Elliptic boundary value problems with λ-dependent boundary conditions

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Abstract

In this paper second order elliptic boundary value problems on bounded domains $\Omega \subset \mathbb{R}^n$ with boundary conditions on $\partial\Omega$ depending nonlinearly on the spectral parameter are investigated in an operator theoretic framework. For a general class of locally meromorphic functions in the boundary condition a solution operator of the boundary value problem is constructed with the help of a linearization procedure. In the special case of rational Nevanlinna or Riesz-Herglotz functions on the boundary the solution operator is obtained in an explicit form in the product Hilbert space $L^2(\Omega) \oplus (L^2(\partial\Omega))^m$, which is a natural generalization of known results on λ-linear elliptic boundary value problems and λ-rational boundary value problems for ordinary second order differential equations.

Key words: boundary value problem, λ-dependent boundary condition, elliptic differential operator, operator function, boundary triple, Weyl function

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 1$, with smooth boundary $\partial\Omega$ and consider a uniformly elliptic differential expression

$$\ell = -\sum_{j,k=1}^{n} \partial_j \, a_{jk} \, \partial_k + a \qquad (1.1)$$

on Ω with coefficients $a_{jk}, a \in C^\infty(\overline{\Omega})$ such that $a_{jk} = \overline{a_{kj}}$ for all $j, k = 1, \ldots, n$ and a is real-valued. The main objective of this paper is to solve the following

eigenparameter dependent boundary value problem: For a given function $g \in L^2(\Omega)$ and λ in some open set $\mathcal{D} \subset \mathbb{C}$ find $f \in L^2(\Omega)$ such that

$$(\ell - \lambda)f = g \qquad \text{and} \qquad \tau(\lambda)f|_{\partial\Omega} = \frac{\partial f_D}{\partial \nu_\ell}\bigg|_{\partial\Omega} \qquad (1.2)$$

holds. Here τ is assumed to be a meromorphic function on $\mathcal{D}$ with values in the space of bounded linear operators on $L^2(\partial\Omega)$, λ is a point of holomorphy of τ, f is a function in the maximal domain $\mathcal{D}_{\max} = \{h \in L^2(\Omega) : \ell h \in L^2(\Omega)\}$ and f_D is the component of f which lies in the domain of the Dirichlet operator.

For the special case of a selfadjoint constant τ in the boundary condition in (1.2) the boundary value problem is uniquely solvable for all λ which belong to the resolvent set of the selfadjoint partial differential operator

$$T_\tau f = \ell f, \qquad \text{dom}\, T_\tau = \left\{ f \in \mathcal{D}_{\max} : \tau f|_{\partial\Omega} = \frac{\partial f_D}{\partial \nu_\ell}\bigg|_{\partial\Omega} \right\}, \qquad (1.3)$$

in $L^2(\Omega)$ and the unique solution of (1.2) is given by $f = (T_\tau - \lambda)^{-1}g$. Similarly, the nontrivial solutions of the associated homogeneous problem, i.e., $g = 0$ in (1.2), are given by the eigenvectors corresponding to the (real) eigenvalues λ of T_τ.

Elliptic problems with λ-linear boundary conditions were already considered by J. Ercolano and M. Schechter in [35,36] and a solution operator $\widetilde{A}$ in the larger space $L^2(\Omega) \oplus L^2(\partial\Omega)$ was constructed and its spectral properties were studied. Again the resolvent of $\widetilde{A}$, or, more precisely, the compression of the resolvent onto the basic space $L^2(\Omega)$,

$$f = P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}\restriction_{L^2(\Omega)} g,$$

yields the unique solution f of (1.2), and the eigenvalues and the (components in $L^2(\Omega)$ of the) eigenvectors of $\widetilde{A}$ are the nontrivial solutions of the homogeneous problem. We emphasize that the solution operator $\widetilde{A}$ in the λ-linear case is selfadjoint with respect to the Hilbert scalar product in $L^2(\Omega) \oplus L^2(\partial\Omega)$ if $\tau(\lambda) = \lambda$ and selfadjoint with respect to an indefinite (Krein space) inner product if $\tau(\lambda) = -\lambda$. The spectral properties of selfadjoint operators in Krein spaces differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces and this affects the solvability of (1.2). E.g., if $\tau(\lambda) = -\lambda$ in (1.2), then the solution operator $\widetilde{A}$ and the homogeneous boundary value problem may have non-real eigenvalues, see [15].

