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Dirichlet-to-Neumann maps, abstract Weyl–Titchmarsh M-functions, and a generalized index of unbounded meromorphic operator-valued functions [☆]

Jussi Behrndt [a], Fritz Gesztesy [b,*], Helge Holden [c], Roger Nichols [d]

[a] *Institut für Numerische Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria*
[b] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*
[c] *Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway*
[d] *Mathematics Department, The University of Tennessee at Chattanooga, 415 EMCS Building, Dept. 6956, 615 McCallie Ave, Chattanooga, TN 37403, USA*

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Dedicated with great pleasure to Werner Kirsch on the occasion of his 60th birthday

Abstract

We introduce a generalized index for certain meromorphic, unbounded, operator-valued functions. The class of functions is chosen such that energy parameter dependent Dirichlet-to-Neumann maps associated to uniformly elliptic partial differential operators, particularly, non-self-adjoint Schrödinger operators, on bounded Lipschitz domains, and abstract operator-valued Weyl–Titchmarsh M-functions and Donoghue-type M-functions corresponding to closed extensions of symmetric operators belong to it.

The principal purpose of this paper is to prove index formulas that relate the difference of the algebraic multiplicities of the discrete eigenvalues of Robin realizations of non-self-adjoint Schrödinger operators, and more abstract pairs of closed operators in Hilbert spaces with the generalized index of the

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* Corresponding author.

E-mail addresses: behrndt@tugraz.at (J. Behrndt), gesztesyf@missouri.edu (F. Gesztesy), holden@math.ntnu.no (H. Holden), Roger-Nichols@utc.edu (R. Nichols).

URLs: http://www.math.tugraz.at/~behrndt/ (J. Behrndt), https://www.math.missouri.edu/people/gesztesy (F. Gesztesy), http://www.math.ntnu.no/~holden/ (H. Holden), http://www.utc.edu/faculty/roger-nichols/index.php (R. Nichols).

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corresponding energy dependent Dirichlet-to-Neumann maps and abstract Weyl–Titchmarsh M-functions, respectively.

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1. Introduction

The principal purpose of this paper is to prove index formulas that relate the algebraic multiplicities of the discrete eigenvalues of closed operators in Hilbert spaces with a certain generalized index of a class of meromorphic, unbounded, closed, operator-valued functions, which have constant domains and are not necessarily Fredholm. In the following, we shall briefly illustrate the index formulas in our main applications and familiarize the reader with the structure of this article.

Let us first consider the Schrödinger differential expression

$$\mathcal{L} = -\Delta + q \tag{1.1}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with a complex-valued, bounded, measurable potential $q \in L^\infty(\Omega)$. Denote by A_D the Dirichlet realization of $\mathcal{L}$ in $L^2(\Omega)$ and let A_Θ be a closed realization of $\mathcal{L}$ subject to Robin-type boundary conditions of the form

$$\Theta \gamma_D f = \gamma_N f, \tag{1.2}$$

where γ_D and γ_N denote the Dirichlet and Neumann trace operator, and Θ is a bounded operator in $L^2(\partial\Omega)$; for precise definitions of the trace maps and the operators A_D and A_Θ we refer to Section 3. We emphasize that the differential expression (1.1) is non-symmetric and hence the Dirichlet and Robin realization A_D and A_Θ are non-self-adjoint, and that, in addition, also the parameter Θ in the Robin boundary condition in (1.2) is non-self-adjoint in general. Since the Lipschitz domain Ω is bounded, the spectra of the operators A_D and A_Θ consist of isolated eigenvalues with finite algebraic multiplicities. As one of our main results we show that the algebraic multiplicities $m_a(z_0; A_D)$ and $m_a(z_0; A_\Theta)$ of an eigenvalue z_0 of A_D and A_Θ satisfy the generalized index formula

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(D(\cdot) - \Theta) = m_a(z_0; A_\Theta) - m_a(z_0; A_D), \tag{1.3}$$

where the generalized index $\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(\cdot)$ is defined below in (1.4), and $D(\cdot)$ denotes the energy parameter-dependent Dirichlet-to-Neumann map associated to the differential expression $\mathcal{L}$. The index formula (1.3) remains valid for points z_0 in the resolvent set of $\rho(A_D)$ or $\rho(A_\Theta)$, in which case $m_a(z_0; A_D) = 0$ or $m_a(z_0; A_\Theta) = 0$, respectively. However, since the values $D(z)$, $z \in \rho(A_D)$, of the Dirichlet-to-Neumann map are unbounded operators in $L^2(\partial\Omega)$, the classical concept of an index for a meromorphic, bounded, Fredholm operator-valued function as introduced in [36] (see also [33, Chapter XI.9] and [35, Chapter 4]) does not apply to $D(\cdot) - \Theta$ on the left-hand side of (1.3).

Instead, it is necessary to specify a suitable class of meromorphic operator-valued functions $M(\cdot)$ with values in the set of unbounded closed operators such that on one hand the function $D(\cdot) - \Theta$ in (1.3) is contained in this class, and on the other hand the generalized index

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) := \operatorname{tr}\left(\frac{1}{2\pi i} \oint_{C(z_0;\varepsilon)} d\zeta\, \overline{M'(\zeta)} M(\zeta)^{-1}\right) \tag{1.4}$$

is well-defined; here $C(z_0;\varepsilon)$ is the counterclockwise oriented circle centered at z_0 with radius $\varepsilon > 0$ sufficiently small, and $\overline{M'(\zeta)}$ denotes the closure of the derivative of $M(\cdot)$ at ζ. This is the main purpose of the preliminary Section 2, which is inspired by considerations in [7] and [26]. Here we collect a set of assumptions and define a class of meromorphic, unbounded, closed, operator-valued functions, which are not necessarily Fredholm, such that the functions $\overline{M'(\cdot)}$ and $M(\cdot)^{-1}$ in the integrand in (1.4) are both finitely meromorphic (see [33,35]), and hence definition (1.4) turns out to be meaningful. Although the generalized index in (1.4) may not be integer-valued in general (in contrast to the classical index, where the operator-valued version of the argument principle from [36] or [35, Theorem 4.4.1] applies) in our main applications (1.3) and (1.6) below it certainly is, since the right-hand side equals an integer.

The main objective of Section 3 is to prove the index formula (1.3) in Theorem 3.10. Besides the differential expression $\mathcal{L} = -\Delta + q$ we also consider the formal adjoint expression $\widetilde{\mathcal{L}} = -\Delta + \overline{q}$ and obtain an analogous index formula for the algebraic multiplicities of the eigenvalues of A_D^* and A_Θ^* in Theorem 3.11. The main ingredient in the proof of the index formula (1.3) is the Krein-type resolvent formula in Theorem 3.10 in which the difference of the resolvents of A_Θ and A_D in $L^2(\Omega)$ is traced back to the boundary space $L^2(\partial\Omega)$ and the perturbation term $D(\cdot) - \Theta$. Such resolvent formulas are well-known for the symmetric case (see, e.g., [1,9,11,15,30,48,58,59]) and in the context of dual pairs related formulas can be found, for instance, in [14] and [49]; the Dirichlet-to-Neumann map $D(\cdot)$ has attracted a lot of attention in the recent past (see, e.g., [1–5,8–12,29,30,57,58], and the references therein). Although formally the index formula (1.3) is an immediate consequence of the Krein-type resolvent formula in Theorem 3.10 we wish to emphasize that it is necessary to verify that the generalized index (1.4) is well-defined for the function $D(\cdot) - \Theta$. In fact, a somewhat subtle analysis is required in this context, and the key difficulty is to show that $(D(\cdot) - \Theta)^{-1}$ is a finitely meromorphic function (cf. Lemma 3.9).

Besides the index formula for Robin realizations of $\mathcal{L}$ in Section 3, we also discuss a slightly more abstract situation in Section 4. Here it is assumed that B_1 and B_2 are closed operators in a Hilbert space $\mathfrak{H}$ which are both extensions of a common underlying densely defined, symmetric operator S. We shall use the abstract concept of boundary triples (see, e.g., [10,17,18,22,23,37,42]) to parametrize B_1 and B_2 in the form

$$B_1 = S^* \upharpoonright \ker(\Gamma_1 - \Theta_1\Gamma_0), \quad B_2 = S^* \upharpoonright \ker(\Gamma_1 - \Theta_2\Gamma_0), \tag{1.5}$$

where Γ_0 and Γ_1 are linear maps from $\operatorname{dom}(S^*)$ into a boundary space $\mathcal{G}$ and Θ_1 and Θ_2 are closed operators in $\mathcal{G}$. Let $M(\cdot)$ denote the Weyl–Titchmarsh function corresponding to the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. Our goal in Section 4 is to prove the index formula

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}\big(\Theta_1 - M(\cdot)\big) - \widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}\big(\Theta_2 - M(\cdot)\big) = m_a\big(z_0; B_1\big) - m_a\big(z_0; B_2\big), \tag{1.6}$$

in which the generalized index of the functions $\Theta_1 - M(\cdot)$ and $\Theta_2 - M(\cdot)$ is related to the algebraic multiplicities of a discrete eigenvalue z_0 of B_1 and B_2 (the formula is also valid for points z_0 in the resolvent set of B_1 or B_2, in which case $m_a(z_0; B_1) = 0$ or $m_a(z_0; B_2) = 0$, respectively). In contrast to the index formula (1.3) in Section 3, here the values of the Weyl–Titchmarsh function $M(\cdot)$ are bounded operators, but the parameters Θ_1 and Θ_2 are in general unbounded, closed operators. However, the strategy and the difficulties in the proof of the index formula in Theorem 4.3 are similar to those in Section 3: One first has to verify that the generalized index is well-defined for the functions $\Theta_1 - M(\cdot)$ and $\Theta_2 - M(\cdot)$ (again the key difficulty is to show that the inverses $(\Theta_1 - M(\cdot))^{-1}$ and $(\Theta_2 - M(\cdot))^{-1}$ are finitely meromorphic at a discrete eigenvalue of B_1 and B_2, respectively) and then a Krein-type resolvent formula (see, e.g., [1,2,9,11,12,14–16,21–24,27–30,32,38,43–47,61], and the references cited therein) yields the index formula (1.6).

To ensure a self-contained presentation in Section 4, we have added a short Appendix A on the abstract concept of boundary triples and their Weyl–Titchmarsh functions. In this appendix we also establish the connection to abstract Donoghue-type M-functions studied in [27,28,31, 32], so that the index formula (1.6) can also be interpreted in the framework of Donoghue-type M-functions.

Finally, we summarize the basic notation used in this paper: $\mathcal{H}$, $\mathfrak{H}$, and $\mathcal{G}$ denote separable complex Hilbert spaces with scalar products $(\cdot\,,\cdot)_{\mathcal{H}}$, $(\cdot\,,\cdot)_{\mathfrak{H}}$, and $(\cdot\,,\cdot)_{\mathcal{G}}$, linear in the first entry, respectively. The Banach spaces of bounded, compact, and trace class (linear) operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_\infty(\mathcal{H})$, and $\mathcal{B}_1(\mathcal{H})$, respectively. The subspace of all finite rank operators will be abbreviated by $\mathcal{F}(\mathcal{H})$. The analogous notation $\mathcal{B}(\mathcal{H}, \mathcal{G})$ will be used for bounded operators between the Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$. The set of densely defined, closed, linear operators in $\mathcal{H}$ will be denoted by $\mathcal{C}(\mathcal{H})$. For a linear operator T we denote by $\operatorname{dom}(T)$, $\operatorname{ran}(T)$ and $\ker(T)$ the domain, range, and kernel, respectively. If T is closable, the closure is denoted by $\overline{T}$. The spectrum, point spectrum, continuous spectrum, residual spectrum, and resolvent set of a closed operator $T \in \mathcal{C}(\mathcal{H})$ will be denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$, and $\rho(T)$; the discrete spectrum of T consists of eigenvalues of T with finite algebraic multiplicity which are isolated in $\sigma(T)$, this set is abbreviated by $\sigma_d(T)$. For the algebraic multiplicity of an eigenvalue $z_0 \in \sigma_d(T)$ we write $m_a(z_0; T)$ and we set $m_a(z_0; T) = 0$ if $z_0 \in \rho(T)$. Furthermore, $\operatorname{tr}_{\mathcal{H}}(T)$ denotes the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$. The symbol $\dot{+}$ denotes a direct (but not necessary orthogonal direct) sum decomposition in connection with subspaces of Banach spaces.

2. On the notion of a generalized index of meromorphic operator-valued functions

Let $\mathcal{H}$ be a separable complex Hilbert space, assume that $\mathcal{D} \subseteq \mathbb{C}$ is an open set, and let $M(\cdot)$ be a $\mathcal{B}(\mathcal{H})$-valued meromorphic function on $\mathcal{D}$ that has the norm convergent Laurent expansion around $z_0 \in \mathcal{D}$ of the form

$$M(z) = \sum_{k=-N_0}^{\infty} (z - z_0)^k M_k(z_0), \quad z \in D(z_0; \varepsilon_0)\backslash\{z_0\}, \tag{2.1}$$

where $M_k(z_0) \in \mathcal{B}(\mathcal{H})$, $k \in \mathbb{Z}$, $k \geq -N_0$ and $\varepsilon_0 > 0$ is sufficiently small such that the punctured open disc

$$D(z_0; \varepsilon_0)\backslash\{z_0\} = \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < \varepsilon_0\} \tag{2.2}$$

is contained in $\mathcal{D}$. The principal part $\operatorname{pp}_{z_0}\{M(z)\}$ of $M(\cdot)$ at z_0 is defined as the finite sum

$$\operatorname{pp}_{z_0}\{M(z)\} = \sum_{k=-N_0}^{-1} (z-z_0)^k M_k(z_0). \tag{2.3}$$

Definition 2.1. Let $\mathcal{D} \subseteq \mathbb{C}$ be an open set and let $M(\cdot)$ be a $\mathcal{B}(\mathcal{H})$-valued meromorphic function on $\mathcal{D}$. Then $M(\cdot)$ is called *finitely meromorphic at* $z_0 \in \mathcal{D}$ if $M(\cdot)$ is analytic on the punctured disk $D(z_0;\varepsilon_0)\backslash\{z_0\} \subset \mathcal{D}$ with sufficiently small $\varepsilon_0 > 0$, and the principal part $\operatorname{pp}_{z_0}\{M(z)\}$ of $M(\cdot)$ at z_0 is of finite rank, that is, the principal part of $M(\cdot)$ is of the type (2.3), and one has

$$M_k(z_0) \in \mathcal{F}(\mathcal{H}), \quad -N_0 \le k \le -1. \tag{2.4}$$

The function $M(\cdot)$ is called *finitely meromorphic on* $\mathcal{D}$ if it is meromorphic on $\mathcal{D}$ and finitely meromorphic at each of its poles.

Assume that $M_j(\cdot)$, $j = 1, 2$, are $\mathcal{B}(\mathcal{H})$-valued meromorphic functions on $\mathcal{D}$ that are both finitely meromorphic at $z_0 \in \mathcal{D}$, choose $\varepsilon_0 > 0$ such that (2.1) and (2.4) hold for both functions $M_j(\cdot)$, and let $0 < \varepsilon < \varepsilon_0$. Then by [33, Lemma XI.9.3] or [35, Proposition 4.2.2] also the functions $M_1(\cdot)M_2(\cdot)$ and $M_2(\cdot)M_1(\cdot)$ are finitely meromorphic at $z_0 \in \mathcal{D}$, the operators

$$\oint_{C(z_0;\varepsilon)} d\zeta\, M_1(\zeta)M_2(\zeta) \quad \text{and} \quad \oint_{C(z_0;\varepsilon)} d\zeta\, M_2(\zeta)M_1(\zeta) \tag{2.5}$$

are both of finite rank and the identity

$$\operatorname{tr}_{\mathcal{H}}\Bigg(\oint_{C(z_0;\varepsilon)} d\zeta\, M_1(\zeta)M_2(\zeta)\Bigg) = \operatorname{tr}_{\mathcal{H}}\Bigg(\oint_{C(z_0;\varepsilon)} d\zeta\, M_2(\zeta)M_1(\zeta)\Bigg) \tag{2.6}$$

holds; here the symbol $\oint$ denotes the contour integral and $C(z_0;\varepsilon) = \partial D(z_0;\varepsilon)$ is the counterclockwise oriented circle with radius ε centered at z_0.

In the next example a standard situation is discussed: the resolvent of a closed operator T in the Hilbert space $\mathcal{H}$ is finitely meromorphic at a discrete eigenvalue (cf. [34] or [41]).

