lambda)^*$ it follows that a strict Nevanlinna family $M(\lambda)$ is a Nevanlinna function. Furthermore, if $M(\lambda)$ is uniformly strict, then $M(\lambda)$ is automatically a $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function. This was shown in [22] and is also implied by later considerations. Note that a $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function $M(\lambda)$ is strict (uniformly strict) if and only if $0 \notin \sigma_p(\operatorname{Im} M(\lambda))$ ($0 \in \rho(\operatorname{Im} M(\lambda))$, respectively) for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

5. Boundary relations and unitary colligations

5.1. A characterization of boundary relations via selfadjoint relations

Let S be a closed symmetric relation in $\mathfrak{H}$ and let $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$, $\operatorname{dom} \Gamma = \mathcal{T}$, be a boundary relation for S^*, cf., Definition 1.1. Then Γ is necessarily closed and $S = \ker \Gamma$ holds. The boundary relation Γ is said to be *minimal* if

$$\mathfrak{H} = \overline{\operatorname{span}} \left\{ \mathfrak{N}_\lambda(\mathcal{T}) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \right\} \tag{5.1}$$

holds, where $\mathfrak{N}_\lambda(\mathcal{T}) = \ker(\mathcal{T} - \lambda)$. Since $\mathfrak{N}_\lambda(\mathcal{T})$ is dense in $\mathfrak{N}_\lambda(S^*)$, it follows that Γ is minimal if and only if S is simple. In this case S is automatically an operator without eigenvalues. Recall the following important result from [22].

Proposition 5.1. *A relation Γ from $\mathfrak{H}^2$ to $\mathcal{H}^2$ with $\ker \Gamma = S$ is a boundary relation for S^* if and only if*

$$\widetilde{A} := \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} \tag{5.2}$$

is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. Moreover, Γ is minimal in the sense of (5.1) *if and only if $\widetilde{A}$ is minimal with respect to $\mathcal{H}$:*

$$\mathfrak{H} \oplus \mathcal{H} = \overline{\operatorname{span}} \left\{ \mathcal{H}, (\widetilde{A} - \lambda)^{-1} \mathcal{H} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}. \tag{5.3}$$

Therefore every selfadjoint relation $\widetilde{A}$ in $\mathfrak{H} \oplus \mathcal{H}$ with $S = \widetilde{A} \cap \mathfrak{H}^2$ yields a boundary relation for S^* and vice versa. Hence for a given symmetric relation S there always exists a Hilbert space $\mathcal{H}$ and a boundary relation $\Gamma \subset \mathfrak{H}^2 \times \mathcal{H}^2$ for S^*. Note that Γ is not unique. Moreover, if $\mathfrak{H}^2$ and $\mathcal{H}^2$ are equipped with the Kreĭn space inner products $(J_\mathfrak{H} \cdot, \cdot)$ and $(J_\mathcal{H} \cdot, \cdot)$, where

$$J_\mathfrak{H} = \begin{pmatrix} 0 & -iI_\mathfrak{H} \\ iI_\mathfrak{H} & 0 \end{pmatrix} \quad \text{and} \quad J_\mathcal{H} = \begin{pmatrix} 0 & -iI_\mathcal{H} \\ iI_\mathcal{H} & 0 \end{pmatrix},$$

then the linear relation Γ from $\mathfrak{H}^2$ to $\mathcal{H}^2$ is a boundary relation for S^* if and only if Γ is a unitary relation from $(\mathfrak{H}^2, (J_\mathfrak{H} \cdot, \cdot))$ to $(\mathcal{H}^2, (J_\mathcal{H} \cdot, \cdot))$ and $\ker \Gamma = S$ holds, cf. [22].

It is easy to see that the Weyl family $M(\lambda)$ of S corresponding to the boundary relation Γ in Definition 1.2 and the selfadjoint relation $\widetilde{A}$ in (5.2) are connected via

$$P_{\mathcal{H}}\big(\widetilde{A}-\lambda\big)^{-1} \upharpoonright \mathcal{H} = -\big(M(\lambda)+\lambda\big)^{-1}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{5.4}$$

a correspondence, which can be regarded as the analogue of the relation (4.11).

Associated with a boundary relation Γ are the linear relations Γ_0 and Γ_1 from $\mathfrak{H}^2$ to $\mathcal{H}$ defined by

$$\Gamma_0 := \big\{\{\widehat{f},h\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{f}=\{f,f'\},\ \widehat{h}=\{h,h'\}\big\} \subset \mathfrak{H}^2 \times \mathcal{H} \tag{5.5}$$

and

$$\Gamma_1 := \big\{\{\widehat{f},h'\} : \{\widehat{f},\widehat{h}\} \in \Gamma,\ \widehat{f}=\{f,f'\},\ \widehat{h}=\{h,h'\}\big\} \subset \mathfrak{H}^2 \times \mathcal{H}. \tag{5.6}$$

It follows immediately from Definition 1.1 (i) that the relations $A_0 := \ker\Gamma_0$ and $A_1 := \ker\Gamma_1$ are symmetric in $\mathfrak{H}$, and, moreover, that $S \subset A_i$, $i=0,1$. Note that if $\rho(A_i) \neq \emptyset$, then

$$\mathcal{T} = A_i \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(\mathcal{T}), \quad \text{direct sum}, \quad \lambda \in \rho(A_i), \quad i=0,1. \tag{5.7}$$

Remark 5.2. *At this stage it is convenient to recall the following definition. Let S be a closed symmetric relation in $\mathfrak{H}$ with equal, not necessarily finite, defect numbers. An* ordinary boundary triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ *for S^* consists of a Hilbert space $\mathcal{H}$ and two linear mappings $\Gamma_0,\Gamma_1 : S^* \to \mathcal{H}$ such that $\Gamma := (\Gamma_0,\Gamma_1) : S^* \to \mathcal{H}\times\mathcal{H}$ is surjective and the abstract Green's identity*

$$(f',g)-(f,g') = (\Gamma_1\widehat{f},\Gamma_0\widehat{g}) - (\Gamma_0\widehat{f},\Gamma_1\widehat{g})$$

holds for all $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\} \in S^*$. See [28] *for the original definition for densely defined symmetric operators and* [25] *for the case of nondensely defined operators. The corresponding Weyl function is defined by*

$$\Gamma_1\widehat{f}_\lambda = M(\lambda)\Gamma_0\widehat{f}_\lambda, \quad f_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{5.8}$$

cf., [25, 26]. *In order to see the connection between boundary triplets and boundary relations, let Γ be a boundary relation for S^* with Weyl family $M(\lambda)$. Then, according to* [22], *the following statements are equivalent:*

(i) $\operatorname{ran}\Gamma = \mathcal{H}^2$;
(ii) $\Gamma = (\Gamma_0,\Gamma_1)$ *is an ordinary boundary triplet for S^*;*
(iii) $M(\lambda)$ *is uniformly strict.*

These equivalences will be further explored in the present section.