The main objective of this paper is to go far beyond the λ-linear case and to investigate the solvability of the boundary value problem (1.2) for a large class of operator-valued functions in the boundary condition. Here it will be assumed that τ is a meromorphic function on some simply connected open set

$\mathcal{D} \subset \mathbb{C}^+$ with values in the space $\mathcal{L}(L^2(\partial\Omega))$ of bounded linear operators on $L^2(\partial\Omega)$ and that τ admits a minimal representation

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + \gamma^+\Big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1}\Big)\gamma \quad (1.4)$$

with the help of the resolvent of a selfadjoint operator or relation A_0 in a Krein or Hilbert space $\mathcal{H}$ and a mapping $\gamma \in \mathcal{L}(L^2(\partial\Omega), \mathcal{H})$. We mention that, e.g., locally holomorphic functions, Nevanlinna and generalized Nevanlinna functions, and so-called definitizable and locally definitizable functions can be represented in the form (1.4), see [2,28,45–48,50,54].

For the construction of a solution operator $\widetilde{A}$ of the boundary value problem (1.2) we make use of the notion of (generalized) boundary triples, and associated Weyl or M-functions, a convenient and useful tool for the spectral analysis of the selfadjoint extensions of an arbitrary symmetric operator with equal deficiency indices, see, e.g., [17,19,20,25,26,40]. Boundary triples for the maximal operator $T_{\max}f = \ell f$, $f \in \mathcal{D}_{\max}$, generated by the elliptic differential expression in $L^2(\Omega)$ were used (also in the non-symmetric case) in [16,39,43] and appear in a slightly different form already in the fundamental paper [41] of G. Grubb. One of the main ingredients in the construction of a solution operator $\widetilde{A}$ of (1.2) is to realize the function τ in the boundary condition as the Weyl function corresponding to some boundary triple, cf. [21] and, e.g., [6,8,10], and [3,9,18,27,29–31,53] for other approaches. So far this is possible only under rather restrictive assumptions on the function τ, e.g., in the special case of an $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function one has to assume that $\operatorname{Im}\tau(\lambda)$ is boundedly invertible, see [25,54], or one has to apply the concept of boundary relations and Weyl families from [22,23]. Therefore, in order to treat the problem (1.2) in a general setting, we extend the existing results on realizations of operator functions as Weyl functions in Section 3. Here a new method is proposed in which an arbitrary operator function τ of the form (1.4) can be realized as the Weyl function corresponding to a generalized boundary triplet associated to a restriction of the selfadjoint operator or relation A_0. The idea is based on a decomposition of τ in a constant part and a "smaller" part which satisfies a special strictness condition, see Definition 3.4 and [7] for the special case of matrix Nevanlinna functions. Although the realization obtained in Theorem 3.1 is in general not minimal it turns out that the connections between the solvability of the boundary value problem (1.2) and the spectral properties of the solution operator $\widetilde{A}$ are not affected at all.

The heart of the paper is Section 4, where the eigenvalue dependent boundary value problem (1.2) is discussed. After recalling some basic properties on elliptic operators associated to (1.1) and a corresponding ordinary boundary triple for $T_{\max}$ in Section 4.1 we construct a solution operator $\widetilde{A}$ of the elliptic boundary value problem (1.2) in a larger Krein or Hilbert space $L^2(\Omega) \times \mathcal{K}$ with the help of the realization result from Section 3. The unique solution

$f \in L^2(\Omega)$ of (1.2) and the compression of the resolvent of $\widetilde{A}$ onto the basic space $L^2(\Omega)$ are then expressed in the form

$$f = P_{L^2(\Omega)}(\widetilde{A} - \lambda)^{-1}\!\restriction_{L^2(\Omega)} g = (T_D - \lambda)^{-1}g - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g,$$

where T_D is the Dirichlet operator asssociated to ℓ in $L^2(\Omega)$, M denotes the Weyl or M-function corresponding to an ordinary boundary triple for $T_{\max}$ and $\gamma(\cdot)$ is the associated γ-field, cf. Proposition 4.1. We point out that for a constant selfadjoint boundary condition τ the solution operator $\widetilde{A}$ coincides with T_τ in (1.3) and the above formula reduces to the well-known Krein formula for canonical selfadjoint extensions in $L^2(\Omega)$ of the minimal operator associated to ℓ, cf. [1,8,16,37–39,44,56–60]. The proof of our main result Theorem 4.2 is based on a coupling technique of ordinary and generalized boundary triples which differs from the methods applied in earlier papers.