Example 2.2. Let T be a closed operator in the Hilbert space $\mathcal{H}$ and let $z_0 \in \sigma_d(T)$. Choose $\varepsilon_0 > 0$ sufficiently small such that the punctured disc $D(z_0;\varepsilon_0)\backslash\{z_0\}$ is contained in $\rho(T)$ and let $0 < \varepsilon < \varepsilon_0$. Then the Riesz projection

$$P(z_0;T) = -\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\, (T - \zeta I_{\mathcal{H}})^{-1}, \tag{2.7}$$

where as above $C(z_0;\varepsilon) = \partial D(z_0;\varepsilon)$, is a finite rank operator in $\mathcal{H}$ and the range of $P(z_0;T)$ coincides with the algebraic eigenspace of T at z_0; in particular, one has

$$\operatorname{tr}_{\mathcal{H}}(P(z_0;T)) = m_a(z_0;T). \tag{2.8}$$

Furthermore, the Hilbert space $\mathcal{H}$ admits the direct sum decomposition

$$\mathcal{H} = \operatorname{ran}(P(z_0; T)) \dotplus \operatorname{ran}(I_{\mathcal{H}} - P(z_0; T)) \tag{2.9}$$

and the spaces $P(z_0; T)\mathcal{H}$ and $(I_{\mathcal{H}} - P(z_0; T))\mathcal{H}$ are both invariant for the closed operators T and $T - z_0 I_{\mathcal{H}}$. Moreover, the restriction $T_1 - z_0 I_{\mathcal{H}}$ of $T - z_0 I_{\mathcal{H}}$ onto the finite-dimensional subspace $P(z_0; T)\mathcal{H}$ is nilpotent, that is, $(T_1 - z_0 I_{\mathcal{H}})^{N_0} = 0$ for some $N_0 \in \mathbb{N}$ and we agree to choose the integer N_0 with this property minimal. The restriction $T_2 - z_0 I_{\mathcal{H}}$ of $T - z_0 I_{\mathcal{H}}$ onto $(I_{\mathcal{H}} - P(z_0; T))\mathcal{H}$ is a boundedly invertible operator in the Hilbert space $(I_{\mathcal{H}} - P(z_0; T))\mathcal{H}$. As in [34, Chapter 1, §2. Proof of Theorem 2.1] one verifies that the resolvent of T in $D(z_0; \varepsilon_0)\backslash\{z_0\}$ admits a norm convergent Laurent expansion of the form

$$\begin{aligned}(T - zI_{\mathcal{H}})^{-1} = &- \sum_{k=-N_0}^{-1} (z - z_0)^k (T_1 - z_0 I_{\mathcal{H}})^{-k-1} P(z_0; T) \\ &+ \sum_{k=0}^{\infty} (z - z_0)^k (T_2 - zI_{\mathcal{H}})^{-(k+1)} (I_{\mathcal{H}} - P(z_0; T)),\end{aligned} \tag{2.10}$$

and, in particular, the operators $(T_1 - z_0 I_{\mathcal{H}})^{-k-1} P(z_0; T)$, $-N_0 \leq k \leq -1$, are of finite rank. Therefore, the resolvent $z \mapsto (T - zI_{\mathcal{H}})$ is finitely meromorphic at z_0. It also follows from the Laurent expansion (2.10) that the derivatives $\frac{d^k}{dz^k}(T - zI_{\mathcal{H}})^{-1}$, $k \in \mathbb{N}$, are finitely meromorphic at z_0.

The following example is a simple generalization and immediate consequence of Example 2.2. The observation below will be used frequently in this paper.

Example 2.3. Let T be a closed operator in the Hilbert space $\mathcal{H}$ and let $z_0 \in \sigma_d(T)$. Assume that $\mathcal{G}$ is an auxiliary Hilbert space and let $\gamma \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. Then the $\mathcal{B}(\mathcal{G})$-valued function

$$z \mapsto \gamma^*(T - zI_{\mathcal{H}})^{-1}\gamma, \quad z \in \rho(T), \tag{2.11}$$

is finitely meromorphic at z_0. Indeed, this simply follows by multiplying the Laurent expansion of the resolvent in (2.10) by $\gamma^* \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ from the left and by $\gamma \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ from the right.

The aim of this preliminary section is to introduce an extended notion of the index applicable to certain non-Fredholm and also unbounded meromorphic operator-valued functions $M(\cdot)$ in Definition 2.5 below. We start by collecting our assumptions on $M(\cdot)$.

Hypothesis 2.4. Let $\mathcal{D} \subseteq \mathbb{C}$ be open and connected, and $\mathcal{D}_0 \subset \mathcal{D}$ a discrete set (i.e., a set without limit points in $\mathcal{D}$). Suppose that the map

$$M : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{C}(\mathcal{H}), \quad z \mapsto M(z), \tag{2.12}$$

takes on values in the set of densely defined, closed operators, $\mathcal{C}(\mathcal{H})$, with the following additional properties:

(i) $\mathcal{M}_0 := \operatorname{dom}(M(z))$ is independent of $z \in \mathcal{D}\backslash\mathcal{D}_0$.

(ii) $M(z)$ is boundedly invertible, $M(z)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \mathcal{D}\backslash\mathcal{D}_0$.

(*iii*) The function

$$M(\cdot)^{-1} : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{B}(\mathcal{H}), \quad z \mapsto M(z)^{-1}, \tag{2.13}$$

is analytic on $\mathcal{D}\backslash\mathcal{D}_0$ and finitely meromorphic on $\mathcal{D}$.
(*iv*) For $\varphi \in \mathcal{M}_0$ the function

$$M(\cdot)\varphi : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{H}, \quad z \mapsto M(z)\varphi, \tag{2.14}$$

is analytic; in particular, the derivative $M'(z)\varphi$ exists for all $\varphi \in \mathcal{M}_0$ and $z \in \mathcal{D}\backslash\mathcal{D}_0$.
(*v*) For $z \in \mathcal{D}\backslash\mathcal{D}_0$, the operators $M'(z)$ defined on $\operatorname{dom}(M'(z)) = \mathcal{M}_0$, admit bounded continuations to operators $\overline{M'(z)} \in \mathcal{B}(\mathcal{H})$, and the operator-valued function

$$\overline{M'(\cdot)} : \mathcal{D}\backslash\mathcal{D}_0 \to \mathcal{B}(\mathcal{H}), \quad z \mapsto \overline{M'(z)}, \tag{2.15}$$

is analytic on $\mathcal{D}\backslash\mathcal{D}_0$ and finitely meromorphic on $\mathcal{D}$.

Granted Hypothesis 2.4 it follows that the maps

$$z \mapsto \overline{M'(z)}M(z)^{-1}, \quad z \mapsto M(z)^{-1}\overline{M'(z)} \tag{2.16}$$

are finitely meromorphic and hence identity (2.6) applies. This leads to the following definition of a generalized index of $M(\cdot)$, which extends the notion of an index for finitely meromorphic $\mathcal{B}(\mathcal{H})$-valued functions employed in [36] and, for instance, in [33,35] (cf. [7, Definition 4.2]).

Definition 2.5. Assume Hypothesis 2.4, let $z_0 \in \mathcal{D}$, and $0 < \varepsilon$ sufficiently small. Then the *generalized index of* $M(\cdot)$ *with respect to the counterclockwise oriented circle* $C(z_0;\varepsilon)$, $\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(M(\cdot))$, is defined by

$$\begin{aligned} \widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) &= \operatorname{tr}_{\mathcal{H}}\left(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, \overline{M'(\zeta)}M(\zeta)^{-1}\right) \\ &= \operatorname{tr}_{\mathcal{H}}\left(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, M(\zeta)^{-1}\overline{M'(\zeta)}\right). \end{aligned} \tag{2.17}$$

(Of course, $\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon_0)}(M(\cdot)) = 0$, unless, $z_0 \in \mathcal{D}_0$.)

The main objective of this paper is to show that this notion of generalized index applies to Dirichlet-to-Neumann maps associated to non-self-adjoint Schrödinger operators in Section 3 and to abstract operator-valued Weyl–Titchmarsh functions or Donoghue-type M-functions in Section 4. It will also turn out that the generalized index is integer-valued in both of these applications.

3. Schrödinger operators with complex potentials and Dirichlet-to-Neumann maps

In this section we discuss applications to Schrödinger operators with bounded, complex-valued potentials on bounded Lipschitz domains. In particular, we consider Krein-type resolvent formulas and compute the generalized index associated to underlying (energy parameter dependent) Dirichlet-to-Neumann maps.

Hypothesis 3.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and let $q \in L^\infty(\Omega)$ be a complex-valued potential.

Assuming Hypothesis 3.1, we consider the Schrödinger differential expression

$$\mathcal{L} = -\Delta + q, \tag{3.1}$$

and its formal adjoint

$$\widetilde{\mathcal{L}} = -\Delta + \overline{q}. \tag{3.2}$$

For our purposes, it is convenient to work with operator realizations of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ in $L^2(\Omega)$ which are defined via boundary conditions on functions from the space

$$H^{3/2}_\Delta(\Omega) := \big\{ f \in H^{3/2}(\Omega) \,\big|\, \Delta f \in L^2(\Omega) \big\}, \tag{3.3}$$

where for each $f \in H^{3/2}(\Omega)$, Δf is understood in the sense of distributions. The space $H^{3/2}_\Delta(\Omega)$ equipped with the scalar product

$$(f,g)_{H^{3/2}_\Delta(\Omega)} = (f,g)_{H^{3/2}(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}, \quad f, g \in H^{3/2}_\Delta(\Omega), \tag{3.4}$$

is a Hilbert space. According to [30, Lemmas 3.1 and 3.2], the Dirichlet trace operator defined on $C^\infty(\overline{\Omega})$ admits a continuous surjective extension

$$\gamma_D : H^{3/2}_\Delta(\Omega) \to H^1(\partial\Omega), \tag{3.5}$$

and the Neumann trace operator defined on $C^\infty(\overline{\Omega})$ admits a continuous surjective extension

$$\gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega). \tag{3.6}$$

For our investigations it is important to note that Green's Second Identity extends to functions in $H^{3/2}_\Delta(\Omega)$, that is,

$$\begin{aligned} (\mathcal{L} f, g)_{L^2(\Omega)} - (f, \widetilde{\mathcal{L}} g)_{L^2(\Omega)} = (\gamma_D f, \gamma_N g)_{L^2(\partial\Omega)} - (\gamma_N f, \gamma_D g)_{L^2(\partial\Omega)}, \\ f, g \in H^{3/2}_\Delta(\Omega). \end{aligned} \tag{3.7}$$

Next, we introduce the Dirichlet operators associated to the differential expressions $\mathcal{L}$ and $\widetilde{\mathcal{L}}$.

Hypothesis 3.2. In addition to the assumptions in Hypothesis 3.1, let A_D and $\widetilde{A}_D$ denote the Dirichlet operators associated to the differential expressions $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ in $L^2(\Omega)$, that is,

$$A_D f = \mathcal{L} f, \quad f \in \operatorname{dom}(A_D) = \big\{ g \in H^{3/2}_{\Delta}(\Omega) \,\big|\, \gamma_D g = 0 \big\}, \tag{3.8}$$

and

$$\widetilde{A}_D f = \widetilde{\mathcal{L}} f, \quad f \in \operatorname{dom}\big(\widetilde{A}_D\big) = \big\{ g \in H^{3/2}_{\Delta}(\Omega) \,\big|\, \gamma_D g = 0 \big\}. \tag{3.9}$$

In the special case $q \equiv 0$, the operator A_D coincides with the self-adjoint free Dirichlet Laplacian on Ω, which we denote by $A_D^{(0)}$:

$$A_D^{(0)} f = -\Delta f, \quad f \in \operatorname{dom}(A_D^{(0)}) = \big\{ g \in H^{3/2}_{\Delta}(\Omega) \,\big|\, \gamma_D g = 0 \big\} \tag{3.10}$$

(cf., e.g., [29, Theorem 2.10 and Lemma 3.4] or [39, Theorem B.2]). Clearly, A_D (resp., $\widetilde{A}_D$) may be viewed as an additive perturbation of $A_D^{(0)}$ by the bounded potential q (resp., $\overline{q}$). These facts lead to the following result.

Proposition 3.3. *Assume Hypothesis 3.2. The Dirichlet operators A_D and $\widetilde{A}_D$ are densely defined, closed operators in $L^2(\Omega)$ which are adjoint to each other,*

$$A_D^* = \widetilde{A}_D. \tag{3.11}$$

In addition, A_D and $\widetilde{A}_D$ have compact resolvents.

We note that (3.11) also implies

$$z \in \rho(A_D) \text{ if and only if } \overline{z} \in \rho\big(\widetilde{A}_D\big). \tag{3.12}$$

In light of the fact that the Dirichlet trace operator γ_D maps $H^{3/2}_{\Delta}(\Omega)$ onto $H^1(\partial\Omega)$, it follows that for $z \in \rho(A_D)$ and $\varphi \in H^1(\partial\Omega)$ the boundary value problem

$$\mathcal{L} f - z f = 0, \quad \gamma_D f = \varphi, \tag{3.13}$$

admits a unique solution $f_z \in H^{3/2}_{\Delta}(\Omega)$. Analogously, for $\widetilde{z} \in \rho\big(\widetilde{A}_D\big)$ and $\psi \in H^1(\partial\Omega)$, the boundary value problem

$$\widetilde{\mathcal{L}} g - \widetilde{z} g = 0, \quad \gamma_D g = \psi, \tag{3.14}$$

admits a unique solution $g_{\widetilde{z}} \in H^{3/2}_{\Delta}(\Omega)$. These observations imply that the solution operators and the Dirichlet-to-Neumann maps in the next definition are well-defined.

Definition 3.4. Assume Hypothesis 3.2 and suppose $z \in \rho(A_D)$ and $\widetilde{z} \in \rho\big(\widetilde{A}_D\big)$. Let $f_z, g_{\widetilde{z}} \in H^{3/2}_{\Delta}(\Omega)$ denote the unique solutions of (3.13) and (3.14) for $\varphi, \psi \in H^1(\partial\Omega)$, respectively.
(i) The solution operators $P(z)$ and $\widetilde{P}(\widetilde{z})$ associated to the boundary value problems (3.13) and (3.14) are defined by

$$P(z)\varphi = f_z, \quad \widetilde{P}(\widetilde{z})\psi = g_{\widetilde{z}}, \tag{3.15}$$

respectively.
(ii) The (energy parameter dependent) Dirichlet-to-Neumann maps $D(z)$ and $\widetilde{D}(\widetilde{z})$ associated to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are defined by

$$D(z)\varphi = \gamma_N f_z, \quad \widetilde{D}(\widetilde{z})\psi = \gamma_N g_{\widetilde{z}}, \tag{3.16}$$

respectively.

In the following, the solution operators $P(z)$ and $\widetilde{P}(\widetilde{z})$ will often be regarded as densely defined operators from $L^2(\partial\Omega)$ into $L^2(\Omega)$, and the Dirichlet-to-Neumann maps will be viewed as densely defined operators in $L^2(\partial\Omega)$. The next lemma collects relevant properties of the solution operators and Dirichlet-to-Neumann maps, and its proof is based primarily on Green's Second Identity, (3.7). The arguments are almost the same as in the self-adjoint case, or in the abstract framework of boundary triples for dual pairs of operators (see [49]), and will not be repeated here. The reader is also referred to Steps 4–6 in the proof of Lemma 3.9 where similar methods are used.