5.2. Boundary relations and unitary colligations

Let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces and let $\mu \in \mathbb{C}_+$ be fixed. The Cayley transform in (2.2) provides via $U = C_\mu(\widetilde{A})$ a one-to-one correspondence between all selfadjoint relations $\widetilde{A}$ in $\mathfrak{H}\oplus\mathcal{H}$ and all unitary colligations $U \in \mathbf{B}(\mathfrak{H}\oplus\mathcal{H})$ as in (4.1):

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} = \left\{\left\{\begin{pmatrix} h \\ f \end{pmatrix}, \begin{pmatrix} Th+Ff \\ Gh+Hf \end{pmatrix}\right\} : h \in \mathfrak{H},\ f \in \mathcal{H}\right\}. \tag{5.9}$$

Recall that the operators $T \in \mathbf{B}(\mathfrak{H})$, $F \in \mathbf{B}(\mathcal{H}, \mathfrak{H})$, $G \in \mathbf{B}(\mathfrak{H}, \mathcal{H})$, and $H \in \mathbf{B}(\mathcal{H})$ have the properties (4.2)–(4.3). Let the selfadjoint relation $\widetilde{A}$ and the unitary colligation U in (5.9) be connected via the Cayley transform: $U = C_\mu(\widetilde{A})$. According to (2.3) the selfadjoint relation $\widetilde{A}$ admits the representation

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} (T-I)h + Ff \\ Gh + (H-I)f \end{pmatrix}, \begin{pmatrix} (\mu T - \bar{\mu})h + \mu F f \\ \mu G h + (\mu H - \bar{\mu}) f \end{pmatrix} \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\}, \tag{5.10}$$

or, since U is unitary, equivalently,

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} (I-T^*)h - G^* f \\ -F^* h + (I-H^*)f \end{pmatrix}, \begin{pmatrix} (\mu - \bar{\mu}T^*)h - \bar{\mu} G^* f \\ -\bar{\mu} F^* h + (\mu - \bar{\mu} H^*) f \end{pmatrix} \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\}. \tag{5.11}$$

Lemma 5.3. *Let the selfadjoint relation $\widetilde{A}$ and the unitary colligation U in* (5.9) *be connected via the Cayley transform: $U = C_\mu(\widetilde{A})$. Then $\widetilde{A}$ is minimal in the sense of* (5.3) *if and only if U is closely connected in the sense of* (4.9).

Proof. Use (5.10) to determine $P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{H}$ for $\lambda \in \mathbb{C}_+$. Note that

$$(\mu - \lambda)Ff + (\mu - \lambda)Th + (\lambda - \bar{\mu})h = 0, \quad \lambda \in \mathbb{C}_+,$$

is equivalent to $h = z(I - zT)^{-1}Ff$, $z = (\lambda - \mu)/(\lambda - \bar{\mu})$. Therefore

$$\begin{aligned} P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{H} &= \operatorname{span}\left\{ Ff + z(T-I)(I-zT)^{-1}Ff : f \in \mathcal{H} \right\} \\ &= \operatorname{span}\left\{ (I - zT)^{-1}Ff : f \in \mathcal{H} \right\} \end{aligned} \tag{5.12}$$

for all $\lambda \in \mathbb{C}_+$. Similarly, use (5.11) to determine $P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{H}$ for $\lambda \in \mathbb{C}_-$. Here

$$(\lambda - \bar{\mu})G^* f + (\lambda - \bar{\mu})T^* h + (\mu - \lambda)h = 0, \quad \lambda \in \mathbb{C}_-,$$

is equivalent to $h = z^{-1}(I - z^{-1}T^*)^{-1}G^* f$. Thus

$$\begin{aligned} P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{H} &= \operatorname{span}\left\{ -G^* f + z^{-1}(I - T^*)(I - z^{-1}T^*)^{-1}G^* f : f \in \mathcal{H} \right\} \\ &= \operatorname{span}\left\{ (I - z^{-1}T^*)^{-1}G^* f : f \in \mathcal{H} \right\} \end{aligned} \tag{5.13}$$

for all $\lambda \in \mathbb{C}_-$. Now (5.12) and (5.13) imply

$$\begin{aligned} &\overline{\operatorname{span}}\left\{ P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{H} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\} \\ &\quad = \overline{\operatorname{span}}\left\{ \operatorname{ran}(I - zT)^{-1}F, \operatorname{ran}(I - wT^*)^{-1}G^* : z, w \in \mathbb{D} \right\}. \end{aligned}$$

Therefore the conditions (5.3) and (4.9) are equivalent. □

Let $\widetilde{A}$ be a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. Define the relation Γ from $\mathfrak{H}^2$ to $\mathcal{H}^2$ by

$$\Gamma := \left\{ \left\{ \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}, \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \varphi' \\ -\psi' \end{pmatrix} \right\} \in \widetilde{A} \right\}, \tag{5.14}$$

cf., (5.2). Furthermore, define the relations $\mathcal{T}$ and S in $\mathfrak{H}$ by

$$\mathcal{T} = \operatorname{dom}\Gamma \quad \text{and} \quad S = \ker\Gamma = \widetilde{A} \cap \mathfrak{H}^2. \tag{5.15}$$

The relations Γ, $\mathcal{T}$, and S will now be expressed in terms of the unitary colligation $U = C_\mu(\widetilde{A})$ in (5.9). In view of (5.10), (5.11) Γ in (5.14) has the representation

$$\Gamma = \left\{ \left\{ \begin{pmatrix} (T-I)h + Ff \\ (\mu T - \bar{\mu})h + \mu F f \end{pmatrix}, \begin{pmatrix} Gh + (H-I)f \\ -\mu G h - (\mu H - \bar{\mu}) f \end{pmatrix} \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\} \tag{5.16}$$

or, equivalently,

$$\Gamma = \left\{ \left\{ \begin{pmatrix} (I-T^*)h - G^* f \\ (\mu - \bar{\mu}T^*)h - \bar{\mu} G^* f \end{pmatrix}, \begin{pmatrix} -F^* h + (I-H^*)f \\ \bar{\mu} F^* h - (\mu - \bar{\mu} H^*) f \end{pmatrix} \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\}. \tag{5.17}$$

The relation $\mathcal{T} = \operatorname{dom}\Gamma$ has the representation

$$\mathcal{T} = \left\{ \{(T-I)h + Ff, (\mu T - \bar{\mu})h + \mu F f\} :\ h \in \mathfrak{H},\ f \in \mathcal{H} \right\}, \tag{5.18}$$

or, equivalently,

$$\mathcal{T} = \left\{ \{(I-T^*)h - G^* f, (\mu - \bar{\mu}T^*)h - \bar{\mu} G^* f\} :\ h \in \mathfrak{H},\ f \in \mathcal{H} \right\}. \tag{5.19}$$

Finally, the relation $S = \widetilde{A} \cap \mathfrak{H}^2$ has the representation

$$S = \left\{ \{(T-I)h, (\mu T - \bar{\mu})h\} :\ h \in \ker G \right\}, \tag{5.20}$$

or, equivalently,

$$S = \left\{ \{(I-T^*)h, (\mu - \bar{\mu}T^*)h\} :\ h \in \ker F^* \right\}. \tag{5.21}$$

All these representations will be used in the proof of the following theorem.

Theorem 5.4. *Let the selfadjoint relation $\widetilde{A}$ and the unitary colligation U in* (5.9) *be connected via the Cayley transform: $U = C_\mu(\widetilde{A})$. Let Γ be defined by* (5.14) *and let $\mathcal{T}$ and S be defined by* (5.15). *Let $\mu \in \mathbb{C}_+$ and let $z(\lambda)$ be as in* (3.2). *Then the following statements hold:*

(i) *The relation $S = \widetilde{A} \cap \mathfrak{H}^2$ is closed and symmetric, and the linear relation $\mathcal{T} = \operatorname{dom}\Gamma$ is dense in S^*, so that $S^* = \overline{\mathcal{T}}$ and $S = \mathcal{T}^*$.*

(ii) *The defect space $\mathfrak{N}_\lambda(\mathcal{T}) = \ker(\mathcal{T} - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is given by*

$$\mathfrak{N}_\lambda(\mathcal{T}) = \begin{cases} \operatorname{ran}(I - zT)^{-1}F, & \lambda \in \mathbb{C}_+, \\ \operatorname{ran}(I - z^{-1}T^*)^{-1}G^*, & \lambda \in \mathbb{C}_-. \end{cases}$$

(iii) *The relation Γ in* (5.14) *is a boundary relation for S^*. It is minimal (or, equivalently, S is simple) if and only if the unitary colligation in $U = C_\mu(\widetilde{A})$ in* (5.9) *is closely connected, i.e., $\widetilde{A}$ is minimal in the sense of* (5.3).