We illustrate our general approach in Section 4.3 in an example where τ is chosen to be a rational $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna (or Riesz-Herglotz) function of the form

$$\tau(\lambda) = \alpha_1 + \lambda\beta_1 + \sum_{i=2}^{m} \beta_i^{1/2}(\alpha_i - \lambda)^{-1}\beta_i^{1/2} \qquad \lambda \in \bigcap_{i=2}^{m} \rho(\alpha_i). \qquad (1.5)$$

Here α_i, β_i are bounded selfadjoint operators on $L^2(\partial\Omega)$ and $\beta_i \geq 0$. In this special case the solution operator from Theorem 4.2 acts in the product space $L^2(\Omega) \oplus (L^2(\partial\Omega))^m$ and can be constructed in a more explicit form, cf. Theorem 4.6 and Corollary 4.7 for the λ-linear problem. We point out that an analogous selfadjoint solution operator in $L^2(I) \oplus \mathbb{C}^m$ of a Sturm-Liouville problem on a bounded interval $I \subset \mathbb{R}$ with a scalar variant of (1.5) in the boundary condition was constructed in [12].

The paper is organized as follows. In Section 2 we give a brief introduction into the theory of ordinary boundary triples and generalized boundary triples associated to symmetric operators and relations in Krein spaces. The corresponding γ-field and Weyl function are defined and some of their basic properties are recalled. In Section 3 it is shown how an arbitrary operator function τ of the form (1.4) can be interpreted as the Weyl function of some generalized boundary triple and some special classes of operator functions are discussed in Section 3.3. Section 4 treats the elliptic boundary value problem (1.2), in particular, a solution operator $\widetilde{A}$ is constructed, it is shown that the compressed resolvent of $\widetilde{A}$ onto the basic space $L^2(\Omega)$ yields the unique nontrivial solution of the inhomogeneous problem (1.2) and that the eigenvalues and eigenvectors of $\widetilde{A}$ solve the homogenous boundary value problem.

2 Generalized boundary triples and Weyl functions of symmetric relations in Krein spaces

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let J be a corresponding fundamental symmetry. We study linear relations in $\mathcal{H}$, that is, linear subspaces of $\mathcal{H} \times \mathcal{H}$. The elements in a linear relation will be denoted by $\hat{f} = \{f, f'\}$, $f, f' \in \mathcal{H}$. For the set of all closed linear relations in $\mathcal{H}$ we write $\widetilde{\mathcal{C}}(\mathcal{H})$. Linear operators in $\mathcal{H}$ are viewed as linear relations via their graphs. The linear space of bounded linear operators defined on a Krein space $\mathcal{H}$ with values in a Krein space $\mathcal{K}$ is denoted by $\mathcal{L}(\mathcal{H}, \mathcal{K})$. If $\mathcal{H} = \mathcal{K}$ we simply write $\mathcal{L}(\mathcal{H})$. We refer the reader to [4,11,32,33] for more details on Krein spaces and linear operators and relations acting therein.

We equip $\mathcal{H} \times \mathcal{H}$ with the Krein space inner product $[\![\cdot,\cdot]\!]$ defined by

$$\left[\!\left[\hat{f}, \hat{g}\right]\!\right] := i\big([f, g'] - [f', g]\big), \qquad \hat{f} = \{f, f'\}, \ \hat{g} = \{g, g'\} \in \mathcal{H} \times \mathcal{H}. \quad (2.1)$$

Then $\left(\begin{smallmatrix} 0 & -iJ \\ iJ & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{H}^2)$ is a corresponding fundamental symmetry. Observe that also in the special case when $(\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space, $[\![\cdot,\cdot]\!]$ is an indefinite metric. In the following we shall often use at the same time inner products $[\![\cdot,\cdot]\!]$ arising from different Krein and Hilbert spaces as in (2.1). Then we shall indicate these forms by subscripts, for example, $[\![\cdot,\cdot]\!]_{\mathcal{H}^2}$, $[\![\cdot,\cdot]\!]_{\mathcal{G}^2}$.