Lemma 3.5. *Assume Hypothesis 3.2. For $z_1, z_2 \in \rho(A_D)$ and $\widetilde{z}_1, \widetilde{z}_2 \in \rho\big(\widetilde{A}_D\big)$ the following identities hold:*
(i) *The solution operator $P(z_1) : L^2(\partial\Omega) \to L^2(\Omega)$ defined on the dense subspace* $\operatorname{dom}(P(z_1)) = H^1(\partial\Omega)$ *is bounded and its adjoint $P(z_1)^* \in \mathcal{B}\big(L^2(\Omega), L^2(\partial\Omega)\big)$ is given by*

$$P(z_1)^* = -\gamma_N\big(\widetilde{A}_D - \overline{z_1} I_{L^2(\Omega)}\big)^{-1}. \tag{3.17}$$

$(\widetilde{i})$ *The solution operator $\widetilde{P}(\widetilde{z}_1) : L^2(\partial\Omega) \to L^2(\Omega)$ defined on the dense subspace* $\operatorname{dom}\big(\widetilde{P}(\widetilde{z}_1)\big) = H^1(\partial\Omega)$ *is bounded and its adjoint $\widetilde{P}(\widetilde{z}_1)^* \in \mathcal{B}\big(L^2(\Omega), L^2(\partial\Omega)\big)$ is given by*

$$\widetilde{P}(\widetilde{z}_1)^* = -\gamma_N(A_D - \overline{\widetilde{z}_1} I_{L^2(\Omega)})^{-1}. \tag{3.18}$$

(ii) *For all $\varphi \in H^1(\partial\Omega)$ one has*

$$P(z_1)\varphi = \big(I_{L^2(\Omega)} + (z_1 - z_2)(A_D - z_1 I_{L^2(\Omega)})^{-1}\big)P(z_2)\varphi. \tag{3.19}$$

$(\widetilde{ii})$ *For all $\psi \in H^1(\partial\Omega)$ one has*

$$\widetilde{P}(\widetilde{z}_1)\psi = \big(I_{L^2(\Omega)} + (\widetilde{z}_1 - \widetilde{z}_2)\big(\widetilde{A}_D - \widetilde{z}_1 I_{L^2(\Omega)}\big)^{-1}\big)\widetilde{P}(\widetilde{z}_2)\psi. \tag{3.20}$$

(iii) *The Dirichlet-to-Neumann map $D(z_1) : L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined on the dense subspace* $\operatorname{dom}(D(z_1)) = H^1(\partial\Omega)$ *is a closed operator in $L^2(\partial\Omega)$ and it satisfies the identity*

$$\big(D(z_1) - D(\overline{z_2})\big)\varphi = (\overline{z_2} - z_1)\widetilde{P}(z_2)^* P(z_1)\varphi, \quad \varphi \in H^1(\partial\Omega). \tag{3.21}$$

In particular, one has

$$D(z_1)\varphi = D(\overline{z_2})\varphi + (\overline{z_2} - z_1)\widetilde{P}(z_2)^*\big(I_{L^2(\Omega)} + (z_1 - z_2)(A_D - z_1 I_{L^2(\Omega)})^{-1}\big)P(z_2)\varphi, \quad (3.22)$$

and for all $\varphi \in H^1(\partial\Omega)$*, the map* $z_1 \mapsto D(z_1)\varphi$ *is holomorphic on* $\rho(A_D)$*.*
$\widetilde{(iii)}$ *The Dirichlet-to-Neumann map* $\widetilde{D}(\widetilde{z}_1) : L^2(\partial\Omega) \to L^2(\partial\Omega)$ *defined on the dense subspace* $\mathrm{dom}\big(\widetilde{D}(\widetilde{z}_1)\big) = H^1(\partial\Omega)$ *is a closed operator in* $L^2(\partial\Omega)$ *and it satisfies the identity*

$$\big(\widetilde{D}(\widetilde{z}_1) - \widetilde{D}(\overline{\widetilde{z}_2})\big)\psi = (\overline{\widetilde{z}_2} - \widetilde{z}_1)P(\widetilde{z}_2)^*\widetilde{P}(\widetilde{z}_1)\psi, \quad \psi \in H^1(\partial\Omega). \quad (3.23)$$

In particular, one has

$$\widetilde{D}(\widetilde{z}_1)\varphi = \widetilde{D}(\overline{\widetilde{z}_2})\psi + (\overline{\widetilde{z}_2} - \widetilde{z}_1)P(\widetilde{z}_2)^*\big(I_{L^2(\Omega)} + (\widetilde{z}_1 - \widetilde{z}_2)\big(\widetilde{A}_D - \widetilde{z}_1 I_{L^2(\Omega)}\big)^{-1}\big)\widetilde{P}(\widetilde{z}_2)\psi, \quad (3.24)$$

and for all $\psi \in H^1(\partial\Omega)$*, the map* $\widetilde{z}_1 \mapsto \widetilde{D}(\widetilde{z}_1)\psi$ *is holomorphic on* $\rho\big(\widetilde{A}_D\big)$*.*

As a useful consequence of Lemma 3.5, one obtains the following result.

Corollary 3.6. *For all* $\varphi, \psi \in H^1(\partial\Omega)$ *one has*

$$\frac{d}{dz}D(z)\varphi = -\widetilde{P}(\overline{z})^*P(z)\varphi, \quad \frac{d}{d\widetilde{z}}\widetilde{D}(\widetilde{z})\psi = -P(\overline{\widetilde{z}})^*\widetilde{P}(\widetilde{z})\psi, \quad (3.25)$$

and the densely defined bounded operators $D'(z) = \frac{d}{dz}D(z)$ *and* $\widetilde{D}'(\widetilde{z}) = \frac{d}{d\widetilde{z}}\widetilde{D}(\widetilde{z})$ *in* $L^2(\partial\Omega)$ *admit continuous extensions*

$$\overline{D'(z)} = -\widetilde{P}(\overline{z})^*\overline{P(z)} \in \mathcal{B}(L^2(\partial\Omega)) \quad (3.26)$$

and

$$\overline{\widetilde{D}'(\widetilde{z})} = -P(\overline{\widetilde{z}})^*\overline{\widetilde{P}(\widetilde{z})} \in \mathcal{B}(L^2(\partial\Omega)). \quad (3.27)$$

The $\mathcal{B}(L^2(\partial\Omega))$*-valued functions* $z \mapsto \overline{D'(z)}$ *and* $\widetilde{z} \mapsto \overline{\widetilde{D}'(\widetilde{z})}$ *are analytic on* $\rho(A_D)$ *and* $\rho\big(\widetilde{A}_D\big)$*, respectively, and finitely meromorphic on* $\mathbb{C}$*.*

Proof. By (3.21) and (3.23), the derivatives $\frac{d}{dz}D(z)\varphi$ and $\frac{d}{d\widetilde{z}}\widetilde{D}(\widetilde{z})\psi$ exist for all $\varphi, \psi \in H^1(\partial\Omega)$ and have the form as in (3.25). It is also clear from Lemma 3.5 that the operators

$$\widetilde{P}(\overline{z})^*P(z), \quad P(\overline{\widetilde{z}})^*\widetilde{P}(\widetilde{z}) \quad (3.28)$$

are defined on the dense subspace $H^1(\partial\Omega)$, and both are bounded. Hence, the continuous extensions onto $L^2(\Omega)$ are given by (3.26) and (3.27), respectively. From (3.26) and Lemma 3.5 we conclude for some $z_0 \in \rho(A_D)$ and all $z \in \rho(A_D)$ that

$$\begin{aligned}
\overline{D'(z)} &= -\Big(\big(I_{L^2(\Omega)} + (\overline{z} - \overline{z_0})\big(\widetilde{A}_D - \overline{z}I_{L^2(\Omega)}\big)^{-1}\big)\widetilde{P}(\overline{z_0})\Big)^* \\
&\quad \times \big(I_{L^2(\Omega)} + (z - z_0)(A_D - zI_{L^2(\Omega)})^{-1}\big)\overline{P(z_0)} \\
&= -\widetilde{P}(\overline{z_0})^*\big(I_{L^2(\Omega)} + (z - z_0)(A_D - zI_{L^2(\Omega)})^{-1}\big) \\
&\quad \times \big(I_{L^2(\Omega)} + (z - z_0)(A_D - zI_{L^2(\Omega)})^{-1}\big)\overline{P(z_0)},
\end{aligned} \quad (3.29)$$

which shows that $z \mapsto \overline{D'(z)}$ is analytic on $\rho(A_D)$ and finitely meromorphic on $\mathbb{C}$ (cf. Examples 2.2 and 2.3). $\square$

Hypothesis 3.7. In addition to the assumptions in Hypothesis 3.1, suppose $\Theta \in \mathcal{B}(L^2(\partial\Omega))$, and let A_Θ and $\widetilde{A}_{\Theta^*}$ denote the Robin realizations of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ in $L^2(\Omega)$,

$$A_\Theta f = -\Delta f + qf, \quad f \in \operatorname{dom}(A_\Theta) = \big\{g \in H^{3/2}_\Delta(\Omega) \,\big|\, \Theta\gamma_D g = \gamma_N g\big\}, \tag{3.30}$$

and

$$\widetilde{A}_{\Theta^*} f = -\Delta f + \overline{q} f, \quad f \in \operatorname{dom}\big(\widetilde{A}_{\Theta^*}\big) = \big\{g \in H^{3/2}_\Delta(\Omega) \,\big|\, \Theta^*\gamma_D g = \gamma_N g\big\}. \tag{3.31}$$

In connection with A_Θ and $\widetilde{A}_{\Theta^*}$, one obtains the following variant of Proposition 3.3:

Proposition 3.8. *Assume Hypothesis 3.7. Then A_Θ and $\widetilde{A}_{\Theta^*}$ are closed operators in $L^2(\Omega)$ which are adjoint to each other,*

$$A_\Theta^* = \widetilde{A}_{\Theta^*}. \tag{3.32}$$

In addition, A_Θ and $\widetilde{A}_{\Theta^*}$ have compact resolvents.

In the next preparatory lemma, we study the operators $D(z) - \Theta$ and $\widetilde{D}(\widetilde{z}) - \Theta^*$ and their inverses in $L^2(\partial\Omega)$. As will turn out, these operators play an important role in the Krein-type resolvent formulas and index formulas at the end of this section.

Lemma 3.9. *Assume Hypothesis 3.7. Let $z \in \rho(A_D) \cap \rho(A_\Theta)$, $\widetilde{z} \in \rho\big(\widetilde{A}_D\big) \cap \rho\big(\widetilde{A}_{\Theta^*}\big)$, and let $D(z)$ and $\widetilde{D}(\widetilde{z})$ be the Dirichlet-to-Neumann maps associated to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, respectively. Then the following assertions hold:*
(i) $D(z) - \Theta$ is boundedly invertible and the inverse is a compact operator in $L^2(\partial\Omega)$,

$$(D(z) - \Theta)^{-1} \in \mathcal{B}_\infty(L^2(\partial\Omega)). \tag{3.33}$$

Furthermore, the map $z \mapsto (D(z) - \Theta)^{-1}$ is analytic on $\rho(A_\Theta)$ and finitely meromorphic on $\mathbb{C}$.
$\widetilde{(i)}$ $\widetilde{D}(\widetilde{z}) - \Theta^*$ is boundedly invertible and the inverse is a compact operator in $L^2(\partial\Omega)$,

$$\big(\widetilde{D}(\widetilde{z}) - \Theta^*\big)^{-1} \in \mathcal{B}_\infty(L^2(\partial\Omega)). \tag{3.34}$$

Furthermore, the map $\widetilde{z} \mapsto \big(\widetilde{D}(\widetilde{z}) - \Theta^*\big)^{-1}$ is analytic on $\rho\big(\widetilde{A}_\Theta\big)$ and finitely meromorphic on $\mathbb{C}$.

Proof. The proof of Lemma 3.9 (i) is divided into seven separate steps. The proof of item $\widetilde{(i)}$ follows precisely the same strategy and is hence omitted.

Step 1. It will be shown first that the operator $D(z) - \Theta$ is injective for any $z \in \rho(A_D) \cap \rho(A_\Theta)$. Assume that for some $\varphi \in H^1(\partial\Omega)$,

$$\big(D(z) - \Theta\big)\varphi = 0 \tag{3.35}$$

and let $f_z \in H^{3/2}_\Delta(\Omega)$ be the unique solution of the boundary value problem

$$\begin{cases} \mathcal{L}f - zf = 0, \\ \gamma_D f = \varphi. \end{cases} \tag{3.36}$$

Then one infers

$$\Theta\gamma_D f_z = \Theta\varphi = D(z)\varphi = D(z)\gamma_D f_z = \gamma_N f_z, \tag{3.37}$$

and hence $f_z \in \mathrm{dom}(A_\Theta)$ with $A_\Theta f_z = zf_z$. As $z \in \rho(A_\Theta)$, one concludes $f_z = 0$, and hence $\varphi = \gamma_D f_z = 0$.

Step 2. In order to see that $D(z) - \Theta$ maps onto $L^2(\partial\Omega)$, one recalls that the inverse of the Dirichlet-to-Neumann map $N(z) = D(z)^{-1}$, the Neumann-to-Dirichlet map, is well-defined for all $z \in \rho(A_D) \cap \rho(A_N)$, where A_N denotes the Neumann realization of $\mathcal{L} = -\Delta + q$,

$$A_N f = -\Delta f + qf, \quad f \in \mathrm{dom}(A_N) = \big\{g \in H^{3/2}_\Delta(\Omega) \,\big|\, \gamma_N g = 0\big\}. \tag{3.38}$$

Moreover, it follows in the same way as in [9, Proposition 4.6] or [8, Lemma 4.6] that

$$N(z) \in \mathcal{B}_\infty(L^2(\partial\Omega)). \tag{3.39}$$

For $z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$, the operator $I_{L^2(\partial\Omega)} - \Theta N(z)$ is injective. In fact, suppose that $\varphi = \Theta N(z)\varphi$ for some $\varphi \in L^2(\partial\Omega)$ and choose $f_z \in H^{3/2}_\Delta(\Omega)$ such that $\mathcal{L}f_z = zf_z$ and $\gamma_N f_z = \varphi$. Then

$$\gamma_N f_z = \varphi = \Theta N(z)\varphi = \Theta N(z)\gamma_N f_z = \Theta\gamma_D f_z, \tag{3.40}$$

and hence $f_z \in \mathrm{dom}(A_\Theta)$. As $z \in \rho(A_\Theta)$, one concludes that $f_z = 0$, and therefore, $\varphi = \gamma_N f_z = 0$.

The fact (3.39) and the assumption $\Theta \in \mathcal{B}(L^2(\partial\Omega))$ imply $\Theta N(z) \in \mathcal{B}_\infty(L^2(\partial\Omega))$ and since $I_{L^2(\partial\Omega)} - \Theta N(z)$ is injective, one concludes

$$\begin{aligned} \big(D(z) - \Theta\big)^{-1} &= N(z)\big((D(z) - \Theta)N(z)\big)^{-1} \\ &= N(z)\big(I_{L^2(\partial\Omega)} - \Theta N(z)\big)^{-1} \in \mathcal{B}(L^2(\partial\Omega)) \end{aligned} \tag{3.41}$$

for all $z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$. Therefore, $(D(z) - \Theta)^{-1}$ is closed as an operator in $L^2(\partial\Omega)$ and since $\mathrm{ran}((D(z) - \Theta)^{-1}) = H^1(\partial\Omega)$, the operator $(D(z) - \Theta)^{-1}$ is also closed as an operator from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$. This implies

$$(D(z) - \Theta)^{-1} \in \mathcal{B}\big(L^2(\partial\Omega), H^1(\partial\Omega)\big), \tag{3.42}$$

and as $H^1(\partial\Omega)$ is compactly embedded in $L^2(\partial\Omega)$, one concludes

$$(D(z) - \Theta)^{-1} \in \mathcal{B}_\infty(L^2(\partial\Omega)), \quad z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N). \tag{3.43}$$

Step 3. Let $z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$ and $\widetilde{z} \in \rho\big(\widetilde{A}_{\Theta^*}\big) \cap \rho\big(\widetilde{A}_D\big) \cap \rho\big(\widetilde{A}_N\big)$. One observes first that for $\varphi \in L^2(\partial\Omega)$ and $\psi \in L^2(\partial\Omega)$ the boundary value problems

$$\begin{cases} \mathcal{L}f - zf = 0, \\ \gamma_N f - \Theta\gamma_D f = \varphi, \end{cases} \tag{3.44}$$

and

$$\begin{cases} \widetilde{\mathcal{L}}g - \widetilde{z}g = 0, \\ \gamma_N g - \Theta^*\gamma_D g = \psi, \end{cases} \tag{3.45}$$

admit unique solutions in $H^{3/2}_\Delta(\Omega)$. In fact, since the operators $(D(z) - \Theta)^{-1}$ and $\big(\widetilde{D}(\widetilde{z}) - \Theta^*\big)^{-1}$ are defined on $L^2(\partial\Omega)$, and map into $H^1(\partial\Omega)$, the boundary value problems

$$\begin{cases} \mathcal{L}f - zf = 0, \\ \gamma_D f = (D(z) - \Theta)^{-1}\varphi, \end{cases} \tag{3.46}$$

and

$$\begin{cases} \widetilde{\mathcal{L}}g - \widetilde{z}g = 0, \\ \gamma_D g = \big(\widetilde{D}(\widetilde{z}) - \Theta^*\big)^{-1}\psi, \end{cases} \tag{3.47}$$

admit unique solutions $f_z \in H^{3/2}_\Delta(\Omega)$ and $g_{\widetilde{z}} \in H^{3/2}_\Delta(\Omega)$. Since

$$\gamma_N f_z - \Theta\gamma_D f_z = (D(z) - \Theta)\gamma_D f_z = \varphi, \tag{3.48}$$

and

$$\gamma_N g_{\widetilde{z}} - \Theta^*\gamma_D g_{\widetilde{z}} = \big(\widetilde{D}(\widetilde{z}) - \Theta^*\big)\gamma_D g_{\widetilde{z}} = \psi, \tag{3.49}$$

it is clear that f_z and $g_{\widetilde{z}}$ solve (3.44) and (3.45), respectively. We shall denote the solution operators corresponding to the boundary value problems (3.44) and (3.45) by $P_\Theta(z)$ and $\widetilde{P}_{\Theta^*}(\widetilde{z})$, respectively, that is,

$$P_\Theta(z) : L^2(\partial\Omega) \to L^2(\Omega), \quad \varphi \mapsto f_z, \tag{3.50}$$

and

$$\widetilde{P}_{\Theta^*}(\widetilde{z}) : L^2(\partial\Omega) \to L^2(\Omega), \quad \psi \mapsto g_{\widetilde{z}}, \tag{3.51}$$

where $f_z \in H^{3/2}_\Delta(\Omega)$ and $g_{\widetilde{z}} \in H^{3/2}_\Delta(\Omega)$ denote the unique solutions of (3.44) and (3.45), respectively.