(iv) *If $\Theta(z)$ is the transfer function of the unitary colligation $U = C_\mu(\widetilde{A})$, then the Weyl family $M(\lambda)$ corresponding to the boundary relation Γ is given by* (3.3). *In particular, $M(\lambda)$ is a Nevanlinna family.*

Proof. (i) holds by Proposition 5.1. However, for the convenience of the reader a short direct proof which makes use of the unitarity of $U = C_\mu(\widetilde{A})$ will be presented.

It follows from the identity $T^*T + G^*G = I$ that S is symmetric:

$$((\mu T - \bar{\mu})h, (T - I)h) \in \mathbb{R}, \quad h \in \ker G.$$

It is straightforward to verify that $\{k, l\} \in \mathcal{T}^*$ if and only if

$$T^*(l - \bar{\mu}k) = l - \mu k \quad \text{and} \quad F^*(l - \bar{\mu}k) = 0. \tag{5.22}$$

Since the colligation (4.1) is unitary, it follows from (4.2) and (5.22) that $S \subset \mathcal{T}^*$. To see the reverse inclusion, let $\{k, l\} \in \mathcal{T}^*$. By the second condition in (5.22) it follows that $l - \bar{\mu}k = Th$ with $h \in \ker G$; cf., Lemma 4.1 (i). But then (4.2) and the first condition in (5.22) imply that $l - \mu k = h$. Hence,

$$(\mu - \bar{\mu})k = (T - I)h, \quad (\mu - \bar{\mu})l = (\mu T - \bar{\mu})h,$$

and now (5.20) shows that $\{k, l\} \in S$. Therefore, $S = \mathcal{T}^*$ and (i) is proven.

(ii) It follows from (5.18) that an element $(T - I)h + Ff$ belongs to $\ker(\mathcal{T} - \lambda)$ if and only if

$$(\mu T - \bar{\mu})h + \mu F f = \lambda(T - I)h + \lambda F f.$$

For $\lambda \in \mathbb{C}_+$ this is equivalent to

$$h = z(I - zT)^{-1} F f \tag{5.23}$$

and therefore

$$(T - I)h + Ff = (1 - z)(I - zT)^{-1} F f, \quad \lambda \in \mathbb{C}_+,$$

i.e., $\mathfrak{N}_\lambda(\mathcal{T}) = \operatorname{span}\{(1 - z(\lambda)T)^{-1} F f : f \in \mathcal{H}\}$ for $\lambda \in \mathbb{C}_+$. Likewise, it follows from (5.19) that $(I - T^*)h - G^* f$ belongs to $\ker(\mathcal{T} - \lambda)$, $\lambda \in \mathbb{C}_-$, if and only if

$$h = z^{-1}(I - z^{-1}T^*)^{-1} G^* f, \quad \lambda \in \mathbb{C}_-. \tag{5.24}$$

This implies the remaining assertions in (ii).

(iii) It follows from Proposition 5.1 that Γ in (5.16) is a boundary relation for S^*. In view of (4.9) and item (ii) Γ is minimal if and only if U is closely connected, or equivalently, by Lemma 5.3 if and only if $\widetilde{A}$ is minimal with respect to $\mathcal{H}$.

(iv) According to part (ii) of the proof an element

$$\big\{(T - I)h + Ff, (\mu T - \bar{\mu})h + \mu F f\big\} \in \mathcal{T}$$

belongs to $\widehat{\mathfrak{N}}_\lambda(\mathcal{T})$, $\lambda \in \mathbb{C}_+$, if and only if (5.23) holds. Hence for $\lambda \in \mathbb{C}_+$ the definitions of the Weyl family in (1.2) and the transfer function in (4.5) together with the form of the boundary relation Γ in (5.16) imply that

$$M(\lambda) = \big\{\{(I - \Theta(z))f, (\mu\Theta(z) - \bar{\mu})f\} : f \in \mathcal{H}\big\}, \quad \lambda \in \mathbb{C}_+.$$

A similar calculation based on (5.24) and (5.17) shows that for $\lambda \in \mathbb{C}_-$ the Weyl family $M(\lambda)$ is given by the second row in (3.3). □

5.3. Some observations concerning boundary relations

The representations of the boundary relation Γ in (5.16) and (5.17) will be useful. First the multi-valued part $\operatorname{mul}\Gamma$ of Γ will be described in terms of the unitary colligation in (5.9).

Lemma 5.5. *The multi-valued part* $\operatorname{mul}\Gamma$ *is given by*

$$\operatorname{mul}\Gamma = \left\{ \begin{pmatrix} (H-I)f \\ (\bar\mu - \mu H)f \end{pmatrix} : f \in \ker F \right\} = \left\{ \begin{pmatrix} (I-H^*)f \\ (\bar\mu H^* - \mu)f \end{pmatrix} : f \in \ker G^* \right\}.$$

Proof. The definition of Γ in (5.16) implies

$$\begin{pmatrix} Gh + (H-I)f \\ -\mu Gh - (\mu H - \bar\mu)f \end{pmatrix} \in \operatorname{mul}\Gamma$$

if and only if $(T-I)h + Ff = 0$ and $(\mu T - \bar\mu)h + \mu Ff = 0$, and this is equivalent to $h = 0$ and $f \in \ker F$. Similarly (5.17) implies the second equality. $\square$

The relations Γ_0 and Γ_1 were defined in (5.5) and (5.6). The representations of the boundary relation Γ in (5.16) and (5.17) lead to representations for Γ_0 and Γ_1:

$$\begin{aligned} \Gamma_0 &= \left\{ \left\{ \begin{pmatrix} (T-I)h + Ff \\ (\mu T - \bar\mu)h + \mu Ff \end{pmatrix}, Gh + (H-I)f \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\} \\ &= \left\{ \left\{ \begin{pmatrix} (I-T^*)h - G^*f \\ (\mu - \bar\mu T^*)h - \bar\mu G^* f \end{pmatrix}, -F^*h + (I-H^*)f \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\}, \end{aligned} \tag{5.25}$$

$$\begin{aligned} \Gamma_1 &= \left\{ \left\{ \begin{pmatrix} (T-I)h + Ff \\ (\mu T - \bar\mu)h + \mu Ff \end{pmatrix}, -\mu Gh - (\mu H - \bar\mu)f \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\} \\ &= \left\{ \left\{ \begin{pmatrix} (I-T^*)h - G^*f \\ (\mu - \bar\mu T^*)h - \bar\mu G^* f \end{pmatrix}, \bar\mu F^*h - (\mu - \bar\mu H^*)f \right\} : h \in \mathfrak{H},\ f \in \mathcal{H} \right\}. \end{aligned} \tag{5.26}$$

The representations of the corresponding kernels $A_0 = \ker\Gamma_0$ and $A_1 = \ker\Gamma_1$ follow from (5.25) and (5.26), respectively:

$$\begin{aligned} A_0 &= \left\{ \{(T-I)h + Ff, (\mu T - \bar\mu)h + \mu Ff\} : Gh + (H-I)f = 0 \right\} \\ &= \left\{ \{(T^*-I)h + G^*f, (\bar\mu T^* - \mu)h + \bar\mu G^* f\} : F^*h + (H^*-I)f = 0 \right\}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} A_1 &= \left\{ \{(T-I)h + Ff, (\mu T - \bar\mu)h + \mu Ff\} : \mu Gh + (\mu H - \bar\mu)f = 0 \right\} \\ &= \left\{ \{(T^*-I)h + G^*f, (\bar\mu T^* - \mu)h + \bar\mu G^* f\} : \bar\mu F^*h + (\bar\mu H^* - \mu)f = 0 \right\}. \end{aligned} \tag{5.28}$$

The next result identifies the Cayley transforms of the relations A_0 and A_1 in (5.27) and (5.28). For this purpose, recall the definition of the isometric operators $V(\zeta)$ and $V_*(\zeta)$, $\zeta \in \mathbb{T}$, in (4.12) and (4.13).