For a linear relation A in the Krein space $\mathcal{H}$ the *adjoint relation* $A^+ \in \widetilde{\mathcal{C}}(\mathcal{H})$ is defined as the orthogonal companion of A in $(\mathcal{H}^2, [\![\cdot,\cdot]\!])$, i.e.,

$$A^+ := A^{[\![\perp]\!]} = \left\{ \hat{f} \in \mathcal{H}^2 : \left[\!\left[\hat{f}, \hat{g}\right]\!\right] = 0 \text{ for all } \hat{g} \in A \right\}.$$

A linear relation A in $\mathcal{H}$ is said to be *symmetric* (*selfadjoint*) if $A \subset A^*$ ($A = A^*$, respectively). We say that a closed symmetric relation $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ is of *defect* $m \in \mathbb{N}_0 \cup \{\infty\}$, if the deficiency indices

$$n_\pm(JA) = \dim \ker\big((JA)^* \mp i\big)$$

of the closed symmetric relation JA in the Hilbert space $(\mathcal{H}, [J\cdot,\cdot])$ are both equal to m. Here * denotes the adjoint with respect to the Hilbert scalar product $[J\cdot,\cdot]$. Note that a symmetric relation $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ is of defect m if and only if there exists a selfadjoint extension of A in $\mathcal{H}$ and each selfadjoint extension A' of A in $\mathcal{H}$ satisfies $\dim(A'/A) = m$.

For symmetric operators in Hilbert spaces the concept of generalized boundary triples or generalized boundary value spaces was introduced by V.A. Derkach and M.M. Malamud in [26], see also [22, §5.2]. We use the same definition in the Krein space case.

Definition 2.1 *Let A be a closed symmetric relation in the Krein space $\mathcal{H}$ and let T be a linear relation in $\mathcal{H}$ such that $\overline{T} = A^+$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a* generalized boundary triple *for A^+, if $\mathcal{G}$ is a Hilbert space and $\Gamma = (\Gamma_0, \Gamma_1)^\top : T \to \mathcal{G} \times \mathcal{G}$ is a linear mapping such that*

$$\left[\hat{f}, \hat{g}\right]_{\mathcal{H}^2} = \left[\Gamma\hat{f}, \Gamma\hat{g}\right]_{\mathcal{G}^2} \tag{2.2}$$

holds for all $\hat{f}, \hat{g} \in T$, $\operatorname{ran}\Gamma_0 = \mathcal{G}$ and $A_0 := \ker\Gamma_0$ is a selfadjoint relation in $\mathcal{H}$.

Let $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ be a closed symmetric relation in $\mathcal{H}$. Then a generalized boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ exists if and only if A admits a selfadjoint extension in $\mathcal{H}$. In this case the defect of A coincides with $\dim \mathcal{G}$. Assume now that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple for A^+. Note that (2.2) can also be written in the form

$$[f', g] - [f, g'] = (\Gamma_1\hat{f}, \Gamma_0\hat{g})_{\mathcal{G}} - (\Gamma_0\hat{f}, \Gamma_1\hat{g})_{\mathcal{G}}, \quad \hat{f} = \{f, f'\},\ \hat{g} = \{g, g'\} \in T. \tag{2.3}$$

and that by (2.2) the operator $\Gamma : T \to \mathcal{G}^2$, $T = \operatorname{dom}\Gamma$, is an isometry from the Krein space $(\mathcal{H}^2, [\![\cdot, \cdot]\!]_{\mathcal{H}^2})$ to the Krein space $(\mathcal{G}^2, [\![\cdot, \cdot]\!]_{\mathcal{G}^2})$, i.e. $\Gamma^{-1} \subset \Gamma^{[+]}$, where $[\![+]\!]$ denotes the adjoint with respect to the Krein space inner products $[\![\cdot, \cdot]\!]_{\mathcal{H}^2}$ in $\mathcal{H}^2$ and $[\![\cdot, \cdot]\!]_{\mathcal{G}^2}$ in $\mathcal{G}^2$, respectively. From $\operatorname{ran}\Gamma_0 = \mathcal{G}$ and the selfadjointness of $A_0 = \ker\Gamma_0$ one concludes that also the inclusion $\Gamma^{[+]} \subset \Gamma^{-1}$ is true (cf. [22, Lemma 5.5]) and therefore Γ is a unitary operator from $(\mathcal{H}^2, [\![\cdot, \cdot]\!]_{\mathcal{H}^2})$ to $(\mathcal{G}^2, [\![\cdot, \cdot]\!]_{\mathcal{G}^2})$. This implies that Γ is closed and from [22, Proposition 2.3] we conclude $A = \ker\Gamma$ and that $\operatorname{ran}\Gamma$ is dense in $\mathcal{G}^2$. Moreover, Γ is surjective if and only if $\operatorname{dom}\Gamma = A^+$ holds.