Step 4. We claim that for $z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$ and $\widetilde{z} \in \rho\big(\widetilde{A}_{\Theta^*}\big) \cap \rho\big(\widetilde{A}_D\big) \cap \rho\big(\widetilde{A}_N\big)$ the operators $P_\Theta(z)$ and $\widetilde{P}_{\Theta^*}(\widetilde{z})$ in (3.50) and (3.51), respectively, are bounded, that is,

$$P_\Theta(z) \in \mathcal{B}\big(L^2(\partial\Omega), L^2(\Omega)\big), \quad \widetilde{P}_{\Theta^*}(\widetilde{z}) \in \mathcal{B}\big(L^2(\partial\Omega), L^2(\Omega)\big). \tag{3.52}$$

In fact, in order to verify the assertion for $P_\Theta(z)$ let $\varphi \in L^2(\partial\Omega)$ and $k \in L^2(\Omega)$. Since $z \in \rho(A_\Theta)$ implies $\overline{z} \in \rho\big(\widetilde{A}_{\Theta^*}\big)$, there exists $h \in \mathrm{dom}\big(\widetilde{A}_{\Theta^*}\big)$ such that

$$k = \big(\widetilde{A}_{\Theta^*} - \overline{z} I_{L^2(\Omega)}\big)h. \tag{3.53}$$

Thus one computes with the help of Green's Second Identity (3.7), the boundary condition $\gamma_N h = \Theta^* \gamma_D h$, and the definition of $P_\Theta(z)$, that

$$\begin{aligned}
(P_\Theta(z)\varphi, k)_{L^2(\Omega)} &= \big(f_z, \big(\widetilde{A}_{\Theta^*} - \overline{z} I_{L^2(\Omega)}\big)h\big)_{L^2(\Omega)} \\
&= \big(f_z, \widetilde{\mathcal{L}}h\big)_{L^2(\Omega)} - (f_z, \overline{z}h)_{L^2(\Omega)} \\
&= \big(f_z, \widetilde{\mathcal{L}}h\big)_{L^2(\Omega)} - (\mathcal{L} f_z, h)_{L^2(\Omega)} \\
&= (\gamma_N f_z, \gamma_D h)_{L^2(\partial\Omega)} - (\gamma_D f_z, \gamma_N h)_{L^2(\partial\Omega)} \\
&= (\gamma_N f_z, \gamma_D h)_{L^2(\partial\Omega)} - (\gamma_D f_z, \Theta^* \gamma_D h)_{L^2(\partial\Omega)} \\
&= \big([\gamma_N f_z - \Theta\gamma_D f_z], \gamma_D h\big)_{L^2(\partial\Omega)} \\
&= \big(\varphi, \gamma_D \big(\widetilde{A}_{\Theta^*} - \overline{z} I_{L^2(\Omega)}\big)^{-1} k\big)_{L^2(\partial\Omega)}.
\end{aligned} \tag{3.54}$$

The above computation implies that $P_\Theta(z)^*$ is defined on all of $L^2(\Omega)$ and given by

$$P_\Theta(z)^* = \gamma_D \big(\widetilde{A}_{\Theta^*} - \overline{z} I_{L^2(\Omega)}\big)^{-1}, \tag{3.55}$$

and since $P_\Theta(z)^*$ is automatically closed it follows that

$$P_\Theta(z)^* \in \mathcal{B}\big(L^2(\Omega), L^2(\partial\Omega)\big). \tag{3.56}$$

Hence $P_\Theta(z)^{**} \in \mathcal{B}(L^2(\partial\Omega), L^2(\Omega))$ and since $\mathrm{dom}(P_\Theta(z)) = L^2(\partial\Omega)$ it follows that $P_\Theta(z)$ and $P_\Theta(z)^{**}$ coincide. Consequently, $P_\Theta(z) \in \mathcal{B}\big(L^2(\partial\Omega), L^2(\Omega)\big)$. The proof of the second assertion in (3.52) is completely analogous.

Step 5. It will be shown that the solution operators in (3.50) and (3.51) satisfy the identities

$$P_\Theta(z) = \big(I_{L^2(\Omega)} + (z - z_0)(A_\Theta - z I_{L^2(\Omega)})^{-1}\big) P_\Theta(z_0) \tag{3.57}$$

for all $z, z_0 \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$, and

$$\widetilde{P}_{\Theta^*}(\widetilde{z}) = \big(I_{L^2(\Omega)} + (\widetilde{z} - \widetilde{z}_0)\big(\widetilde{A}_{\Theta^*} - \widetilde{z} I_{L^2(\Omega)}\big)^{-1}\big) \widetilde{P}_{\Theta^*}(\widetilde{z}_0) \tag{3.58}$$

for all $\widetilde{z}, \widetilde{z}_0 \in \rho\big(\widetilde{A}_{\Theta^*}\big) \cap \rho\big(\widetilde{A}_D\big) \cap \rho\big(\widetilde{A}_N\big)$, respectively. We verify (3.57) and omit details of the analogous proof of (3.58). Let $\varphi \in L^2(\partial\Omega)$ and let $f_{z_0} \in H^{3/2}_\Delta(\Omega)$ be the unique solution of the boundary value problem

$$\begin{cases} \mathcal{L}f - z_0 f = 0, \\ \gamma_N f - \Theta\gamma_D f = \varphi, \end{cases} \tag{3.59}$$

so that $P_\Theta(z_0)\varphi = f_{z_0}$. Since $z \in \rho(A_\Theta)$, one can make use of the direct sum decomposition

$$H^{3/2}_\Delta(\Omega) = \operatorname{dom}(A_\Theta) \dot{+} \big\{ f \in H^{3/2}_\Delta(\Omega) \,\big|\, \mathcal{L}f - zf = 0 \big\} \tag{3.60}$$

and write f_{z_0} in the form

$$f_{z_0} = f_\Theta + f_z, \tag{3.61}$$

where $f_\Theta \in \operatorname{dom}(A_\Theta)$ and $f_z \in H^{3/2}_\Delta(\Omega)$ satisfies $\mathcal{L}f_z - zf_z = 0$. Since $\gamma_N f_\Theta - \Theta\gamma_D f_\Theta = 0$, it follows from (3.61) that

$$\gamma_N f_z - \Theta\gamma_D f_z = \gamma_N f_{z_0} - \Theta\gamma_D f_{z_0} = \varphi, \tag{3.62}$$

and hence f_z in (3.61) is the unique solution of the boundary value problem

$$\begin{cases} \mathcal{L}f - zf = 0, \\ \gamma_N f - \Theta\gamma_D f = \varphi, \end{cases} \tag{3.63}$$

so that $P_\Theta(z)\varphi = f_z$. As $f_z - f_{z_0} = -f_\Theta \in \operatorname{dom}(A_\Theta)$, one can choose $g \in L^2(\Omega)$ such that

$$f_z - f_{z_0} = (A_\Theta - zI_{L^2(\Omega)})^{-1} g, \tag{3.64}$$

and then one computes

$$\begin{aligned} (z - z_0) f_{z_0} &= z\big(f_z - (A_\Theta - zI_{L^2(\Omega)})^{-1} g\big) - z_0 f_{z_0} \\ &= \mathcal{L}(f_z - f_{z_0}) - z(A_\Theta - zI_{L^2(\Omega)})^{-1} g \\ &= \mathcal{L}(A_\Theta - zI_{L^2(\Omega)})^{-1} g - z(A_\Theta - zI_{L^2(\Omega)})^{-1} g \\ &= g, \end{aligned} \tag{3.65}$$

which yields

$$\begin{aligned} P_\Theta(z)\varphi &= f_z \\ &= f_{z_0} + (A_\Theta - zI_{L^2(\Omega)})^{-1} g \\ &= f_{z_0} + (z - z_0)(A_\Theta - zI_{L^2(\Omega)})^{-1} f_{z_0} \\ &= \big(I_{L^2(\Omega)} + (z - z_0)(A_\Theta - zI_{L^2(\Omega)})^{-1}\big) P_\Theta(z_0)\varphi. \end{aligned} \tag{3.66}$$

This establishes (3.57); the proof of (3.58) is analogous.

Step 6. Let $z \in \rho(A_\Theta) \cap \rho(A_D) \cap \rho(A_N)$ and $\widetilde{z} \in \rho\big(\widetilde{A}_{\Theta^*}\big) \cap \rho\big(\widetilde{A}_D\big) \cap \rho\big(\widetilde{A}_N\big)$. In this step we verify the identity

$$\big(D(z)-\Theta\big)^{-1}=\big(D(\overline{\widetilde{z}})-\Theta\big)^{-1}+(z-\overline{\widetilde{z}})\big(\widetilde{P}_{\Theta^*}(\widetilde{z})\big)^*P_\Theta(z). \tag{3.67}$$

Let $\varphi,\psi\in L^2(\partial\Omega)$ and let $f_z=P_\Theta(z)\varphi$ and $g_{\widetilde{z}}=\widetilde{P}_{\Theta^*}(\widetilde{z})\psi$. Then f_z satisfies

$$\begin{cases}\mathcal{L}f_z-zf_z=0,\\ \gamma_N f_z-\Theta\gamma_D f_z=\varphi,\end{cases} \tag{3.68}$$

$g_{\widetilde{z}}$ satisfies

$$\begin{cases}\widetilde{\mathcal{L}}g_{\widetilde{z}}-\widetilde{z}g_{\widetilde{z}}=0,\\ \gamma_N g_{\widetilde{z}}-\Theta^*\gamma_D g_{\widetilde{z}}=\psi,\end{cases} \tag{3.69}$$

and

$$\gamma_D f_z=(D(z)-\Theta)^{-1}\varphi,\quad \gamma_D g_{\widetilde{z}}=\big(\widetilde{D}(\widetilde{z})-\Theta^*\big)^{-1}\psi. \tag{3.70}$$

Hence, one infers

$$\begin{aligned}
&\big((D(z)-\Theta)^{-1}\varphi,\psi\big)_{L^2(\partial\Omega)}-\big(\varphi,\big(\widetilde{D}(\widetilde{z})-\Theta^*\big)^{-1}\psi\big)_{L^2(\partial\Omega)}\\
&\quad=\big(\gamma_D f_z,[\gamma_N g_{\widetilde{z}}-\Theta^*\gamma_D g_{\widetilde{z}}]\big)_{L^2(\partial\Omega)}-\big([\gamma_N f_z-\Theta\gamma_D f_z],\gamma_D g_{\widetilde{z}}\big)_{L^2(\partial\Omega)}\\
&\quad=(\gamma_D f_z,\gamma_N g_{\widetilde{z}})_{L^2(\partial\Omega)}-(\gamma_N f_z,\gamma_D g_{\widetilde{z}})_{L^2(\partial\Omega)}\\
&\quad=\big(\mathcal{L}f_z,g_{\widetilde{z}}\big)_{L^2(\Omega)}-\big(f_z,\widetilde{\mathcal{L}}g_{\widetilde{z}}\big)_{L^2(\Omega)}\\
&\quad=(zf_z,g_{\widetilde{z}})_{L^2(\Omega)}-(f_z,\widetilde{z}g_{\widetilde{z}})_{L^2(\Omega)}\\
&\quad=(z-\overline{\widetilde{z}})\big(P_\Theta(z)\varphi,\widetilde{P}_{\Theta^*}(\widetilde{z})\psi\big)_{L^2(\Omega)}\\
&\quad=(z-\overline{\widetilde{z}})\big(\big(\widetilde{P}_{\Theta^*}(\widetilde{z})\big)^*P_\Theta(z)\varphi,\psi\big)_{L^2(\partial\Omega)}.
\end{aligned} \tag{3.71}$$

In particular, for $z=\overline{\widetilde{z}}$,

$$\big((D(\overline{\widetilde{z}})-\Theta)^{-1}\varphi,\psi\big)_{L^2(\partial\Omega)}=\big(\varphi,\big(\widetilde{D}(\widetilde{z})-\Theta^*\big)^{-1}\psi\big)_{L^2(\partial\Omega)}, \tag{3.72}$$

and hence

$$(D(\overline{\widetilde{z}})-\Theta)^{-1}=\big(\big(\widetilde{D}(\widetilde{z})-\Theta^*\big)^{-1}\big)^*. \tag{3.73}$$

Together with (3.73), (3.71) implies that

$$(D(z)-\Theta)^{-1}-(D(\overline{\widetilde{z}})-\Theta)^{-1}=(z-\overline{\widetilde{z}})\big(\widetilde{P}_{\Theta^*}(\widetilde{z})\big)^*P_\Theta(z), \tag{3.74}$$

yielding (3.67).

Step 7. For $z\in\rho(A_\Theta)\cap\rho(A_D)\cap\rho(A_N)$ and $\widetilde{z}\in\rho\big(\widetilde{A}_{\Theta^*}\big)\cap\rho\big(\widetilde{A}_D\big)\cap\rho\big(\widetilde{A}_N\big)$ one obtains via (3.57) and (3.67) the identity

$$\begin{aligned}\big(D(z)-\Theta\big)^{-1} &= \big(D(\overline{\widetilde{z}})-\Theta\big)^{-1} \\ &\quad + (z-\overline{\widetilde{z}})\big(\widetilde{P}_{\Theta^*}(\widetilde{z})\big)^*\big(I_{L^2(\Omega)}+(z-\overline{\widetilde{z}})(A_\Theta - zI_{L^2(\Omega)})^{-1}\big)P_\Theta(\overline{\widetilde{z}}).\end{aligned} \tag{3.75}$$

Here, the fact that $\overline{\widetilde{z}} \in \rho(A_\Theta)\cap\rho(A_D)\cap\rho(A_N)$ has been used. It follows from (3.75) that the map

$$z \mapsto (D(z)-\Theta)^{-1} \tag{3.76}$$

is holomorphic on the set $\rho(A_\Theta)\cap\rho(A_D)\cap\rho(A_N)$ and that it admits an analytic continuation to the set $\rho(A_\Theta)$. One also infers from (3.43) that the values of this analytic continuation are compact operators in $L^2(\partial\Omega)$. Moreover, the fact that $z\mapsto (A_\Theta - zI_{L^2(\Omega)})^{-1}$ is finitely meromorphic on $\mathbb{C}$ implies that the map in (3.76) is finitely meromorphic on $\mathbb{C}$ (cf. Example 2.3), completing the proof of Lemma 3.9. □

The next theorems contain the index formulas that constitute the main results in this section. To set the stage, we also verify Krein-type resolvent formulas which relate the inverses $(A_\Theta - zI_{L^2(\Omega)})^{-1}$ and $\big(\widetilde{A}_{\Theta^*} - \widetilde{z}I_{L^2(\Omega)}\big)^{-1}$ with the resolvents of the Dirichlet realizations A_D and $\widetilde{A}_D$, respectively. For the self-adjoint case, such formulas are well-known and can be found, for example, in [1,8,9,11,15,29,30,48,58,59]. For dual pairs of elliptic differential operators we refer to [14], and for a more abstract operator theory framework, see [49] and [50]. The present version is partly inspired by [10, Theorem 6.16] and can be regarded as a non-self-adjoint variant for dual pairs of Schrödinger operators with complex-valued potentials.