Lemma 5.6. *Let $\mu \in \mathbb{C}_+$. Then the Cayley transforms of A_0 and A_1 are given by*

$$C_\mu(A_0) = V(1), \quad C_{\bar\mu}(A_0) = V_*(1), \tag{5.29}$$

and by

$$C_\mu(A_1) = V(\bar\mu/\mu), \quad C_{\bar\mu}(A_1) = V_*(\bar\mu/\mu). \tag{5.30}$$

5.4. Special boundary relations

Various subclasses of the class of Weyl families have been characterized in terms of the boundary relation in [22]. Here the connection with the corresponding unitary colligation is explored. Let the selfadjoint relation $\widetilde{A}$ and the unitary colligation U in (5.9) be connected via the Cayley transform $U = C_\mu(\widetilde{A})$, where $\mu \in \mathbb{C}_+$. Let Γ be defined by (5.14) and let $\mathcal{T}$ and S be defined by (5.15). Let $M(\lambda)$ and $\Theta(z)$ be connected via (3.3) and (3.4), where $z(\lambda)$ is as in (3.2).

Proposition 5.7. *The following statements are equivalent:*

(i) $\operatorname{mul}\Gamma = \{0\}$ *or, equivalently,* $\operatorname{ran}\Gamma$ *is dense;*
(ii) $\ker F = \{0\}$ *or, equivalently,* $\ker G^* = \{0\}$*;*
(iii) $M(\lambda)$ *is strict for some, and hence for all,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. (i) $\Leftrightarrow$ (ii) This follows immediately from Lemma 5.5.
(ii) $\Leftrightarrow$ (iii) This holds by Proposition 4.5 (v). □

Observe, that the equivalence (i) $\Leftrightarrow$ (iii) was proved in a different manner in [22, Proposition 4.5]. Similar comments hold for the next two propositions; see in particular [22, Propositions 4.15, 4.16 and Corollary 4.17]. For completeness, full proofs are presented here using the present approach.

Proposition 5.8. *The following statements are equivalent:*

(i) $\operatorname{ran}\Gamma_0$ *is closed and* $A_0 = \ker\Gamma_0$ *is selfadjoint;*
(ii) $\operatorname{ran}(I-H)$ *is closed;*
(iii) *the operator part* $M_0(\lambda)$ *in* (4.31) *is bounded.*

In this case $\operatorname{ran}\Gamma_0 = \operatorname{dom} M(\lambda)$ *for all* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. (i) $\Leftrightarrow$ (ii) If A_0 is selfadjoint then $\mathcal{T} = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(\mathcal{T})$ holds for all $\lambda \in \rho(A_0)$; see (5.7). This implies that $\operatorname{ran}\Gamma_0 = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(\mathcal{T})) = \operatorname{dom} M(\lambda)$. This subspace is closed if and only if $\operatorname{ran}(I-H)$ is closed by Proposition 4.5 (iii). Thus, (i) implies (ii).

Conversely, assume that $\operatorname{ran}(I-H)$ is closed. Then by Lemma 4.2 the isometric operator $V(1)$ is unitary and Lemma 5.6 shows that its inverse Cayley transform A_0 is selfadjoint. Moreover, the first part of the proof shows that $\operatorname{ran}\Gamma_0$ is closed.

(ii) $\Leftrightarrow$ (iii) By Proposition 4.5 (iii) $\operatorname{dom} M(\lambda)$ is closed if and only if $\operatorname{ran}(I-H)$ is closed. In view of (4.31) this means that the operator part $M_0(\lambda)$ of $M(\lambda)$ is a closed bounded operator on $\operatorname{dom} M(\lambda)$. □

In the next proposition bounded Nevanlinna functions are characterized. The equivalence between (ii) and (iii) goes back to [17]; the equivalence between (i) and (iii) is known from [22].

Proposition 5.9. *The following statements are equivalent:*

(i) $\operatorname{ran}\Gamma_0 = \mathcal{H}$ *and* $A_0 = \ker\Gamma_0$ *is selfadjoint;*
(ii) $\operatorname{ran}(I-H) = \mathcal{H}$ *or, equivalently,* $(I-H)^{-1} \in \mathbf{B}(\mathcal{H})$*;*
(iii) $\operatorname{dom} M(\lambda) = \mathcal{H}$ *or, equivalently,* $M(\lambda) \in \mathbf{B}(\mathcal{H})$*, for some, and hence for all,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. It follows from (i) and (iii) in Proposition 4.5 that $\operatorname{dom} M(\lambda) = \mathcal{H}$ if and only if $\operatorname{ran}(I-H) = \mathcal{H}$. Hence, the result is obtained from Proposition 5.8. □

Proposition 5.10. *The following statements are equivalent:*

(i) $\operatorname{dom}\Gamma$ *is closed or, equivalently,* $\operatorname{ran}\Gamma$ *is closed;*
(ii) $\operatorname{ran} G$ *is closed or, equivalently,* $\operatorname{ran} F^*$ *is closed;*
(iii) $M(\lambda) \,\widehat{+}\, M(\lambda)^*$ *is closed for some, and hence for all,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$*;*
(iv) $\mathfrak{N}_\lambda(\mathcal{T})$ *is closed for some, and hence for all,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. (i) ⇔ (ii) It follows directly from the description of Γ in (5.16) that $\mathcal{T} = \operatorname{dom}\Gamma$ is closed if and only if $\operatorname{ran} F$ is closed; and that $\operatorname{ran}\Gamma$ is closed if and only if $\operatorname{ran} G$ is closed. Now apply Lemma 4.1 (v).

(ii) ⇔ (iii) This holds by Proposition 4.5 (vi).

(ii) ⇔ (iv) By Theorem 5.4 (ii) $\mathfrak{N}_\lambda(\mathcal{T})$ is closed for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) if and only if $\operatorname{ran} F$ ($\operatorname{ran} G^*$, respectively) is closed. It remains to apply Lemma 4.1 (v). □

Again the equivalence (i) ⇔ (iii) was established in a different manner in [22, Lemma 4.4]. By combining Propositions 5.7 and 5.8 one obtains

Corollary 5.11. *The following statements are equivalent:*

(i) $\operatorname{ran}\Gamma$ *is dense,* $\operatorname{ran}\Gamma_0$ *is closed, and* $A_0 = \ker\Gamma_0$ *is selfadjoint;*
(ii) $(I-H)^{-1} \in \mathbf{B}(\mathcal{H})$ *and* $\ker F = \{0\}$ *or, equivalently,* $\ker G^* = \{0\}$*;*
(iii) $M(\lambda)$ *is strict and belongs to* $\mathbf{B}(\mathcal{H})$.

The next result describes the situation of an ordinary boundary triplet, cf., Remark 5.2. It is obtained here by combining Propositions 5.7 and 5.10; see also Proposition 4.5 (vii).

Corollary 5.12. *The following statements are equivalent:*

(i) $\operatorname{ran}\Gamma = \mathcal{H}^2$*;*
(ii) $\operatorname{ran} G = \mathcal{H}$*, or equivalently,* $\operatorname{ran} F^* = \mathcal{H}$*;*
(iii) $M(\lambda)$ *is uniformly strict (and belongs to* $\mathbf{B}(\mathcal{H})$*) for some, and hence for all,* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

In the case that $M(\lambda) \in \mathbf{B}(\mathcal{H})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, one obtains the following operator representation, which is based on the unitary colligation in (4.1) and the operator $V(1)$ in Lemma 4.2.