Generalized boundary triples are a generalization of the well-known concept of (ordinary) boundary triples, see, e.g., [17,19,20,25,26,40], and both notions coincide if the defect of the symmetric relation is finite. In short, a generalized boundary triple with a surjective Γ is an ordinary boundary triple. The following definition from [20] reads slightly different.

Definition 2.2 *Let A be a closed symmetric relation in the Krein space $\mathcal{H}$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be an* ordinary boundary triple *for A^+, if $\mathcal{G}$ is a Hilbert space and $\Gamma = (\Gamma_0, \Gamma_1)^\top : A^+ \to \mathcal{G} \times \mathcal{G}$ is a surjective linear mapping such that*

$$\left[\hat{f}, \hat{g}\right]_{\mathcal{H}^2} = \left[\Gamma\hat{f}, \Gamma\hat{g}\right]_{\mathcal{G}^2} \tag{2.4}$$

holds for all $\hat{f}, \hat{g} \in A^+$.

Let again $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ be symmetric and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triple for A^+, $T = \operatorname{dom}\Gamma$. If the resolvent set $\rho(A_0)$ of the selfadjoint

relation $A_0 = \ker \Gamma_0$ is nonempty, then it is not difficult to see that

$$A^+ = A_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\lambda,A^+}, \quad \widehat{\mathcal{N}}_{\lambda,A^+} = \left\{ \{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathcal{N}_{\lambda,A^+} = \ker(A^+ - \lambda) \right\},$$

holds for all $\lambda \in \rho(A_0)$. Here $\widehat{+}$ denotes the direct sum of subspaces. Since $\overline{T} = A^+$ and $A_0 \subset T$ it follows that

$$\widehat{\mathcal{N}}_{\lambda,T} = \left\{ \{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \mathcal{N}_{\lambda,T} = \ker(T - \lambda) \right\}$$

is dense in $\widehat{\mathcal{N}}_{\lambda,A^+}$ and T can be decomposed as

$$T = A_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\lambda,T} = \ker \Gamma_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\lambda,T}, \qquad \lambda \in \rho(A_0). \tag{2.5}$$

Associated to a generalized boundary triple are the so-called γ-field and Weyl function. For symmetric operators in Hilbert spaces the following definition can be found in [26].

Definition 2.3 *Let A be a closed symmetric relation in the Krein space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, $A_0 = \ker \Gamma_0$, be a generalized boundary triple for A^+. Assume $\rho(A_0) \neq \emptyset$ and denote the projection in $\mathcal{H} \times \mathcal{H}$ onto the first component by π_1. The γ-field γ and Weyl function M corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are defined by*

$$\gamma(\lambda) = \pi_1 \Big(\Gamma_0 \!\upharpoonright\! \widehat{\mathcal{N}}_{\lambda,T} \Big)^{-1} \quad and \quad M(\lambda) = \Gamma_1 \Big(\Gamma_0 \!\upharpoonright\! \widehat{\mathcal{N}}_{\lambda,T} \Big)^{-1}, \quad \lambda \in \rho(A_0).$$

In the following proposition we collect some properties of the γ-field and the Weyl function associated to a generalized boundary triple. For γ-fields and Weyl functions of ordinary boundary triples the statements in Proposition 2.4 are well known (see, e.g., [20]) and in our slightly more general situation the proofs are similar and in essence included in [8, § 2.3].