Theorem 3.10. *Assume Hypotheses 3.2 and 3.7. For $z\in\rho(A_D)\cap\rho(A_\Theta)$ the Krein-type resolvent formula*

$$(A_\Theta - zI_{L^2(\Omega)})^{-1} = (A_D - zI_{L^2(\Omega)})^{-1} + P(z)(D(z)-\Theta)^{-1}\widetilde{P}(\overline{z})^* \tag{3.77}$$

holds, and

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(D(\cdot)-\Theta) = m_a(z_0;A_\Theta) - m_a(z_0;A_D), \quad z_0\in\mathbb{C}. \tag{3.78}$$

Theorem 3.11. *Assume Hypotheses 3.2 and 3.7. For $\widetilde{z}\in\rho\big(\widetilde{A}_D\big)\cap\rho\big(\widetilde{A}_{\Theta^*}\big)$ the Krein-type resolvent formula*

$$\big(\widetilde{A}_{\Theta^*} - \widetilde{z}I_{L^2(\Omega)}\big)^{-1} = \big(\widetilde{A}_D - \widetilde{z}I_{L^2(\Omega)}\big)^{-1} + \widetilde{P}(\widetilde{z})\big(\widetilde{D}(\widetilde{z})-\Theta^*\big)^{-1}P(\overline{\widetilde{z}})^* \tag{3.79}$$

holds and

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}\big(\widetilde{D}(\cdot)-\Theta^*\big) = m_a\big(z_0;\widetilde{A}_{\Theta^*}\big) - m_a\big(z_0;\widetilde{A}_D\big), \quad z_0\in\mathbb{C}. \tag{3.80}$$

Proof of Theorem 3.10. Fix $z\in\rho(A_D)\cap\rho(A_\Theta)$. One recalls that according to Lemma 3.9,

$$(D(z)-\Theta)^{-1} \in \mathcal{B}_\infty(L^2(\partial\Omega)). \tag{3.81}$$

Moreover, since

$$\operatorname{dom}(P(z)) = \operatorname{dom}\big(D(z) - \Theta\big) = \operatorname{ran}\big((D(z) - \Theta)^{-1}\big), \tag{3.82}$$

the perturbation term

$$P(z)\big(D(z) - \Theta\big)^{-1} \widetilde{P}(\overline{z})^* \tag{3.83}$$

on the right-hand side of (3.77) is well-defined. Next, let $f \in L^2(\Omega)$ and consider the function

$$h = (A_D - zI_{L^2(\Omega)})^{-1} f + P(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f. \tag{3.84}$$

We claim that $h \in H^{3/2}_{\Delta}(\Omega)$ satisfies the boundary condition

$$\Theta\gamma_D h = \gamma_N h. \tag{3.85}$$

First of all, it is clear that $h \in H^{3/2}_{\Delta}(\Omega)$ since $\operatorname{dom}(A_D) \subset H^{3/2}_{\Delta}(\Omega)$ by (3.8) and $\operatorname{ran}(P(z)) \subset H^{3/2}_{\Delta}(\Omega)$ by Definition 3.4. In order to check (3.85) one observes that

$$\begin{aligned} \gamma_D h &= \gamma_D(A_D - zI_{L^2(\Omega)})^{-1} f + \gamma_D P(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f \\ &= (D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f, \end{aligned} \tag{3.86}$$

and

$$\begin{aligned} \gamma_N h &= \gamma_N(A_D - zI_{L^2(\Omega)})^{-1} f + \gamma_N P(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f \\ &= -\widetilde{P}(\overline{z})^* f + D(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f \\ &= \Theta(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f, \end{aligned} \tag{3.87}$$

where we have used Lemma 3.5 $\widetilde{(i)}$ and the definition of the Dirichlet-to-Neumann map. At this point it is clear from (3.86) and (3.87) that (3.85) holds. Thus, one concludes $h \in \operatorname{dom}(A_\Theta)$ and hence it follows from

$$\begin{aligned} &(A_\Theta - zI_{L^2(\Omega)})h \\ &\quad = (A_\Theta - zI_{L^2(\Omega)})\big((A_D - zI_{L^2(\Omega)})^{-1} f + P(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f\big) \\ &\quad = (\mathcal{L} - zI_{L^2(\Omega)})(A_D - zI_{L^2(\Omega)})^{-1} f + (\mathcal{L} - zI_{L^2(\Omega)})P(z)(D(z) - \Theta)^{-1} \widetilde{P}(\overline{z})^* f \\ &\quad = f \end{aligned} \tag{3.88}$$

that (3.77) holds as well.

Next we will verify that the map

$$z \mapsto M(z) = D(z) - \Theta, \quad z \in \rho(A_D), \tag{3.89}$$

satisfies the assumptions in Hypothesis 2.4 with $\mathcal{D} = \mathbb{C}$ and $\mathcal{D}_0 = \sigma_p(A_D) \cup \sigma_p(A_\Theta)$. First, one recalls that the values of $D(\cdot)$ in (3.89) are closed operators in $L^2(\partial\Omega)$ according to

Lemma 3.5 (*iii*) and since $\Theta \in \mathcal{B}(L^2(\partial\Omega))$ the same is true for the values of $M(\cdot)$. It is also clear from Lemma 3.5 (*iii*) that

$$\operatorname{dom}(M(z)) = \operatorname{dom}(D(z)) = H^1(\partial\Omega) \tag{3.90}$$

is independent of z, that is, Hypothesis 2.4 (i) holds. Furthermore, it follows from Lemma 3.9 (i) that $M(z)^{-1} \in \mathcal{B}(L^2(\partial\Omega))$ for all $z \in \mathbb{C}\backslash\mathcal{D}_0$, and that $M(\cdot)^{-1}$ is analytic on $\mathbb{C}\backslash\mathcal{D}_0$ and finitely meromorphic on $\mathbb{C}$. Hence, items (*ii*) and (*iii*) in Hypothesis 2.4 are satisfied as well. Finally, the validity of items (*iv*) and (*v*) in Hypothesis 2.4 follow from Lemma 3.5 (*iii*) and Corollary 3.6.

It remains to prove the index formula (3.78). Making use of Corollary 3.6 one obtains for $0 < \varepsilon$ sufficiently small,

$$\begin{aligned}
\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) &= \operatorname{tr}_{L^2(\partial\Omega)}\bigg(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, M(\zeta)^{-1}\overline{M'(\zeta)}\bigg) \\
&= \operatorname{tr}_{L^2(\partial\Omega)}\bigg(\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, \big(D(\zeta)-\Theta\big)^{-1}\overline{D'(\zeta)}\bigg) \\
&= \operatorname{tr}_{L^2(\partial\Omega)}\bigg(-\frac{1}{2\pi i}\ointctrclockwise_{C(z_0;\varepsilon)} d\zeta\, \big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^*\overline{P(\zeta)}\bigg).
\end{aligned} \tag{3.91}$$

Next, we fix a point $\zeta' \in \rho(A_D)$ and use Lemma 3.5 to rewrite

$$\overline{P(\zeta)} = \big(I_{L^2(\Omega)} + (\zeta-\zeta')(A_D-\zeta)^{-1}\big)\overline{P(\zeta')} \tag{3.92}$$

and

$$\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^* = \big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta'})^*\big(I_{L^2(\Omega)} + (\zeta-\zeta')(A_D-\zeta)^{-1}\big). \tag{3.93}$$

Since $\zeta \mapsto I_{L^2(\Omega)} + (\zeta-\zeta')(A_D-\zeta)^{-1}$ and $\zeta \mapsto (D(\zeta)-\Theta)^{-1}$ are finitely meromorphic functions by Example 2.2 and Lemma 3.9, the functions in (3.92) and (3.93) admit the representations

$$\overline{P(\zeta)} = \sum_{k=-N_0}^{\infty} (\zeta-z_0)^k G_k(z_0), \tag{3.94}$$

and

$$\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^* = \sum_{k=-N_0}^{\infty} (\zeta-z_0)^k H_k(z_0), \tag{3.95}$$

for $\zeta \in D(z_0;\varepsilon)\backslash\{z_0\}$, where $N_0 \in \mathbb{N}$, $G_k(z_0) \in \mathcal{B}(L^2(\partial\Omega), L^2(\Omega))$, $H_k(z_0) \in \mathcal{B}(L^2(\Omega), L^2(\partial\Omega))$ for all $k \geq 0$, and

$$G_k(z_0) \in \mathcal{F}\big(L^2(\partial\Omega), L^2(\Omega)\big), \quad H_k(z_0) \in \mathcal{F}\big(L^2(\Omega), L^2(\partial\Omega)\big), \quad -N_0 \leq k \leq -1. \tag{3.96}$$

Now we argue in the same manner as in the proof of [35, Proposition 4.2.2]. From (3.94) and (3.95) one first concludes with the help of Cauchy's theorem that

$$\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^*\overline{P(\zeta)} = \sum_{-N_0\le k\le N_0-1} H_k(z_0)G_{-k-1}(z_0), \tag{3.97}$$

$$\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\overline{P(\zeta)}\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^* = \sum_{-N_0\le k\le N_0-1} G_k(z_0)H_{-k-1}(z_0), \tag{3.98}$$

where

$$H_k(z_0)G_{-k-1}(z_0)\in\mathcal{F}\big(L^2(\partial\Omega)\big), \qquad -N_0\le k\le N_0-1, \tag{3.99}$$

and

$$G_k(z_0)H_{-k-1}(z_0)\in\mathcal{F}\big(L^2(\Omega)\big), \qquad -N_0\le k\le N_0-1. \tag{3.100}$$

Using the cyclicity of the trace (see, e.g. [62, Theorem 7.11 (b)]) one finds

$$\begin{aligned}
&\operatorname{tr}_{L^2(\partial\Omega)}\bigg(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^*\overline{P(\zeta)}\bigg)\\
&\quad=\operatorname{tr}_{L^2(\partial\Omega)}\bigg(-\sum_{-N_0\le k\le N_0-1} H_k(z_0)G_{-k-1}(z_0)\bigg)\\
&\quad=\operatorname{tr}_{L^2(\Omega)}\bigg(-\sum_{-N_0\le k\le N_0-1} G_k(z_0)H_{-k-1}(z_0)\bigg)\\
&\quad=\operatorname{tr}_{L^2(\Omega)}\bigg(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\overline{P(\zeta)}\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^*\bigg).
\end{aligned} \tag{3.101}$$

At this point one concludes from (3.91), (3.101) and (3.77) that

$$\begin{aligned}
\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(M(\cdot)) &= \operatorname{tr}_{L^2(\Omega)}\bigg(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\overline{P(\zeta)}\big(D(\zeta)-\Theta\big)^{-1}\widetilde{P}(\overline{\zeta})^*\bigg)\\
&= \operatorname{tr}_{L^2(\Omega)}\bigg(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\,\big((A_\Theta-\zeta I_{L^2(\Omega)})^{-1}-(A_D-\zeta I_{L^2(\Omega)})^{-1}\big)\bigg)\\
&= \operatorname{tr}_{L^2(\Omega)}(P(z_0;A_\Theta))-\operatorname{tr}_{L^2(\Omega)}(P(z_0;A_D))\\
&= m_a(z_0;A_\Theta)-m_a(z_0;A_D),
\end{aligned} \tag{3.102}$$

where $P(z_0; A_\Theta)$ and $P(z_0; A_D)$ denote the Riesz projections onto the algebraic eigenspaces of A_Θ and A_D corresponding to z_0 (cf. Example 2.2). □

4. Closed extensions of symmetric operators and abstract Weyl–Titchmarsh M-functions

Let B_1 and B_2 be densely defined closed operators in a separable complex Hilbert space $\mathfrak{H}$ such that $\rho(B_1) \cap \rho(B_2) \neq \emptyset$ and consider the intersection $S = B_1 \cap B_2$ of B_1 and B_2, which is a closed operator of the form

$$Sf = B_1 f = B_2 f, \quad \operatorname{dom}(S) = \{f \in \operatorname{dom}(B_1) \cap \operatorname{dom}(B_2) \,|\, B_1 f = B_2 f\}. \tag{4.1}$$

Hypothesis 4.1. Assume that S in (4.1) is densely defined and symmetric in $\mathfrak{H}$ with equal deficiency indices. Let A_0 be a fixed self-adjoint extension of S in $\mathfrak{H}$, and assume that for $j = 1, 2$ the operators A_0 and B_j, as well as A_0 and B_j^*, are disjoint extensions of S, that is,

$$S = A_0 \cap B_1 = A_0 \cap B_2 = A_0 \cap B_1^* = A_0 \cap B_2^*. \tag{4.2}$$

It follows from Hypothesis 4.1 that both operators B_1 and B_2 are closed restrictions of the adjoint S^* of S, and hence B_1 and B_2 can be parametrized with the help of a boundary triple for S^* and closed parameters Θ_1 and Θ_2 in $\mathcal{G}$ (cf. (A.3)–(A.4)). In the same manner, B_1^* and B_2^* are closed restrictions of S^* and by (A.5) they correspond to the parameters Θ_1^* and Θ_2^* in $\mathcal{G}$. The assumption that for $j = 1, 2$ the operators A_0 and B_j, and A_0 and B_j^* are disjoint extensions of S implies that Θ_j and Θ_j^*, $j = 1, 2$, are closed operators, and hence their domains are dense in $\mathcal{G}$. We refer the reader to Appendix A for a brief introduction to the theory of boundary triples.

The following lemma is an immediate consequence of Proposition A.4, (A.3)–(A.5), and (A.7).

Lemma 4.2. *Assume that B_1, B_2, S and A_0 satisfy Hypothesis 4.1. Then there exists a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^*, and densely defined closed operators $\Theta_1, \Theta_2, \Theta_1^*, \Theta_2^* \in \mathcal{C}(\mathcal{G})$, such that $A_0 = S^* \upharpoonright \ker(\Gamma_0)$ and*

$$\begin{aligned} B_1 &= S^* \upharpoonright \ker(\Gamma_1 - \Theta_1\Gamma_0), \quad & B_2 &= S^* \upharpoonright \ker(\Gamma_1 - \Theta_2\Gamma_0), \\ B_1^* &= S^* \upharpoonright \ker(\Gamma_1 - \Theta_1^*\Gamma_0), \quad & B_2^* &= S^* \upharpoonright \ker(\Gamma_1 - \Theta_2^*\Gamma_0). \end{aligned} \tag{4.3}$$

The next theorem is the main result of this section. Here we make use of the boundary triple in Lemma 4.2 and express the difference of the algebraic multiplicities of a discrete eigenvalue of B_1 and B_2 with the help of the corresponding Weyl–Titchmarsh function $M(\cdot)$ and the parameters Θ_1 and Θ_2. Theorem 4.3 can be viewed as an abstract variant of Theorem 3.10.

Theorem 4.3. *Assume that B_1, B_2, S and A_0 satisfy Hypothesis 4.1, choose the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ in Lemma 4.2, and Θ_1, Θ_2 such that (4.3) holds. Let $M(\cdot)$ be the Weyl–Titchmarsh function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and assume that*

$$z_0 \in \sigma_d(B_j) \cup \rho(B_j), \quad j = 1, 2, \ \textit{and} \ z_0 \in \sigma_d(A_0) \cup \rho(A_0). \tag{4.4}$$

Then there exists $\varepsilon_0 > 0$ such that both functions $\Theta_j - M(\cdot)$, $j = 1, 2$, satisfy Hypothesis 2.4 with $\mathcal{D} = D(z_0; \varepsilon_0)$ and $\mathcal{D}_0 = \{z_0\}$, and the index formula

$$\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(\Theta_1 - M(\cdot)) - \widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(\Theta_2 - M(\cdot)) = m_a(z_0; B_1) - m_a(z_0; B_2) \tag{4.5}$$

holds.

Proof. We verify that the functions $\Theta_j - M(\cdot)$, $j = 1, 2$, satisfy items (i)–(ii) and (iv)–(v) of Hypothesis 2.4 with $\mathcal{D} = D(z_0; \varepsilon_0)$ and $\mathcal{D}_0 = \{z_0\}$. The proof of item (iii) is more involved and will be given separately in Proposition 4.7. First, one observes that by the assumptions in (4.4) one can choose $\varepsilon_0 > 0$ such that the punctured disc $D(z_0; \varepsilon_0)\backslash\{z_0\}$ is contained in the set $\rho(B_1) \cap \rho(B_2) \cap \rho(A_0)$. As Θ_j, $j = 1, 2$ are densely defined closed operators by Lemma 4.2 and the values of the Weyl–Titchmarsh function $M(\cdot)$ are bounded operators in $\mathcal{G}$, the functions

$$\Theta_j - M(\cdot) : D(z_0; \varepsilon_0)\backslash\{z_0\} \to \mathcal{C}(\mathcal{G}), \quad z \mapsto \Theta_j - M(z), \quad j = 1, 2, \tag{4.6}$$

are well-defined and of the form as in Hypothesis 2.4 with $\mathcal{D} = D(z_0; \varepsilon_0)$ and $\mathcal{D}_0 = \{z_0\}$. It is also clear that $\operatorname{dom}(\Theta_j - M(z)) = \operatorname{dom}(\Theta_j)$ is independent of $z \in D(z_0; \varepsilon_0)\backslash\{z_0\}$ and that $(\Theta_j - M(z))^{-1} \in \mathcal{B}(\mathcal{G})$ for all $z \in D(z_0; \varepsilon_0)\backslash\{z_0\}$ by Theorem A.5 (i). Hence items (i) and (ii) in Hypothesis 2.4 are satisfied. Since the Weyl–Titchmarsh function $M(\cdot)$ is analytic on $\rho(A_0)$ one infers

$$\frac{d}{dz}(\Theta_j - M(z))\varphi = -\frac{d}{dz}M(z)\varphi, \quad \varphi \in \operatorname{dom}(\Theta_j), \quad z \in D(z_0; \varepsilon_0)\backslash\{z_0\}, \tag{4.7}$$

and hence

$$\overline{(\Theta_j - M(z))'} = -M'(z), \tag{4.8}$$

that is, items (iv) and (v) in Hypothesis 2.4 hold (see (A.15) and Lemma A.3 for the fact that $M'(\cdot)$ is analytic on $D(z_0; \varepsilon_0)\backslash\{z_0\}$ and finitely meromorphic on $D(z_0; \varepsilon_0)$).