Proposition 5.13. *If any of the conditions in Proposition* 5.9 *is satisfied, then the Weyl function* $M(\lambda) \in \mathbf{B}(\mathcal{H})$ *has the following representation:*

$$M(\lambda) = M(\mu)^* + (\mu - \bar{\mu})(I - H^*)^{-1} F^* \big(I - zV(1)\big)^{-1} F (I - H)^{-1}. \tag{5.31}$$

Here the unitary operator $V(1) = T + F(I - H)^{-1} G$ *is the Cayley transform of the selfadjoint relation* $A_0 = \ker \Gamma_0$, $V(1) = C_\mu(A_0)$, *so that*

$$M(\lambda) = M(\mu)^* + (I - H^*)^{-1} F^* (\lambda - \bar{\mu}) \big(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\big) F (I - H)^{-1}. \tag{5.32}$$

Proof. It follows from the description of $M(\lambda)$ in (3.3) that for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} M(\lambda) - M(\mu)^* &= (H^* - I)^{-1} (\bar{\mu} H^* - \mu) - (\mu \Theta(z) - \bar{\mu})(\Theta(z) - I)^{-1} \\ &= (\mu - \bar{\mu})(I - H^*)^{-1} \big(H^* \Theta(z) - I\big)(\Theta(z) - I)^{-1} \\ &= -(\mu - \bar{\mu})(I - H^*)^{-1} F^* (I - zT)^{-1} F (\Theta(z) - I)^{-1}, \end{aligned} \tag{5.33}$$

where the last identity is obtained by using (4.2). Note that

$$(\Theta(z) - I)^{-1} = -\left(I - z(I - H)^{-1} G (I - zT)^{-1} F\right)^{-1} (I - H)^{-1}. \tag{5.34}$$

Now the term $(I - zT)^{-1} F$ in (5.33) will be rewritten. By the definition of $V(1)$

$$\begin{aligned} I - zV(1) &= I - zT - zF(I - H)^{-1} G \\ &= \big(I - zF(I - H)^{-1} G (I - zT)^{-1}\big)(I - zT), \end{aligned}$$

so that

$$(I - zV(1))(I - zT)^{-1} = I - zF(I - H)^{-1} G (I - zT)^{-1}.$$

Hence

$$\begin{aligned} (I - zV(1))(I - zT)^{-1} F &= \big(I - zF(I - H)^{-1} G (I - zT)^{-1}\big) F \\ &= F\big(I - z(I - H)^{-1} G (I - zT)^{-1} F\big). \end{aligned} \tag{5.35}$$

Now, combine (5.33), (5.34), and (5.35) to obtain (5.31).

In order to obtain (5.32), note that it follows from (5.27) that

$$A_0 - \lambda = \left\{ \, (V(1) - I)h, (\lambda - \bar{\mu})(I - z(\lambda)V(1))h\} : \, h \in \mathfrak{H} \, \right\},$$

which leads to

$$(\mu - \bar{\mu})(I - z(\lambda)V(1))^{-1} = (\lambda - \bar{\mu})\big(I + (\lambda - \mu)(A_0 - \lambda)^{-1}\big). \tag{5.36}$$

Now (5.32) is obtained by substituting (5.36) in (5.31). □

6. Boundary relations and Nevanlinna families

The connection between boundary relations and their Weyl families on the one hand, and unitary colligations and their transfer functions on the other hand is now used in conjunction with the realization of Schur functions as transfer functions via the de Branges-Rovnyak model.

6.1. A functional model for Nevanlinna families

It will be shown that each Nevanlinna family can be realized as the Weyl family of the multiplication operator in a reproducing kernel Hilbert space $\mathfrak{H}(A,B)$ and a certain boundary relation for its adjoint.

Theorem 6.1. *Let $M(\lambda)$ be a Nevanlinna family in $\mathcal{H}$, let $\{A(\lambda),B(\lambda)\}$ be a symmetric Nevanlinna pair such that $M(\lambda)=\{\{A(\lambda)g,B(\lambda)g\}: g\in\mathcal{H}\}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and let $\mathfrak{H}(A,B)$ be the corresponding reproducing kernel Hilbert space. Then:*

(i) *the linear relation*

$$S=\big\{\,\{\varphi,\psi\}\in\mathfrak{H}(A,B)^2:\,\psi(\lambda)=\lambda\varphi(\lambda)\,\big\} \tag{6.1}$$

is a closed simple symmetric operator in $\mathfrak{H}(A,B)$;

(ii) *the linear relation*

$$\mathcal{T}=\big\{\,\{\varphi,\psi\}\in\mathfrak{H}(A,B)^2:\,\psi(\lambda)-\lambda\varphi(\lambda)=A(\lambda)c_1+B(\lambda)c_2,\ c_1,c_2\in\mathcal{H}\,\big\}$$

is dense in S^*;

(iii) *the linear relation*

$$\Gamma=\left\{\left\{\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\begin{pmatrix}c_2\\ -c_1\end{pmatrix}\right\}:\,\{\varphi,\psi\}\in\mathcal{T}\right\} \tag{6.2}$$

is a minimal boundary relation for S^* whose Weyl family is equal to M.

Proof. The proof involves a reduction via the Cayley transform to the case of Schur functions. It consists of several steps. In the following $\mu\in\mathbb{C}_+$ is fixed.

Step 1. To the Nevanlinna family M associate a Schur function Θ via the formula (3.4), so that M can be recovered from Θ via (3.3). It suffices to prove Theorem 6.1 for the case that the Nevanlinna pair $\{A,B\}$ is given by (3.12). In fact, if $\{A',B'\}$ is a second symmetric Nevanlinna pair representing M then there exists an $\mathbf{B}(\mathcal{H})$-valued function χ, $\chi(\lambda)^{-1}\in\mathbf{B}(\mathcal{H})$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, such that $A'(\lambda)=\chi(\lambda)A(\lambda)$ and $B'(\lambda)=\chi(\lambda)B(\lambda)$. Then the considerations in Section 3.2 (see (3.25)) imply that $S'=\{\{\varphi',\psi'\}\in\mathfrak{H}(A',B'):\psi'(\lambda)=\lambda\varphi'(\lambda)\}$ is a closed simple symmetric operator in $\mathfrak{H}(A',B')$ if and only if S in (6.1) is a closed simple symmetric operator in $\mathfrak{H}(A,B)$. Similarly,

$$\mathcal{T}'=\big\{\,\{\varphi',\psi'\}\in\mathfrak{H}(A',B')^2:\,\psi'(\lambda)-\lambda\varphi'(\lambda)=A'(\lambda)c_1+B'(\lambda)c_2,\ c_1,c_2\in\mathcal{H}\,\big\}$$

is dense in S'^* if and only if $\mathcal{T}$ in (ii) is dense in S^*, and

$$\Gamma'=\left\{\left\{\begin{pmatrix}\varphi'\\ \psi'\end{pmatrix},\begin{pmatrix}c_2\\ -c_1\end{pmatrix}\right\}:\,\{\varphi',\psi'\}\in\mathcal{T}'\right\}$$

is a minimal boundary relation for S'^* if and only if Γ in (6.2) is a minimal boundary relation for S^* in (6.1). Furthermore, the corresponding Weyl families coincide.