Proposition 2.4 *Let $A \in \widetilde{\mathcal{C}}(\mathcal{H})$ be symmetric, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triple for A^+ and assume $\rho(A_0) \neq \emptyset$, $A_0 = \ker \Gamma_0$. Then the γ-field $\lambda \mapsto \gamma(\lambda) \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ and Weyl function $\lambda \mapsto M(\lambda) \in \mathcal{L}(\mathcal{G})$ of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are holomorphic on $\rho(A_0)$ and the identities*

$$\gamma(\lambda) = \Big(I + (\lambda - \mu)(A_0 - \lambda)^{-1} \Big) \gamma(\mu) \tag{2.6}$$

and

$$\gamma(\bar{\lambda})^+ h = \Gamma_1 \Big\{ (A_0 - \lambda)^{-1} h, (I + \lambda(A_0 - \lambda)^{-1}) h \Big\}, \qquad h \in \mathcal{H}, \tag{2.7}$$

as well as

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu}) \gamma(\mu)^+ \gamma(\lambda) \tag{2.8}$$

and

$$M(\lambda) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+ \Big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1} \Big) \gamma(\lambda_0)$$

hold for all $\lambda, \mu \in \rho(A_0)$ and any fixed $\lambda_0 \in \rho(A_0)$.

3 Realization of operator functions as Weyl functions

Let $\mathcal{D} \subset \mathbb{C}^+$ be a simply connected open set, let $\mathcal{G}$ be a Hilbert space and let τ be a piecewise meromorphic $\mathcal{L}(\mathcal{G})$-valued function on $\mathcal{D} \cup \mathcal{D}^*$, $\mathcal{D}^* = \{\lambda \in \mathbb{C} : \bar{\lambda} \in \mathcal{D}\}$, which admits the representation

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + \gamma^+ \Big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1} \Big) \gamma, \quad (3.1)$$

with some selfadjoint relation A_0 in a Krein space $\mathcal{H}$ and a mapping $\gamma \in \mathcal{L}(\mathcal{G}, \mathcal{H})$. It is assumed that $\rho(A_0)$ is nonempty, that (3.1) holds for a fixed $\lambda_0 \in \mathcal{O} \cup \mathcal{O}^*$ and all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$, where $\mathcal{O}$ is an open subset of $\rho(A_0) \cap \mathcal{D}$, $\mathcal{O}^* = \{\lambda \in \mathbb{C} : \bar{\lambda} \in \mathcal{O}\}$, and that the minimality condition

$$\mathcal{H} = \operatorname{clsp}\Big\{ \big(I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}\big)\gamma x : \lambda \in \mathcal{O} \cup \mathcal{O}^*, \, x \in \mathcal{G} \Big\} \quad (3.2)$$

is satisfied. It is clear that τ is holomorphic on $\mathcal{O} \cup \mathcal{O}^*$ and that $\tau(\lambda)^* = \tau(\bar{\lambda})$ holds for all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. The set of points of holomorphy of τ will be denoted by $\mathfrak{h}(\tau)$.

The following theorem is the main result of this section. The proof of Theorem 3.1 will be given after some preparations at the end of in Section 3.2.

Theorem 3.1 *Let $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a piecewise meromorphic operator function which is represented in the form (3.1)-(3.2). Then there exists a Krein space $\mathcal{K}$, a closed symmetric operator S in $\mathcal{K}$ and a generalized boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+ such that the corresponding Weyl function coincides with τ on $\mathcal{O} \cup \mathcal{O}^*$.*

Since generalized boundary triples reduce to ordinary boundary triples if $\dim \mathcal{G}$ is finite we obtain the following corollary.

Corollary 3.2 *Let $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a piecewise meromorphic operator function which is represented in the form (3.1)-(3.2) and assume, in addition, that $\dim \mathcal{G}$ is finite. Then there exists a Krein space $\mathcal{K}$, a closed symmetric operator S in $\mathcal{K}$ and an ordinary boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+ such that the corresponding Weyl function coincides with τ on $\mathcal{O} \cup \mathcal{O}^*$.*

Remark 3.3 *Many important classes of $\mathcal{L}(\mathcal{G})$-valued functions satisfy the above assumptions, cf. Section 3.3. E.g., for Nevanlinna functions or generalized Nevanlinna functions one chooses $\mathcal{D} = \mathbb{C}^+$, A_0 becomes a selfadjoint relation in a Hilbert or Pontryagin space, respectively, and (3.1) holds for all $\lambda \in \rho(A_0)$, cf. [45,50]. So-called definitizable and locally definitizable functions can be represented in the form (3.1)-(3.2) with the help of definitizable and locally definitizable selfadjoint relations A_0 in Krein spaces, see [46–48]. For operator functions piecewise holomorphic in $\mathcal{D} \cup \mathcal{D}^*$ and a given open subset $\mathcal{O}$, $\overline{\mathcal{O}} \subset \mathcal{D}$, a Krein space $\mathcal{H}$ and a selfadjoint relation A_0 with $\overline{\mathcal{O}} \cup \overline{\mathcal{O}^*} \subset \rho(A_0)$ such that (3.1)-(3.2) holds for all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ was constructed in [2,28,48].*