Next, we turn to the proof of the index formula. According to Theorem A.5 one infers $z \in \rho(B_j) \cap \rho(A_0)$ if and only if $(\Theta_j - M(z))^{-1} \in \mathcal{B}(\mathcal{G})$ for $j = 1, 2$ and Krein's formula

$$(B_j - zI_{\mathfrak{H}})^{-1} - (A_0 - zI_{\mathfrak{H}})^{-1} = \gamma(z)(\Theta_j - M(z))^{-1}\gamma(\overline{z})^* \tag{4.9}$$

is valid for all $z \in \rho(B_j) \cap \rho(A_0)$, $j = 1, 2$; here $\gamma(\cdot)$ denotes the γ-field corresponding to the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. Let $P(z_0; B_j)$, $j = 1, 2$, and $P(z_0; A_0)$ be the Riesz projections onto the algebraic eigenspaces of B_j and A_0 corresponding to z_0; since A_0 is self-adjoint the range of $P(z_0; A_0)$ coincides with $\ker(A_0 - z_0 I_{\mathfrak{H}})$. Then it follows from Definition 2.5, (4.8), and (A.17) that for $0 < \varepsilon$ sufficiently small,

$$\begin{aligned}
&\widetilde{\operatorname{ind}}_{C(z_0;\varepsilon)}(\Theta_j - M(\cdot)) \\
&\quad = \operatorname{tr}_{\mathcal{G}}\left(\frac{1}{2\pi i} \oint_{C(z_0;\varepsilon)} d\zeta\, (\Theta_j - M(\zeta))^{-1}\overline{(\Theta_j - M(\zeta))'}\right) \\
&\quad = \operatorname{tr}_{\mathcal{G}}\left(-\frac{1}{2\pi i} \oint_{C(z_0;\varepsilon)} d\zeta\, (\Theta_j - M(\zeta))^{-1} M'(\zeta)\right) \\
&\quad = \operatorname{tr}_{\mathcal{G}}\left(-\frac{1}{2\pi i} \oint_{C(z_0;\varepsilon)} d\zeta\, (\Theta_j - M(\zeta))^{-1}\gamma(\overline{\zeta})^*\gamma(\zeta)\right)
\end{aligned} \tag{4.10}$$

and the same argument as in the proof of Theorem 3.10 shows that

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{G}}\Big(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\, (\Theta_j - M(\zeta))^{-1}\gamma(\overline{\zeta})^*\gamma(\zeta)\Big)\\
&\qquad = \mathrm{tr}_{\mathfrak{H}}\Big(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\, \gamma(\zeta)(\Theta_j - M(\zeta))^{-1}\gamma(\overline{\zeta})^*\Big)
\end{aligned} \tag{4.11}$$

holds. Hence it follows from (4.10), (4.11), and (4.9) that

$$\begin{aligned}
&\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(\Theta_j - M(\cdot))\\
&\quad = \mathrm{tr}_{\mathfrak{H}}\Big(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\, \gamma(\zeta)(\Theta_j - M(\zeta))^{-1}\gamma(\overline{\zeta})^*\Big)\\
&\quad = \mathrm{tr}_{\mathfrak{H}}\Big(-\frac{1}{2\pi i}\oint_{C(z_0;\varepsilon)} d\zeta\, \big((B_j - \zeta I_{\mathfrak{H}})^{-1} - (A_0 - \zeta I_{\mathfrak{H}})^{-1}\big)\Big)\\
&\quad = \mathrm{tr}_{\mathfrak{H}}(P(z_0; B_j)) - \mathrm{tr}_{\mathfrak{H}}(P(z_0; A_0))\\
&\quad = m_a(z_0; B_j) - m_a(z_0; A_0), \quad j = 1, 2,
\end{aligned} \tag{4.12}$$

and this implies

$$\begin{aligned}
&\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(\Theta_1 - M(\cdot)) - \widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}(\Theta_2 - M(\cdot))\\
&\quad = m_a(z_0; B_1) - m_a(z_0; A_0) - m_a(z_0; B_2) + m_a(z_0; A_0)\\
&\quad = m_a(z_0; B_1) - m_a(z_0; B_2). \quad \square
\end{aligned} \tag{4.13}$$

In the next corollary, we discuss the special case that the closed operator B_1 is self-adjoint in $\mathfrak{H}$. In this case we set $A_0 = B_1$ and instead of Hypothesis 4.1 it suffices to assume that the closed symmetric operator $S = A_0 \cap B_2$ in (4.1) is densely defined and that $S = A_0 \cap B_2^*$ holds. Following Lemma 4.2 and Proposition A.4 one obtains a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^*, and densely defined closed operators $\Theta_2, \Theta_2^* \in \mathcal{C}(\mathcal{G})$, such that $A_0 = S^* \upharpoonright \ker(\Gamma_0)$ and

$$B_2 = S^* \upharpoonright \ker(\Gamma_1 - \Theta_2\Gamma_0), \quad B_2^* = S^* \upharpoonright \ker(\Gamma_1 - \Theta_2^*\Gamma_0). \tag{4.14}$$

Corollary 4.4. *Let $B_1 = A_0$, B_2, and $S = A_0 \cap B_2$ be as above, and choose a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and Θ_2 such that $A_0 = S^* \upharpoonright \ker(\Gamma_0)$ and (4.14) holds. Let $M(\cdot)$ be the Weyl–Titchmarsh function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and assume that*

$$z_0 \in \sigma_d(B_2) \cup \rho(B_2) \cup \sigma_d(A_0) \cup \rho(A_0). \tag{4.15}$$

Then there exists $\varepsilon_0 > 0$ such that the function $\Theta_2 - M(\cdot)$ satisfies Hypothesis 2.4 with $\mathcal{D} = D(z_0; \varepsilon_0)$ and $\mathcal{D}_0 = \{z_0\}$, and the index formula

$$\widetilde{\mathrm{ind}}_{C(z_0;\varepsilon)}\big(\Theta_2 - M(\cdot)\big) = m_a\big(z_0; B_2\big) - m_a\big(z_0; A_0\big) \tag{4.16}$$

holds.

It remains to show that the functions $\Theta_j - M(\cdot)$, $j = 1, 2$, satisfy Hypothesis 2.4 (*iii*). In the following considerations we discuss the general situation of unbounded closed operators Θ_1 and Θ_2 in Lemma 4.2 such that

$$B_j = S^* \upharpoonright \ker(\Gamma_1 - \Theta_j \Gamma_0), \quad B_j^* = S^* \upharpoonright \ker(\Gamma_1 - \Theta_j^* \Gamma_0), \quad j = 1, 2. \tag{4.17}$$

For the special case of bounded operators $\Theta_1, \Theta_2 \in \mathcal{B}(\mathcal{G})$ the considerations simplify slightly and we refer the reader to Remark 4.8 for more details. We start with the following preliminary lemma.

Lemma 4.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be the boundary triple in Lemma 4.2, let $\Theta_j \in \mathcal{C}(\mathcal{G})$, $j = 1, 2$, be densely defined closed operators such that (4.17) holds, and consider the map*

$$\Gamma_0^{\Theta_j} = \Gamma_1 - \Theta_j \Gamma_0, \quad \mathrm{dom}\big(\Gamma_0^{\Theta_j}\big) = \{f \in \mathrm{dom}(S^*) \,|\, \Gamma_0 f \in \mathrm{dom}(\Theta_j)\}. \tag{4.18}$$

Then the following assertions hold for $j = 1, 2$:

(*i*) $\mathrm{dom}(B_j) = \ker\big(\Gamma_0^{\Theta_j}\big)$;

(*ii*) $\mathrm{ran}(\Gamma_0^{\Theta_j})$ *is dense in* $\mathcal{G}$;

(*iii*) *the direct sum decomposition*

$$\mathrm{dom}\big(\Gamma_0^{\Theta_j}\big) = \mathrm{dom}(B_j) \dotplus \big(\ker(S^* - zI_{\mathfrak{H}}) \cap \mathrm{dom}\big(\Gamma_0^{\Theta_j}\big)\big) \tag{4.19}$$

holds for all $z \in \rho(B_j)$;

(*iv*) $\mathrm{dom}\big(\Gamma_0^{\Theta_j}\big)$ *is dense in* $\mathrm{dom}(S^*)$ *with respect to the graph norm.*

Proof. (*i*) is a direct consequence of the definition of $\Gamma_0^{\Theta_j}$ in (4.18) and (4.17).

(*ii*) In order to verify that $\mathrm{ran}\big(\Gamma_0^{\Theta_j}\big)$ is dense in $\mathcal{G}$ consider first the row operator $[-\Theta_j \;\, I_{\mathcal{G}}] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on the $\mathrm{dom}(\Theta_j) \times \mathcal{G}$ and note that by

$$\mathrm{ran}\big([-\Theta_j \;\, I_{\mathcal{G}}]\big)^{\perp} = \ker\left(\begin{bmatrix} -\Theta_j^* \\ I_{\mathcal{G}} \end{bmatrix}\right) = \{0\} \tag{4.20}$$

the range of $[-\Theta_j \;\, I_{\mathcal{G}}]$ is dense in $\mathcal{G}$. Hence it follows from

$$\Gamma_0^{\Theta_j} = \Gamma_1 - \Theta_j \Gamma_0 = [-\Theta_j \;\, I_{\mathcal{G}}] \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} \text{ and } \mathrm{ran}\left(\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}\right) = \mathcal{G} \times \mathcal{G} \tag{4.21}$$

that $\mathrm{ran}\big(\Gamma_0^{\Theta_j}\big)$ is dense in $\mathcal{G}$.

(*iii*) The inclusion ($\supset$) in (4.19) is clear from (*i*). In order to verify the inclusion ($\subset$) in (4.19), let $z \in \rho(B_j)$ and $h \in \mathrm{dom}\big(\Gamma_0^{\Theta_j}\big) \subset \mathrm{dom}(S^*)$, and choose $k \in \mathrm{dom}(B_j)$ such that

$$(S^* - zI_{\mathfrak{H}})h = (B_j - zI_{\mathfrak{H}})k. \tag{4.22}$$

Since S^* is an extension of B_j it follows that $h-k \in \ker(S^* - zI_{\mathfrak{H}})$ and as $h \in \operatorname{dom}(\Gamma_0^{\Theta_j})$ and $k \in \ker(\Gamma_0^{\Theta_j}) \subset \operatorname{dom}(\Gamma_0^{\Theta_j})$, hence also $h-k \in \operatorname{dom}(\Gamma_0^{\Theta_j})$. Thus,

$$h = k + (h-k), \quad \text{where } k \in \operatorname{dom}(B_j), \quad h-k \in \ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j}), \tag{4.23}$$

and hence the inclusion ($\subset$) in (4.19) is shown. The fact that the sum in (4.19) is direct follows from the assumption $z \in \rho(B_j)$.

(iv) Since $\Gamma_0 : \ker(S^* - zI_{\mathfrak{H}}) \to \mathcal{G}$, $z \in \rho(A_0)$, is an isomorphism with respect to the graph norm in $\ker(S^* - zI_{\mathfrak{H}})$ (which is equivalent to the norm in $\mathfrak{H}$), and since $\operatorname{dom}(\Theta_j)$ is dense in $\mathcal{G}$ we conclude that $\ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j})$ is dense in $\ker(S^* - zI_{\mathfrak{H}})$ with respect to the graph norm. It follows from (i) and the direct sum decomposition (4.19) that $\operatorname{dom}(\Gamma_0^{\Theta_j})$ is dense in $\operatorname{dom}(S^*)$ with respect to the graph norm. $\square$

One observes that by Lemma 4.5 the map

$$\Gamma_0^{\Theta_j} \upharpoonright \big(\ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j})\big) \to \mathcal{G}, \quad z \in \rho(B_j), \tag{4.24}$$

is injective and maps onto the dense subspace $\operatorname{ran}(\Gamma_0^{\Theta_j})$. Hence, for $z \in \rho(B_j)$ fixed, and every $\varphi \in \operatorname{ran}(\Gamma_0^{\Theta_j})$, there exists a unique $f_z \in \ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j})$ such that $\Gamma_0^{\Theta_j} f_z = \varphi$. In analogy to the γ-field corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ we define for $z \in \rho(B_j)$ the map

$$\gamma_{\Theta_j}(z)\varphi = f_z, \quad \operatorname{dom}(\gamma_{\Theta_j}(z)) = \operatorname{ran}(\Gamma_0^{\Theta_j}), \tag{4.25}$$

where $f_z \in \operatorname{dom}(\Gamma_0^{\Theta_j}) \cap \ker(S^* - zI_{\mathfrak{H}})$ satisfies $\Gamma_0^{\Theta_j} f_z = \varphi$. In the next lemma some important properties of the operators $\gamma_{\Theta_j}(z)$ are collected. The methods in the proof are abstract analogs of the computations in Step 4 and Step 5 in the proof of Lemma 3.9.

Lemma 4.6. *For all $z \in \rho(B_j)$ the operator $\gamma_{\Theta_j}(z)$ is densely defined and bounded from $\mathcal{G}$ into $\mathfrak{H}$. Furthermore, the identity*

$$\gamma_{\Theta_j}(z)\varphi = \big(I_{\mathfrak{H}} + (z-\zeta)(B_j - zI_{\mathfrak{H}})^{-1}\big)\gamma_{\Theta_j}(\zeta)\varphi, \quad z, \zeta \in \rho(B_j), \tag{4.26}$$

holds for all $\varphi \in \operatorname{dom}(\gamma_{\Theta_j}(z)) = \operatorname{dom}(\gamma_{\Theta_j}(\zeta)) = \operatorname{ran}(\Gamma_0^{\Theta_j})$, and extends by continuity to

$$\overline{\gamma_{\Theta_j}(z)} = \big(I_{\mathfrak{H}} + (z-\zeta)(B_j - zI_{\mathfrak{H}})^{-1}\big)\overline{\gamma_{\Theta_j}(\zeta)}, \quad z, \zeta \in \rho(B_j). \tag{4.27}$$

Proof. First of all it is clear from the definition of $\gamma_{\Theta_j}(z)$, $z \in \rho(B_j)$, in (4.25) and Lemma 4.5 (ii) that the operator $\gamma_{\Theta_j}(z)$ is densely defined in $\mathcal{G}$ and maps into $\mathfrak{H}$. Next we verify the identity (4.26). Thus, let $z, \zeta \in \rho(B_j)$ and consider $\varphi \in \operatorname{dom}(\gamma_{\Theta_j}(z)) = \operatorname{dom}(\gamma_{\Theta_j}(\zeta))$. Then

$$f_z = \gamma_{\Theta_j}(z)\varphi \in \ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j}), \tag{4.28}$$

and

$$f_\zeta = \gamma_{\Theta_j}(\zeta)\varphi \in \ker(S^* - \zeta I_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j}), \tag{4.29}$$

and it follows from (4.19) that there exists $f_j \in \operatorname{dom}(B_j)$ such that

$$f_\zeta = f_j + f_z. \tag{4.30}$$

As $f_z - f_\zeta = -f_j \in \operatorname{dom}(B_j)$ there exists $h \in \mathfrak{H}$ such that $f_z - f_\zeta = (B_j - zI_{\mathfrak{H}})^{-1}h$. It follows that

$$\begin{aligned}(z-\zeta)f_\zeta &= z\big(f_z - (B_j - zI_{\mathfrak{H}})^{-1}h\big) - \zeta f_\zeta \\ &= S^*(f_z - f_\zeta) - z(B_j - zI_{\mathfrak{H}})^{-1}h \\ &= S^*(B_j - zI_{\mathfrak{H}})^{-1}h - z(B_j - zI_{\mathfrak{H}})^{-1}h \\ &= h,\end{aligned} \tag{4.31}$$

and this implies

$$f_z = f_\zeta + (B_j - zI_{\mathfrak{H}})^{-1}h = \big(I_{\mathfrak{H}} + (z-\zeta)(B_j - zI_{\mathfrak{H}})^{-1}\big) f_\zeta. \tag{4.32}$$

Together with (4.28)–(4.29) we conclude (4.26).