Let $\mathfrak{S}(\Theta)$ be the reproducing kernel Hilbert space corresponding to the Schur function Θ in (3.4) and let

$$U = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{S}(\Theta) \\ \mathcal{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{S}(\Theta) \\ \mathcal{H} \end{pmatrix}$$

be the closely connected unitary colligation constructed in Theorem 4.4 such that Θ is the transfer function of U. In the following the linear relations in (5.20) and (5.18) will be denoted by $S_{\mathfrak{S}(\Theta)}$ and $\mathcal{T}_{\mathfrak{S}(\Theta)}$ instead of S and $\mathcal{T}$, respectively. Making use of the particular form of the operators T, F, G, and H in Theorem 4.4 it follows that the elements

$$\big\{\{(T-I)h + Ff, (\mu T - \bar{\mu})h + \mu F f\} : h \in \mathfrak{S}(\Theta),\ f \in \mathcal{H}\big\}$$

belonging to $\mathcal{T}_{\mathfrak{S}(\Theta)}$ have the form

$$\big((T-I)h + Ff\big)(z) = \begin{cases} \frac{h(z)-h(0)}{z} - h(z) + \frac{\Theta(z)-\Theta(0)}{z} f, & z \in \mathbb{D}, \\ \frac{1}{z} h(z) - \Theta(z)h(0) - h(z) + (I - \Theta(z)\Theta(0))\, f, & z \in \mathbb{D}^*, \end{cases}$$

and

$$\begin{aligned} &\big((\mu T - \bar{\mu})h + \mu F f\big)(z) \\ &\qquad = \begin{cases} \mu \frac{h(z)-h(0)}{z} - \bar{\mu} h(z) + \mu \frac{\Theta(z)-\Theta(0)}{z} f, & z \in \mathbb{D}, \\ \mu \frac{1}{z} h(z) - \mu\Theta(z)h(0) - \bar{\mu} h(z) + \mu\,(I - \Theta(z)\Theta(0))\, f, & z \in \mathbb{D}^*. \end{cases} \end{aligned}$$

The elements $\{(T-I)h, (\mu T - \bar{\mu})h\}$, $h \in \ker G$, in the closed symmetric relation $S_{\mathfrak{S}(\Theta)}$ satisfy $h(0) = 0$ and they have the form

$$\big((T-I)h\big)(z) = \frac{h(z)}{z} - h(z), \qquad z \in \mathbb{D} \cup \mathbb{D}^*,$$

$$\big((\mu T - \bar{\mu})h\big)(z) = \mu \frac{h(z)}{z} - \bar{\mu} h(z), \qquad z \in \mathbb{D} \cup \mathbb{D}^*.$$

Step 2. Let $\{A, B\}$ be the Nevanlinna pair in (3.12), let $\varphi, \psi \in \mathfrak{H}(A, B)$ and assume that

$$\psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{6.3}$$

holds for some $c_1, c_2 \in \mathcal{H}$. Then define

$$\Phi(z) := r(\lambda)\varphi(\lambda), \quad \Psi(z) := r(\lambda)\psi(\lambda), \qquad z = \frac{\lambda - \mu}{\lambda - \bar{\mu}}, \tag{6.4}$$

where r is the unitary mapping from $\mathfrak{H}(A, B)$ onto $\mathfrak{S}(\Theta)$ from (3.22). It will be shown that the formula

$$\big((I-T)\Psi\big)(z) + \big((\mu T - \bar{\mu})\Phi\big)(z) = \big(F(c_1 - \mu c_2)\big)(z) \tag{6.5}$$

holds for all $z \in \mathbb{D} \cup \mathbb{D}^*$.

In fact, for $z \in \mathbb{D}$ (6.3), (6.4), (3.12), together with $r(\lambda)$ in (3.22) imply

$$(z-1)\Psi(z) - (\bar{\mu}z - \mu)\Phi(z) = (\Theta(z) - I)c_1 - (\mu\Theta(z) - \bar{\mu})c_2, \quad z \in \mathbb{D}. \tag{6.6}$$

For a fixed $w \in \mathbb{D}$ (6.6) becomes

$$(w-1)\Psi(w) - (\bar{\mu}w - \mu)\Phi(w) = (\Theta(w) - I)c_1 - (\mu\Theta(w) - \bar{\mu})c_2. \tag{6.7}$$

Subtract (6.7) from (6.6) and divide by $z - w$. This gives

$$\begin{aligned}\frac{\Theta(z)-\Theta(w)}{z-w}(c_1 - \mu c_2) &= \frac{z\Psi(z) - w\Psi(w)}{z-w} - \frac{\Psi(z)-\Psi(w)}{z-w} \\ &\quad - \bar{\mu}\frac{z\Phi(z) - w\Phi(w)}{z-w} + \mu\frac{\Phi(z)-\Phi(w)}{z-w}\end{aligned} \tag{6.8}$$

for $z \in \mathbb{D}$. Observe that with the help of (4.20), (4.21), and (4.22) the equation (6.8) can be written as

$$\begin{aligned}&\big((I-wT)^{-1}F(c_1-\mu c_2)\big)(z) \\ &\qquad = \big((I-wT)^{-1}\Psi\big)(z) - \big((I-wT)^{-1}T\Psi\big)(z) \\ &\qquad\quad - \bar{\mu}\big((I-wT)^{-1}\Phi\big)(z) + \mu\big((I-wT)^{-1}T\Phi\big)(z)\end{aligned} \tag{6.9}$$

for $z \in \mathbb{D}$. Multiplication of (6.9) by $I - wT$ from the left yields (6.5) for $z \in \mathbb{D}$.

Now assume that $z \in \mathbb{D}^*$. Then (6.3), (6.4), and (3.22) imply that

$$\big(1 - \tfrac{1}{z}\big)\Psi(z) - \big(\bar{\mu} - \mu\tfrac{1}{z}\big)\Phi(z) = (I - \Theta(z))c_1 + (\bar{\mu}\Theta(z) - \mu)c_2, \quad z \in \mathbb{D}^*. \tag{6.10}$$

Now multiply (6.7) by $\Theta(z)$, $z \in \mathbb{D}^*$, subtract this from (6.10), and divide by $1 - w/z$. This gives

$$\begin{aligned}\frac{I-\Theta(z)\Theta(w)}{1-w/z}(c_1-\mu c_2) &= \frac{\Psi(z) - \Theta(z)w\Psi(w)}{1-w/z} - \frac{z^{-1}\Psi(z) - \Theta(z)\Psi(w)}{1-w/z} \\ &\quad - \bar{\mu}\frac{\Phi(z) - \Theta(z)w\Phi(w)}{1-w/z} + \mu\frac{z^{-1}\Phi(z) - \Theta(z)\Phi(w)}{1-w/z}\end{aligned} \tag{6.11}$$

and, again, (4.20), (4.21), and (4.22) yield (6.9) for $z \in \mathbb{D}^*$. Hence (6.5) holds for $z \in \mathbb{D}^*$.

Step 3. In this step it will be verified that the relation $\mathcal{T}$ coincides with $r^{-1}\mathcal{T}_{\mathfrak{S}(\Theta)}$ and that $S = r^{-1}S_{\mathfrak{S}(\Theta)}$.

In order to check $r^{-1}\mathcal{T}_{\mathfrak{S}(\Theta)} \subset \mathcal{T}$, define

$$\varphi(\lambda) := r^{-1}(\lambda)\big((T-I)h + Ff\big)(z), \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

$$\psi(\lambda) := r^{-1}(\lambda)\big((\mu T - \bar{\mu})h + \mu Ff\big)(z), \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

where $z = (\lambda - \mu)/(\lambda - \bar{\mu})$. Making use of the specific form of these elements from step 1 a straightforward calculation shows that

$$\psi(\lambda) - \lambda\varphi(\lambda) = \begin{cases}(\mu - \bar{\mu})(h(0) - (\Theta(z) - \Theta(0))f), & \lambda \in \mathbb{C}_+ \\ (\bar{\mu} - \mu)(\Theta(z)h(0) - (I - \Theta(z)\Theta(0))f), & \lambda \in \mathbb{C}_-.\end{cases} \tag{6.12}$$

On the other hand, setting

$$c_1 := \mu h(0) + (\mu\Theta(0) - \bar{\mu})f \quad \text{and} \quad c_2 = h(0) + (\Theta(0) - I)f$$

one verifies easily that $A(\lambda)c_1 + B(\lambda)c_2$ coincides with the right hand side of (6.12). Therefore $\psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2$, which shows that $r^{-1}\mathcal{T}_{\mathfrak{S}(\Theta)} \subset \mathcal{T}$.