Note that (4.27) follows from (4.26) and the fact that $\gamma_{\Theta_j}(z)$ and $\gamma_{\Theta_j}(\zeta)$ are both continuous. In order to show the continuity of $\gamma_{\Theta_j}(z)$, $z \in \rho(B_j)$, it suffices to check that $\gamma_{\Theta_j}(z)^* \in \mathcal{B}(\mathfrak{H}, \mathcal{G})$ since this yields $\overline{\gamma_{\Theta_j}(z)} = \gamma_{\Theta_j}(z)^{**} \in \mathcal{B}(\mathcal{G}, \mathfrak{H})$. Fix $z \in \rho(B_j)$ and recall from Lemma 4.2 that $B_j^* = S^* \upharpoonright \ker(\Gamma_1 - \Theta_j^*\Gamma_0)$ and $\overline{z} \in \rho(B_j^*)$. Let $\varphi \in \operatorname{dom}(\gamma_{\Theta_j}(z))$,

$$f_z = \gamma_{\Theta_j}(z)\varphi \in \ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}(\Gamma_0^{\Theta_j}) \tag{4.33}$$

and $h \in \mathfrak{H}$, and choose $g \in \operatorname{dom}(B_j^*)$ such that $h = (B_j^* - \overline{z}I_{\mathfrak{H}})g$. Then one computes

$$\begin{aligned}(\gamma_{\Theta_j}(z)\varphi, h)_{\mathfrak{H}} &= \big(f_z, (B_j^* - \overline{z}I_{\mathfrak{H}})g\big)_{\mathfrak{H}} \\ &= (f_z, B_j^* g)_{\mathfrak{H}} - (zf_z, g)_{\mathfrak{H}} \\ &= (f_z, S^* g)_{\mathfrak{H}} - (S^* f_z, g)_{\mathfrak{H}} \\ &= (\Gamma_0 f_z, \Gamma_1 g)_{\mathcal{G}} - (\Gamma_1 f_z, \Gamma_0 g)_{\mathcal{G}} \\ &= (\Gamma_0 f_z, \Theta_j^* \Gamma_0 g)_{\mathcal{G}} - (\Gamma_1 f_z, \Gamma_0 g)_{\mathcal{G}} \\ &= \big(\Gamma_1 f_z - \Theta_j \Gamma_0 f_z, -\Gamma_0 g\big)_{\mathcal{G}} \\ &= \big(\Gamma_0^{\Theta_j} f_z, -\Gamma_0 g\big)_{\mathcal{G}} \\ &= \big(\varphi, -\Gamma_0 (B_j^* - \overline{z}I_{\mathfrak{H}})^{-1}h\big)_{\mathcal{G}},\end{aligned} \tag{4.34}$$

and concludes $\gamma_{\Theta_j}(z)^* h = -\Gamma_0(B_j^* - \overline{z}I_{\mathfrak{H}})^{-1}h$, $h \in \mathfrak{H}$. In particular, since the adjoint operator $\gamma_{\Theta_j}(z)^*$ is closed and defined on the whole space $\mathfrak{H}$ it follows that $\gamma_{\Theta_j}(z)^* \in \mathcal{B}(\mathfrak{H}, \mathcal{G})$. This completes the proof of Lemma 4.6. $\square$

With the preparations in Lemma 4.5 and Lemma 4.6 we will now verify condition (*iii*) in Hypothesis 2.4 for the functions $\Theta_j - M(\cdot)$, $j = 1, 2$. The proof of Proposition 4.7 is an abstract variant of the considerations in Step 6 and 7 in the proof of Lemma 3.9.

Proposition 4.7. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be the boundary triple in Lemma 4.2 with $A_0 = S^* \upharpoonright \ker(\Gamma_0)$ and corresponding Weyl–Titchmarsh function $M(\cdot)$, and let $\Theta_j \in \mathcal{C}(\mathcal{G})$, $j = 1, 2$, be densely defined closed operators such that (4.3) holds. Assume that*

$$z_0 \in \sigma_d(B_j) \cup \rho(B_j), \quad j = 1, 2, \text{ and } z_0 \in \sigma_d(A_0) \cup \rho(A_0). \tag{4.35}$$

Then there exists $\varepsilon_0 > 0$ such that both functions $\Theta_j - M(\cdot)$, $j = 1, 2$, satisfy Hypothesis 2.4 (iii) with $\mathcal{D} = D(z_0; \varepsilon_0)$ and $\mathcal{D}_0 = \{z_0\}$, that is, the functions

$$\big(\Theta_j - M(\cdot)\big)^{-1} : D(z_0; \varepsilon_0)\backslash\{z_0\} \to \mathcal{B}(\mathcal{G}), \quad z \mapsto \big(\Theta_j - M(z)\big)^{-1} \tag{4.36}$$

are analytic on $D(z_0; \varepsilon_0)\backslash\{z_0\}$ and finitely meromorphic on $D(z_0; \varepsilon_0)$.

Proof. Choose $\varepsilon_0 > 0$ as in the proof of Theorem 4.3, so that the punctured disc $D(z_0; \varepsilon_0)\backslash\{z_0\}$ is contained in the set $\rho(B_j) \cap \rho(A_0)$, $j = 1, 2$. Let $z \in D(z_0; \varepsilon_0)\backslash\{z_0\}$ and fix $\zeta \in \rho(B_j)$.

Consider the map $\Gamma_0^{\Theta_j}$ in (4.18), let $\gamma_{\Theta_j}(z)$ be as in (4.25) and let $\varphi \in \operatorname{ran}\big(\Gamma_0^{\Theta_j}\big)$. Then

$$f_z = \gamma_{\Theta_j}(z)\varphi \in \ker(S^* - zI_{\mathfrak{H}}) \cap \operatorname{dom}\big(\Gamma_0^{\Theta_j}\big) \tag{4.37}$$

satisfies $\Gamma_0^{\Theta_j} f_z = \varphi$ and since $M(z)\Gamma_0 f_z = \Gamma_1 f_z$ (see Definition A.2) one finds

$$-\big(\Theta_j - M(z)\big)\Gamma_0 f_z = -\Theta_j \Gamma_0 f_z + \Gamma_1 f_z = \Gamma_0^{\Theta_j} f_z = \varphi, \tag{4.38}$$

which implies

$$\big(\Theta_j - M(z)\big)^{-1}\varphi = -\Gamma_0 f_z; \tag{4.39}$$

recall that $(\Theta_j - M(z))^{-1} \in \mathcal{B}(\mathcal{G})$ for $z \in D(z_0; \varepsilon_0)\backslash\{z_0\}$ by Theorem A.5 (*i*).

Similarly, as $\overline{\zeta} \in \rho(B_j^*)$ and $B_j^* = S^* \upharpoonright \ker(\Gamma_1 - \Theta_j^* \Gamma_0)$ the same argument as in the proof of Lemma 4.5 (*ii*) shows that the range of

$$\Gamma_0^{\Theta_j^*} = \Gamma_1 - \Theta_j^* \Gamma_0, \quad \operatorname{dom}\big(\Gamma_0^{\Theta_j^*}\big) = \{f \in \operatorname{dom}(S^*) \,|\, \Gamma_0 f \in \operatorname{dom}(\Theta_j^*)\}, \tag{4.40}$$

is dense in $\mathcal{G}$. The direct sum decomposition

$$\operatorname{dom}\big(\Gamma_0^{\Theta_j^*}\big) = \operatorname{dom}(B_j^*) \dotplus \big(\ker(S^* - \overline{\zeta} I_{\mathfrak{H}}) \cap \operatorname{dom}\big(\Gamma_0^{\Theta_j^*}\big)\big) \tag{4.41}$$

and $\operatorname{dom}(B_j^*) = \ker\big(\Gamma_0^{\Theta_j^*}\big)$ imply that for all $\psi \in \operatorname{ran}\big(\Gamma_0^{\Theta_j^*}\big)$ there exists a unique

$$g_{\overline{\zeta}} \in \ker\big(S^* - \overline{\zeta} I_{\mathfrak{H}}\big) \cap \operatorname{dom}\big(\Gamma_0^{\Theta_j^*}\big) \text{ such that } \Gamma_0^{\Theta_j^*} g_{\overline{\zeta}} = \psi. \tag{4.42}$$

As in Lemma 4.6 one verifies that the map $\gamma_{\Theta_j^*}(\overline{\zeta}) : \mathcal{G} \to \mathfrak{H}$, $\psi \mapsto g_{\overline{\zeta}}$ is densely defined and bounded, and, in particular, the adjoint operator is bounded, that is, $(\gamma_{\Theta_j^*}(\overline{\zeta}))^* \in \mathcal{B}(\mathfrak{H}, \mathcal{G})$. The same argument as in (4.38) shows that

$$\big(\Theta_j^* - M(\overline{\zeta})\big)^{-1}\psi = -\Gamma_0 g_{\overline{\zeta}} \tag{4.43}$$

and a straightforward calculation using (4.39), (4.43), (4.18), (4.40) yields

$$\begin{aligned}
&\big((\Theta_j - M(z))^{-1}\varphi, \psi\big)_{\mathcal{G}} - \big((\Theta_j - M(\zeta))^{-1}\varphi, \psi\big)_{\mathcal{G}} \\
&\quad = \big((\Theta_j - M(z))^{-1}\varphi, \psi\big)_{\mathcal{G}} - \big(\varphi, (\Theta_j^* - M(\overline{\zeta}))^{-1}\psi\big)_{\mathcal{G}} \\
&\quad = (-\Gamma_0 f_z, \Gamma_0^{\Theta_j^*} g_{\overline{\zeta}})_{\mathcal{G}} - (\Gamma_0^{\Theta_j} f_z, -\Gamma_0 g_{\overline{\zeta}})_{\mathcal{G}} \\
&\quad = \big(-\Gamma_0 f_z, (\Gamma_1 - \Theta_j^*\Gamma_0) g_{\overline{\zeta}}\big)_{\mathcal{G}} - \big((\Gamma_1 - \Theta_j\Gamma_0) f_z, -\Gamma_0 g_{\overline{\zeta}}\big)_{\mathcal{G}} \\
&\quad = (\Gamma_1 f_z, \Gamma_0 g_{\overline{\zeta}})_{\mathcal{G}} - (\Gamma_0 f_z, \Gamma_1 g_{\overline{\zeta}})_{\mathcal{G}} \\
&\quad = (S^* f_z, g_{\overline{\zeta}})_{\mathfrak{H}} - (f_z, S^* g_{\overline{\zeta}})_{\mathfrak{H}} \\
&\quad = (z f_z, g_{\overline{\zeta}})_{\mathfrak{H}} - (f_z, \overline{\zeta} g_{\overline{\zeta}})_{\mathfrak{H}} \\
&\quad = (z - \zeta)(\gamma_{\Theta_j}(z)\varphi, \gamma_{\Theta_j^*}(\overline{\zeta})\psi)_{\mathfrak{H}}.
\end{aligned} \tag{4.44}$$

Hence

$$(\Theta_j - M(z))^{-1}\varphi - (\Theta_j - M(\zeta))^{-1}\varphi = (z - \zeta)\big(\gamma_{\Theta_j^*}(\overline{\zeta})\big)^* \gamma_{\Theta_j}(z)\varphi \tag{4.45}$$

holds for all $\varphi \in \operatorname{ran}\big(\Gamma_0^{\Theta_j}\big)$ and with the help of the identities (4.26) and (4.27) in Lemma 4.6 one obtains

$$\begin{aligned}
&(\Theta_j - M(z))^{-1} \\
&\quad = (\Theta_j - M(\zeta))^{-1} + (z - \zeta)\big(\gamma_{\Theta_j^*}(\overline{\zeta})\big)^* \big(I_{\mathfrak{H}} + (z - \zeta)(B_j - zI_{\mathfrak{H}})^{-1}\big)\overline{\gamma_{\Theta_j}(\zeta)}
\end{aligned} \tag{4.46}$$

for all $z \in D(z_0; \varepsilon_0)\backslash\{z_0\}$. Since $z_0 \in \sigma_d(B_j) \cup \rho(B_j)$ the $\mathcal{B}(\mathfrak{H})$-valued map $z \mapsto (B_j - zI_{\mathfrak{H}})^{-1}$ is analytic on $D(z_0; \varepsilon_0)\backslash\{z_0\}$ and finitely meromorphic on $D(z_0; \varepsilon_0)$ by Example 2.2. As the operators $\overline{\gamma_{\Theta_j}(\zeta)}$ and $(\gamma_{\Theta_j^*}(\overline{\zeta}))^*$ are bounded it follows from Example 2.3 that the same is true for the map

$$z \mapsto \big(\gamma_{\Theta_j^*}(\overline{\zeta})\big)^* (B_j - zI_{\mathfrak{H}})^{-1}\overline{\gamma_{\Theta_j}(\zeta)}. \tag{4.47}$$

Hence it follows that also the map $z \mapsto (\Theta_j - M(z))^{-1}$ is analytic on $D(z_0; \varepsilon_0)\backslash\{z_0\}$ and finitely meromorphic on $D(z_0; \varepsilon_0)$. This completes the proof of Proposition 4.7. □

Remark 4.8. Assume that the closed operators Θ_j, $j = 1, 2$, in (4.3) are bounded; this happens if and only if $\operatorname{dom}(S^*) = \operatorname{dom}(B_j) + \operatorname{dom}(A_0)$ holds (see (A.8)). In this case some of the previous considerations in Lemma 4.5 and Lemma 4.6 slightly simplify. In particular, the map $\Gamma_0^{\Theta_j}$ in

(4.18) is defined on $\mathrm{dom}(S^*)$ and maps onto $\mathcal{G}$. As a consequence, the operators $\gamma_{\Theta_j}(z)$, $z \in \rho(B_j)$, are defined on $\mathcal{G}$ and the identities (4.26) and (4.27) are the same.

Remark 4.9. A typical situation in which the closed operators Θ_j, $j = 1, 2$, in (4.3) are unbounded is the following: Suppose that the deficiency indices of S are infinite and that the resolvent difference

$$(B_j - zI_{\mathfrak{H}})^{-1} - (A_0 - zI_{\mathfrak{H}})^{-1}, \quad z \in \rho(A_0) \cap \rho(B_j), \tag{4.48}$$

is a compact operator. Then $\mathcal{G}$ is an infinite dimensional Hilbert space and it follows from [22, Theorem 2] that the closed operator Θ_j in $\mathcal{G}$ has a compact resolvent, and hence is unbounded.

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Appendix A. Boundary triples, Weyl–Titchmarsh functions, and abstract Donoghue-type *M*-functions

The aim of this appendix is to give a brief introduction to boundary triples and their Weyl–Titchmarsh functions, and to establish the connection to abstract Donoghue-type M-functions that were studied, for instance, in [25,27,28,31,32,43,44], and [45]. In addition, we refer the reader to [1,2,9–16,18,20–23,37,38,48–60], for more details, applications, and references on boundary triples and their Weyl–Titchmarsh functions.

Let $\mathfrak{H}$ be a separable complex Hilbert space, let S be a densely defined closed symmetric operator in $\mathfrak{H}$ and let S^* be the adjoint of S. The notion of boundary triple (or boundary value space) appeared first in [17] and [42].

Definition A.1. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *boundary triple* for S^* if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(S^*) \to \mathcal{G}$ are linear operators such that

$$(S^* f, g)_{\mathfrak{H}} - (f, S^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}} \tag{A.1}$$

holds for all $f, g \in \mathrm{dom}(S^*)$ and the map $\Gamma = \left[\begin{smallmatrix}\Gamma_0\\ \Gamma_1\end{smallmatrix}\right] : \mathrm{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$ is onto.