In order to verify the reverse inclusion, let $\varphi, \psi \in \mathfrak{H}(A,B)$ and assume that

$$\psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2, \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

for some $c_1, c_2 \in \mathcal{H}$. Set $\Phi(z) = r(\lambda)\varphi(\lambda)$ and $\Psi(z) = r(\lambda)\psi(\lambda)$, cf., (6.4), and define the function h on $\mathbb{D} \cup \mathbb{D}^*$ and the vector $f \in \mathcal{H}$ by

$$h(z) := \frac{\Psi(z) - \mu\Phi(z)}{\mu - \bar{\mu}}, \; z \in \mathbb{D} \cup \mathbb{D}^*, \quad f := \frac{c_1 - \mu c_2}{\mu - \bar{\mu}}. \tag{6.13}$$

Then $h \in \mathfrak{S}(\Theta)$ and by means of (6.5) it is easy to see that

$$\Phi(z) = \big((T - I)h\big)(z) + (Ff)(z), \quad \Psi(z) = \big((\mu T - \bar{\mu})h\big)(z) + \mu(Ff)(z), \tag{6.14}$$

so that $\{\Phi, \Psi\} \in \mathcal{T}_{\mathfrak{S}(\Theta)}$ and hence $\mathcal{T} \subset r^{-1}\mathcal{T}_{\mathfrak{S}(\Theta)}$.

Therefore $\mathcal{T} = r^{-1}\mathcal{T}_{\mathfrak{S}(\Theta)}$. The proof of the identity $S = r^{-1}S_{\mathfrak{S}(\Theta)}$ is similar but simpler: if fact, in the formulas given above one can put $f = 0$, $h(0) = 0$ in the proof of the first inclusion, and $c_1 = c_2 = 0$ in the proof of the reverse inclusion, in which case the function h in (6.13) satisfies $h(0) = 0$. Finally, apply Theorem 5.4 (i) to see that $S = \mathcal{T}^*$ and $S^* = \overline{\mathcal{T}}$ hold.

Step 4. It remains to show that Γ in (6.2) is a minimal boundary relation for S^* such that M is the associated Weyl family.

Let $\{\varphi, \psi\} \in \mathcal{T}$, that is, $\varphi, \psi \in \mathfrak{H}(A,B)$ and

$$\psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2$$

for some $c_1, c_2 \in \mathcal{H}$, let $\{\Phi, \Psi\} \in \mathcal{T}_{\mathfrak{S}(\Theta)}$, $\Phi(z) = r(\lambda)\varphi(\lambda)$, $\Psi(z) = r(\lambda)\psi(\lambda)$, and let $h \in \mathfrak{S}(\Theta)$ and $f \in \mathcal{H}$ be as in (6.13) such that (6.14) holds. Then

$$\begin{aligned} Gh + (H - I)f &= h(0) + (\Theta(0) - I)f \\ &= \frac{1}{\mu - \bar{\mu}} r(\mu)\big(\psi(\mu) - \mu\varphi(\mu)\big) + (\Theta(0) - I)f = c_2 \end{aligned}$$

and

$$-\mu Gh - (\mu H - \bar{\mu})f = -c_1,$$

and the fact that (5.16) is a boundary relation for $S^*_{\mathfrak{S}(\Theta)}$ implies that Γ in (6.2) is a boundary relation for S^*. Furthermore, Γ is minimal since U is closely connected, cf., Theorem 4.4 and Theorem 5.4 (iii).

Now assume that $\{\varphi, \psi\} \in \widehat{\mathfrak{N}}_w(\mathcal{T})$ for some $w \in \mathbb{C} \setminus \mathbb{R}$, i.e., $\psi(\lambda) = w\varphi(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then there exist $c_1, c_2 \in \mathcal{H}$ such that

$$(w - \lambda)\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This implies that $A(w)c_1 + B(w)c_2 = 0$ and together with the symmetry property $A(\bar{w})^* = A(w)$, $B(\bar{w})^* = B(w)$ of the Nevanlinna pair one concludes that the Weyl family $M_\Gamma(w)$ corresponding to the boundary relation Γ belongs to

$$\bigl\{\{c_2, -c_1\} : A(w)c_1 + B(w)c_2 = 0\bigr\} = \bigl\{\{A(w)g, B(w)g\} : g \in \mathcal{H}\bigr\} = M(w),$$

i.e., $M_\Gamma(w) \subset M(w)$ holds for all $w \in \mathbb{C}\setminus\mathbb{R}$. By taking adjoints and using the symmetry property of Nevanlinna families (see Definition 3.1 (ii)) one gets the reverse inclusion $M_\Gamma(\bar{w}) \supset M(\bar{w})$, $w \in \mathbb{C}\setminus\mathbb{R}$. Therefore equality prevails here. This completes the proof of Theorem 6.1. □

6.2. Some special cases

There are several consequences and special cases concerning the model established in Theorem 6.1. First the multi-valued part of Γ, described also via the unitary colligation in Lemma 5.5, is expressed in terms of the above model.

Corollary 6.2. *Let $M(\lambda)$, $\{A(\lambda), B(\lambda)\}$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and Γ be as in Theorem 6.1. Then*

$$\operatorname{mul}\Gamma = \left\{ \begin{pmatrix} A(\bar\lambda)g \\ B(\bar\lambda)g \end{pmatrix} : \, g \in \ker \mathsf{N}_{A,B}(\lambda,\lambda) \right\} = M(\lambda) \cap M(\lambda)^*. \tag{6.15}$$

In particular, M is a strict Nevanlinna function if and only if $\ker \mathsf{N}_{A,B}(\lambda,\lambda) = \{0\}$.

Proof. The second equality in (6.15) is immediate from Lemma 3.3 (iv) and the identity (3.21). It follows from (6.2) that

$$\operatorname{mul}\Gamma = \left\{ \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} : \, A(\lambda)c_1 + B(\lambda)c_2 = 0, \; c_1, c_2 \in \mathcal{H}, \; \lambda \in \mathbb{C}\setminus\mathbb{R} \right\}.$$

The symmetry propoerty of the Nevanlinna pair together with (3.7) implies $\{c_2, -c_1\} \in M(\lambda)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and hence

$$\operatorname{mul}\Gamma \subset M(\lambda) \cap M(\bar\lambda) = M(\lambda) \cap M(\lambda)^*.$$

The inclusion $M(\lambda) \cap M(\lambda)^* \subset \operatorname{mul}\Gamma$ follows directly from the basic identities (1.1) and (1.2).

According to Proposition 5.7 M is strict if and only if $\operatorname{mul}\Gamma = \{0\}$. Hence the last statement is obtained from (6.15). □

The equality $\operatorname{mul}\Gamma = M(\lambda) \cap M(\lambda)^*$ in Corollary 6.2 has been proved also in [22, Lemma 4.1]. Observe that by (3.21), (3.16) and Proposition 4.3 (iii) $\ker \mathsf{N}_{A,B}(\lambda,\lambda)$ is constant on each halfplane $\mathbb{C}_+$ and $\mathbb{C}_-$, but in general $\ker \mathsf{N}_{A,B}(\lambda,\lambda) \neq \ker \mathsf{N}_{A,B}(\bar\lambda,\bar\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$. However, according to Corollary 6.2

$$\left\{ \begin{pmatrix} A(\bar\lambda)g \\ B(\bar\lambda)g \end{pmatrix} : \, g \in \ker \mathsf{N}_{A,B}(\lambda,\lambda) \right\}$$

does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and it coincides with $\operatorname{mul}\Gamma$.