We note that a boundary triple for S^* exists if and only if the deficiency indices of S coincide, or, equivalently, if S admits self-adjoint extensions in $\mathfrak{H}$. A boundary triple (if it exists) is not unique (except in the trivial case $S = S^*$). Assume in the following that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triple for S^*. Then

$$S = S^* \upharpoonright \ker(\Gamma) = S^* \upharpoonright (\ker(\Gamma_0) \cap \ker(\Gamma_1)) \tag{A.2}$$

holds and the maps $\Gamma_0, \Gamma_1 : \mathrm{dom}(S^*) \to \mathcal{G}$ are continuous with respect to the graph norm of S^*. A key feature of a boundary triple is that all closed extensions of S can be parametrized in an efficient way. More precisely, there is a one-to-one correspondence between the closed extensions $A_\Theta \subset S^*$ of S and the closed linear subspaces (relations) $\Theta \subset \mathcal{G} \times \mathcal{G}$ given by

$$\Theta \mapsto A_\Theta = S^* \upharpoonright \{f \in \mathrm{dom}(S^*) \,|\, \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \tag{A.3}$$

We refer the reader to [6] for the notion, basic operations, and properties of linear relations. In the case where Θ in (A.3) is (the graph of) an operator, the extension A_Θ is given by

$$A_\Theta = S^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \tag{A.4}$$

A particularly convenient feature is that the adjoint of A_Θ in (A.3)–(A.4) is given by the extension that corresponds to the parameter Θ^*, that is, the identity

$$(A_\Theta)^* = A_{\Theta^*} \tag{A.5}$$

holds; here the adjoint of linear relation Θ is defined in the same manner as the adjoint of a densely defined operator. It follows, in particular, that A_Θ is self-adjoint in $\mathfrak{H}$ if and only if the parameter Θ is self-adjoint in $\mathcal{G}$. In the following the self-adjoint extension

$$A_0 = S^* \upharpoonright \ker(\Gamma_0) \tag{A.6}$$

of S will play the role of a fixed extension. One notes that A_0 corresponds to the subspace $\Theta_0 = \{0\} \times \mathcal{G}$ in (A.3); in addition, one observes that the index 0 corresponds to the subspace Θ_0 and not to the zero operator in $\mathcal{G}$.

For our purposes it is convenient to have criteria available which ensure that Θ in (A.3)–(A.4) is a (bounded) operator. We recall from [22,23] that Θ is a closed operator if and only A_Θ and A_0 are disjoint, that is,

$$S = A_\Theta \cap A_0, \tag{A.7}$$

and that $\Theta \in \mathcal{B}(\mathcal{G})$ if and only if A_Θ and A_0 are disjoint and

$$\mathrm{dom}(S^*) = \mathrm{dom}(A_\Theta) \dotplus \mathrm{dom}(A_0) \tag{A.8}$$

holds.

Next we recall the definition of the γ-field and Weyl–Titchmarsh function corresponding to a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. For this purpose consider the self-adjoint operator $A_0 = S^* \upharpoonright \ker(\Gamma_0)$ and note that for any $z \in \rho(A_0)$ the direct sum decomposition

$$\mathrm{dom}(S^*) = \mathrm{dom}(A_0) \dotplus \ker(S^* - zI_{\mathfrak{H}}) = \ker(\Gamma_0) \dotplus \ker(S^* - zI_{\mathfrak{H}}) \tag{A.9}$$

holds. This implies, in particular, that the restriction of the boundary map Γ_0 onto $\ker(S^* - zI_{\mathfrak{H}})$ is injective for all $z \in \rho(A_0)$. Moreover, the surjectivity of $\Gamma : \mathrm{dom}(S^*) \to \mathcal{G} \times \mathcal{G}$ and (A.9) yield that the restriction $\Gamma_0 \upharpoonright \ker(S^* - zI_{\mathfrak{H}})$ maps onto $\mathcal{G}$ and hence the inverse $(\Gamma_0 \upharpoonright \ker(S^* - zI_{\mathfrak{H}}))^{-1}$ is a bounded operator defined on $\mathcal{G}$. This observation shows that the γ-field and Weyl–Titchmarsh function in the next definition are well-defined and their values are bounded operators for all $z \in \rho(A_0)$.

Definition A.2. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* and let $A_0 = S^* \upharpoonright \ker(\Gamma_0)$. The γ-field $\gamma(\cdot)$ corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is defined by

$$\gamma : \rho(A_0) \to \mathcal{B}(\mathcal{G}, \mathfrak{H}), \quad z \mapsto \gamma(z) = (\Gamma_0 \upharpoonright \ker(S^* - zI_{\mathfrak{H}}))^{-1}, \tag{A.10}$$

and the Weyl–Titchmarsh function $M(\cdot)$ corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is defined by

$$M : \rho(A_0) \to \mathcal{B}(\mathcal{G}), \quad z \mapsto M(z) = \Gamma_1(\Gamma_0 \upharpoonright \ker(S^* - zI_{\mathfrak{H}}))^{-1}. \tag{A.11}$$

In the following let $\gamma(\cdot)$ and $M(\cdot)$ be the γ-field and Weyl–Titchmarsh function corresponding to a boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^*. We recall some important properties of the functions $\gamma(\cdot)$ and $M(\cdot)$ which can be found, for instance, in [10,18,22,23]. First of all we note that $\gamma(\cdot)$ and $M(\cdot)$ are both analytic operator functions on $\rho(A_0)$ with values in $\mathcal{B}(\mathcal{G}, \mathfrak{H})$ and $\mathcal{B}(\mathcal{G})$, respectively. The adjoint of $\gamma(z)$ is a bounded operator from $\mathfrak{H}$ into $\mathcal{G}$ of the form

$$\gamma(z)^* = \Gamma_1(A_0 - \overline{z}I_{\mathfrak{H}})^{-1} \in \mathcal{B}(\mathfrak{H}, \mathcal{G}). \tag{A.12}$$

Furthermore, the important identities

$$\gamma(z) = \big(I_{\mathfrak{H}} + (z - \zeta)(A_0 - zI_{\mathfrak{H}})^{-1}\big)\gamma(\zeta) \tag{A.13}$$

and

$$M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z) \tag{A.14}$$

hold for all $z, \zeta \in \rho(A_0)$. A combination of (A.13) and (A.14) shows

$$\begin{aligned} M(z) &= M(\zeta)^* + (z - \overline{\zeta})\gamma(\zeta)^*\big(I_{\mathfrak{H}} + (z - \zeta)(A_0 - zI_{\mathfrak{H}})^{-1}\big)\gamma(\zeta) \\ &= M(\zeta)^* + (z - \overline{\zeta})\gamma(\zeta)^*\gamma(\zeta) + (z - \zeta)(z - \overline{\zeta})\gamma(\zeta)^*(A_0 - zI_{\mathfrak{H}})^{-1}\gamma(\zeta). \end{aligned} \tag{A.15}$$

One observes that (A.14) implies

$$M(z)^* = M(\overline{z}), \quad z \in \rho(A_0), \tag{A.16}$$

$$\frac{d}{dz}M(z) = \gamma(\overline{z})^*\gamma(z), \quad z \in \rho(A_0), \tag{A.17}$$

and that

$$\operatorname{Im}(M(z)) = \frac{1}{2i}\big(M(z) - M(z)^*\big) = (\operatorname{Im}(z))\gamma(z)^*\gamma(z) \in \mathcal{B}(\mathcal{G}) \tag{A.18}$$

is a uniformly positive (resp., uniformly negative) operator for $z \in \mathbb{C}^+$ (resp., $z \in \mathbb{C}^-$). Therefore, the Weyl–Titchmarsh function $M(\cdot)$ is a $\mathcal{B}(\mathcal{G})$-valued Riesz–Herglotz or Nevanlinna function (see [22,31,40,45]), which, in addition, is uniformly strict (cf. [20]). In particular, there exists a self-adjoint operator $\alpha \in \mathcal{B}(\mathcal{G})$ and a non-decreasing self-adjoint operator map $t \mapsto \Sigma(t) \in \mathcal{B}(\mathcal{G})$ on $\mathbb{R}$ such that $M(\cdot)$ admits the integral representation

$$M(z) = \alpha + \int_{\mathbb{R}} d\Sigma(t) \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right), \quad z \in \rho(A_0), \tag{A.19}$$

where $\int_{\mathbb{R}} d\Sigma(t)(1+t^2)^{-1} \in \mathcal{B}(\mathcal{G})$.

The next lemma follows from (A.15) and the fact that the resolvent of A_0 and its derivatives are finitely meromorphic at a discrete eigenvalue z_0 of A_0 (cf. Examples 2.2 and 2.3).

Lemma A.3. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* with $A_0 = S^* \upharpoonright \ker(\Gamma_0)$, and let $M(\cdot)$ be the corresponding Weyl–Titchmarsh function. If $z_0 \in \sigma_d(A_0) \cup \rho(A_0)$ then M and its derivatives $M^{(l)}(\cdot)$, $l \in \mathbb{N}$, are finitely meromorphic at z_0.*

In the next proposition we provide a particular boundary triple for S^* such that the corresponding Weyl–Titchmarsh function coincides with the abstract Donoghue-type M-function that was studied, for instance, in [27,28,31]. The construction in Proposition A.4 can be found, for instance, in [19, Proposition 4.1]. For the convenience of the reader we provide a short proof.

Proposition A.4. *Let S be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices, fix a self-adjoint extension A of S in $\mathfrak{H}$ and decompose the elements $f \in \operatorname{dom}(S^*)$ according to the direct sum decomposition*

$$\operatorname{dom}(S^*) = \operatorname{dom}(A) \dotplus \mathcal{N}_i, \quad \mathcal{N}_i = \ker(S^* - iI_{\mathfrak{H}}), \tag{A.20}$$

in the form $f = f_A + f_i$, $f_A \in \operatorname{dom}(A)$, $f_i \in \mathcal{N}_i$. Let $P_{\mathcal{N}_i} : \mathfrak{H} \to \mathcal{N}_i$ be the orthogonal projection onto $\mathcal{N}_i$ and let $\iota_{\mathcal{N}_i} : \mathcal{N}_i \to \mathfrak{H}$ be the canonical embedding of $\mathcal{N}_i$ into $\mathfrak{H}$.

Then $\{\mathcal{N}_i, \Gamma_0, \Gamma_1\}$, where the boundary maps $\Gamma_0, \Gamma_1 : \operatorname{dom}(S^*) \to \mathcal{N}_i$ are defined by

$$\Gamma_0 f = f_i \text{ and } \Gamma_1 f = P_{\mathcal{N}_i}(A + iI_{\mathfrak{H}}) f_A + i f_i, \tag{A.21}$$

is a boundary triple for S^* with $A_0 = S^* \upharpoonright \ker(\Gamma_0) = A$ and the corresponding Weyl–Titchmarsh function $M(\cdot)$ is given by

$$M(z) = zI_{\mathcal{N}_i} + (z^2+1) P_{\mathcal{N}_i}(A - zI_{\mathfrak{H}})^{-1} \iota_{\mathcal{N}_i}, \quad z \in \rho(A). \tag{A.22}$$

Proof. Let $f, g \in \operatorname{dom}(S^*)$ be decomposed in the form $f = f_A + f_i$ and $g = g_A + g_i$, where $f_A, g_A \in \operatorname{dom}(A)$ and $f_i, g_i \in \mathcal{N}_i$. Since A is self-adjoint in $\mathfrak{H}$ we have $(Af_A, g_A)_{\mathfrak{H}} = (f_A, Ag_A)_{\mathfrak{H}}$ and it follows that

$$\begin{aligned}
&(S^* f, g)_{\mathfrak{H}} - (f, S^* g)_{\mathfrak{H}} \\
&\quad = \big(Af_A + if_i, g_A + g_i\big)_{\mathfrak{H}} - \big(f_A + f_i, Ag_A + ig_i\big)_{\mathfrak{H}} \\
&\quad = \big(Af_A + if_i, g_i\big)_{\mathfrak{H}} + (if_i, g_A)_{\mathfrak{H}} - \big(f_i, Ag_A + ig_i\big)_{\mathfrak{H}} - (f_A, ig_i)_{\mathfrak{H}} \\
&\quad = \big((A + iI_{\mathfrak{H}})f_A + if_i, g_i\big)_{\mathfrak{H}} - \big(f_i, (A + iI_{\mathfrak{H}})g_A + ig_i\big)_{\mathfrak{H}}.
\end{aligned} \tag{A.23}$$

Moreover, it follows from the definition of the boundary maps in (A.21) that

$$\begin{aligned}(\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}_i} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}_i} \\ &= \big(P_{\mathcal{N}_i}(A + iI_{\mathfrak{H}})f_A + if_i, g_i\big)_{\mathcal{N}_i} - \big(f_i, P_{\mathcal{N}_i}(A + iI_{\mathfrak{H}})g_A + ig_i\big)_{\mathcal{N}_i} \\ &= \big((A + iI_{\mathfrak{H}})f_A + if_i, g_i\big)_{\mathfrak{H}} - \big(f_i, (A + iI_{\mathfrak{H}})g_A + ig_i\big)_{\mathfrak{H}}.\end{aligned} \tag{A.24}$$

Therefore, by combining (A.23) and (A.24) we conclude

$$(S^* f, g)_{\mathfrak{H}} - (f, S^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{N}_i} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{N}_i}, \tag{A.25}$$

and hence the abstract Green's identity (A.1) in Definition A.1 is satisfied.

Next we verify that the map

$$\Gamma = \begin{bmatrix}\Gamma_0 \\ \Gamma_1\end{bmatrix} : \mathrm{dom}(S^*) \to \mathcal{G} \times \mathcal{G} \tag{A.26}$$

is surjective. To see this consider $\varphi, \psi \in \mathcal{N}_i$, choose $f_A \in \mathrm{dom}(A)$ such that

$$(A + iI_{\mathfrak{H}})f_A = \psi - i\varphi, \tag{A.27}$$

and let $f = f_A + \varphi \in \mathrm{dom}(S^*)$. It follows from (A.21) that

$$\Gamma_0 f = \varphi \text{ and } \Gamma_1 f = P_{\mathcal{N}_i}(A + iI_{\mathfrak{H}})f_A + i\varphi = \psi. \tag{A.28}$$

Hence the map in (A.26) is onto and it follows that $\{\mathcal{N}_i, \Gamma_0, \Gamma_1\}$ is a boundary triple for S^*. It is clear from the construction that $A_0 = S^* \upharpoonright \ker(\Gamma_0) = A$ holds.

It remains to show that the Weyl–Titchmarsh function corresponding to the boundary triple $\{\mathcal{N}_i, \Gamma_0, \Gamma_1\}$ has the asserted form. For this consider first $f_i \in \mathcal{N}_i$ and note that for $f = f_i$ the abstract boundary values in (A.21) are given by

$$\Gamma_0 f_i = f_i \text{ and } \Gamma_1 f_i = if_i. \tag{A.29}$$

Therefore, Definition A.2 implies

$$\gamma(i) : \mathcal{N}_i \to \mathfrak{H}, \quad f_i \mapsto \gamma(i)f_i = f_i, \tag{A.30}$$

that is, $\gamma(i)$ is the canonical embedding of $\mathcal{N}_i$ into $\mathfrak{H}$,

$$\gamma(i) = \iota_{\mathcal{N}_i}, \tag{A.31}$$

and $\gamma(i)^* : \mathfrak{H} \to \mathcal{N}_i$ is the orthogonal projection onto $\mathcal{N}_i$, that is, $\gamma(i)^* = P_{\mathcal{N}_i}$. Furthermore, Definition A.2 also implies

$$M(i) : \mathcal{G} \to \mathcal{G}, \quad f_i \mapsto M(i)f_i = if_i, \tag{A.32}$$

that is, $M(i) = iI_{\mathcal{N}_i}$. Next, it follows from (A.15) with $\zeta = i$ that

$$M(z) = zI_{\mathcal{N}_i} + (z^2 + 1)P_{\mathcal{N}_i}(A - zI_{\mathfrak{H}})^{-1}\iota_{\mathcal{N}_i} \tag{A.33}$$

holds for all $z \in \rho(A)$, completing the proof of Proposition A.4. □

Finally, we recall a useful version of Krein's resolvent formula for the resolvents of the closed extensions A_Θ in (A.3)–(A.4), which also provides a correspondence between the spectrum of A_Θ inside the set $\rho(A_0)$ and the spectrum of $\Theta - M(\cdot)$.

Theorem A.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* with $A_0 = S^* \upharpoonright \ker(\Gamma_0)$, and let $\gamma(\cdot)$ and $M(\cdot)$ be the corresponding γ-field and Weyl–Titchmarsh function, respectively. Let $A_\Theta \subset S^*$ be a closed extension of S which corresponds to a closed operator or subspace Θ as in (A.3)–(A.4). Then the following assertions hold for all $z \in \rho(A_0)$:*

(i) $z \in \rho(A_\Theta)$ if and only if $0 \in \rho(\Theta - M(z))$.

(ii) $z \in \sigma_j(A_\Theta)$ if and only if $0 \in \sigma_j(\Theta - M(z))$, $j \in \{p, c, r\}$.

(iii) for all $z \in \rho(A_\Theta) \cap \rho(A_0)$,

$$(A_\Theta - zI_{\mathfrak{H}})^{-1} = (A_0 - zI_{\mathfrak{H}})^{-1} + \gamma(z)\big(\Theta - M(z)\big)^{-1}\gamma(\overline{z})^*. \tag{A.34}$$

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