Next the special case that $M(\lambda) \in \mathbf{B}(\mathcal{H})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, will be characterized. It follows from Theorem 6.1 that Γ_0 in (5.5) is given by

$$\Gamma_0 = \left\{ \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, c_2 \right\} : \{\varphi, \psi\} \in \mathcal{T} \right\}$$

where $\mathcal{T}$ is as in (ii) of Theorem 6.1. Hence,

$$\operatorname{ran} \Gamma_0 = \left\{ c_2 \in \mathcal{H} : \psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2,\ \varphi, \psi \in \mathfrak{H}(A, B),\ c_1 \in \mathcal{H} \right\},$$

and the symmetric extension $A_0 = \ker \Gamma_0$ of S is given by

$$A_0 = \ker \Gamma_0 = \left\{ \{\varphi, \psi\} \in \mathfrak{H}(A, B)^2 : \psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1,\ c_1 \in \mathcal{H} \right\}. \quad (6.16)$$

An application of Proposition 5.9 now leads to the following corollary.

Corollary 6.3. *Let $M(\lambda)$, $\{A(\lambda), B(\lambda)\}$ and Γ be as in Theorem* 6.1*. Then $M(\lambda)$ is an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function if and only if*

$$\mathcal{H} = \left\{ c_2 \in \mathcal{H} : \psi(\lambda) - \lambda\varphi(\lambda) = A(\lambda)c_1 + B(\lambda)c_2,\ \varphi, \psi \in \mathfrak{H}(A, B),\ c_1 \in \mathcal{H} \right\}$$

and the relation $A_0 = \ker \Gamma_0$ in (6.16) *is selfadjoint.*

Assume that the Nevanlinna family $M(\lambda)$ is an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function. In this case the Nevanlinna pair $\{I, M(\lambda)\}$ is chosen and the reproducing kernel Hilbert space is $\mathfrak{H}(M)$, cf., Section 3.2. Then Theorem 6.1 reads as follows.

Corollary 6.4. *Let $M(\lambda)$ be an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function. Then:*

(i) *the linear relation $S = \{\{\varphi, \psi\} \in \mathfrak{H}(M)^2 : \psi(\lambda) = \lambda\varphi(\lambda)\}$ is a closed simple symmetric operator in $\mathfrak{H}(M)$;*

(ii) *the linear relation*

$$\mathcal{T} = \left\{ \{\varphi, \psi\} \in \mathfrak{H}(M)^2 : \psi(\lambda) - \lambda\varphi(\lambda) = c_1 + M(\lambda)c_2,\ c_1, c_2 \in \mathcal{H} \right\}$$

is dense in S^* and $A_0 = \{\{\varphi, \psi\} \in \mathfrak{H}(M)^2 : \psi(\lambda) - \lambda\varphi(\lambda) = c_1,\ c_1 \in \mathcal{H}\}$ is a selfadjoint extension of S in $\mathfrak{H}(M)$;

(iii) *the linear relation Γ in* (6.2) *is a minimal boundary relation for S^* with corresponding Weyl function $M(\lambda)$.*

It is well known that strict (uniformly strict) $\mathbf{B}(\mathcal{H})$-valued Nevanlinna functions can be realized as Weyl functions of generalized boundary triplets (ordinary boundary triplets, respectively), cf., [22, 25, 26]. For these functions Theorem 6.1 and the above considerations yield the following corollary.

Corollary 6.5. *Let $M(\lambda)$ be a strict (uniformly strict) $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function and let $\mathcal{T}$ and Γ be as in Corollary* 6.4*. Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where*

$$\Gamma_0\{\varphi, \psi\} = c_2 \quad \text{and} \quad \Gamma_1\{\varphi, \psi\} = -c_1, \qquad \{\varphi, \psi\} \in \mathcal{T},$$

is a generalized boundary triplet (ordinary boundary triplet, respectively) for S^*.

Proof. If $M(\lambda) \in \mathbf{B}(\mathcal{H})$ is strict, then $\operatorname{mul}\Gamma = \{0\}$, $\operatorname{ran}\Gamma_0 = \mathcal{H}$, and $A_0 = \ker\Gamma_0$ is selfadjoint; see Proposition 5.7, Proposition 5.9 and Corollary 5.11. This means that (see [26, Definition 6.1], [22, Definition 5.6]) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triplet for S^*. If M is uniformly strict, then according to Corollary 5.12 $\Gamma = \{\Gamma_0, \Gamma_1\}$ is surjective and hence $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^*. □

6.3. Some historical remarks

A functional model for bounded uniformly strict operator-valued Nevanlinna functions as Q-functions of symmetric and selfadjoint operators or relations in Hilbert spaces goes back to M.G. Kreĭn, H. Langer, and B. Textorius [32, 33, 34, 35, 36, 37]. This model was given by means of the so-called ε-method. The notion of Q-function "coincides" with the present terminology of Weyl function associated with an ordinary boundary triplet for symmetric operators or relations. The notion of ordinary boundary triplet was extended in [26] to the notion of generalized boundary triplet which in terms of Weyl functions now corresponds to strict $\mathbf{B}(\mathcal{H})$-valued Nevanlinna functions M, that is, $0 \notin \sigma_p(\operatorname{Im} M(\lambda))$ holds for some, and hence for all, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The operator representation of bounded, not necessarily strict, Nevanlinna functions can be found in [17, 30, 37], see also Proposition 5.9 and Corollary 6.3. Functional models in terms of reproducing kernel spaces have been considered in several papers, see, e.g., [1, 4, 5, 6]

Recall that an $\mathbf{B}(\mathcal{H})$-valued Nevanlinna function $M(\lambda)$ admits the integral representation

$$M(\lambda) = \alpha + \beta\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma(t), \tag{6.17}$$

where $\alpha, \beta \in \mathbf{B}(\mathcal{H})$ are selfadjoint, $\beta \geq 0$, and $\Sigma(t)$ is a nondecreasing $\mathbf{B}(\mathcal{H})$-valued function satisfying $\int_{\mathbb{R}} (1+t^2)^{-1} d\Sigma(t) \in \mathbf{B}(\mathcal{H})$. In [26] V.A. Derkach and M.M. Malamud in their treatment of ordinary boundary triplets use the integral representation (6.17) to construct a model in the orthogonal sum of $L^2(\mathbb{R}, \Sigma)$ and a space which takes care of the linear term $\beta\lambda$. With the help of Stieltjes transform they also obtain a representation in terms of a reproducing kernel Hilbert space analogous to that in Corollary 6.5, cf. [2]. Their considerations are valid for the case of matrix-valued Nevanlinna functions; for the operator-valued case see the more recent treatment given in [38].

Assume that $M(\lambda)$, $\lambda \in \mathbb{C}_+$, is a family of maximal dissipative relations in a Hilbert space $\mathcal{H}$, such that for some, and hence for all, $\nu \in \mathbb{C}_+$ the $\mathbf{B}(\mathcal{H})$-valued function $\lambda \mapsto (M(\lambda) + \nu)^{-1} \in \mathcal{H}$ is holomorphic on $\mathbb{C}_+$. Then

$$\Theta(z) = I - (\mu - \bar{\mu})(M(\lambda) + \mu)^{-1}, \quad \lambda \in \mathbb{C}_+,$$

where $z = (\lambda - \mu)/(\lambda - \bar{\mu})$, is a Schur function defined on $\mathbb{D}$. It is now possible to use the coisometric representation for such a Schur function, cf., [3]. Its counterpart is an operator representation of $M(\lambda)$ in the upper halfplane. The scalar version of such results can be found in [35, 36].

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