Fix some $\mu_0 \in \mathfrak{h}(\tau)$ and define the closed subspace $\widehat{\mathcal{G}}$ of $\mathcal{G}$ by

$$\widehat{\mathcal{G}} := \bigcap_{\lambda \in \mathfrak{h}(\tau)} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \bar{\mu}_0}. \tag{3.3}$$

It is not difficult to see that $\widehat{\mathcal{G}}$ does not depend on the choice of $\mu_0 \in \mathfrak{h}(\tau)$ and that the set $\mathfrak{h}(\tau)$ in the intersection in (3.3) can be replaced by the union of an open subset in $\mathcal{D}$ and an open subset in $\mathcal{D}^*$, e.g., $\mathcal{O} \cup \mathcal{O}^*$.

Definition 3.4 *A piecewise meromorphic function $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ is called* strict *if the space $\widehat{\mathcal{G}}$ in (3.3) is trivial.*

3.1   Realization of strict operator functions

In this subsection we prove that every strict $\mathcal{L}(\mathcal{G})$-valued operator function τ of the form (3.1)-(3.2) can be realized as the Weyl function of a generalized boundary triple. We start with a simple observation.

Lemma 3.5 *Let $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a meromorphic function represented in the form (3.1)-(3.2) with some $\gamma \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ and let $\widehat{\mathcal{G}}$ be as in (3.3). Then $\widehat{\mathcal{G}} = \ker \gamma$ and, in particular, τ is strict if and only if γ is injective.*

Proof. For $x \in \ker \gamma$ we conclude from (3.1) $\tau(\lambda)x = \operatorname{Re} \tau(\lambda_0)x$ for all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ and therefore x belongs to

$$\widehat{\mathcal{G}} = \bigcap_{\lambda \in \mathfrak{h}(\tau)} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \bar{\mu}_0}. \tag{3.4}$$

Conversely, if $x \in \widehat{\mathcal{G}}$, then x belongs also to the right hand side of (3.4) with μ_0 replaced by $\bar{\lambda}_0$. Making use of (3.1) for $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ we obtain

$$
\begin{aligned}
0 &= \left(\frac{\tau(\lambda) - \tau(\lambda_0)}{\lambda - \lambda_0} x, y \right) = \left(\gamma^+ \left(I + (\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1} \right) \gamma x, y \right) \\
&= \left[\gamma x, (I + (\bar{\lambda} - \lambda_0)(A_0 - \bar{\lambda})^{-1}) \gamma y \right]
\end{aligned}
$$

for all $y \in \mathcal{G}$ and all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. The minimality condition (3.2) implies $\gamma x = 0$. $\qquad \square$

The following theorem is a generalization of [6, Theorem 3.3], [24, Proposition 3.1] and [26, §3].

Theorem 3.6 *Let τ be a strict $\mathcal{L}(\mathcal{G})$-valued function represented in the form (3.1)-(3.2). Then there exists a closed symmetric operator A in the Krein space $\mathcal{H}$ and a generalized boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that τ is the corresponding Weyl function on $\mathcal{O} \cup \mathcal{O}^*$. Furthermore, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple if and only if $\operatorname{ran} \gamma$ is closed.*

Proof. Let τ be represented by the selfadjoint relation A_0 in $\mathcal{H}$ as in (3.1). For all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ and the fixed $\lambda_0 \in \mathcal{O} \cup \mathcal{O}^*$ we define the mapping

$$
\gamma(\lambda) := \left(I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1} \right) \gamma \ \in \mathcal{L}(\mathcal{G}, \mathcal{H}). \tag{3.5}
$$

Then we have $\gamma(\lambda_0) = \gamma$, $\gamma(\zeta) = (1 + (\zeta - \eta)(A_0 - \zeta)^{-1})\gamma(\eta)$ and

$$
\tau(\zeta) - \tau(\eta)^* = (\zeta - \bar{\eta})\gamma(\eta)^+ \gamma(\zeta) \tag{3.6}
$$

for all $\zeta, \eta \in \mathcal{O} \cup \mathcal{O}^*$. For some $\xi \in \mathcal{O} \cup \mathcal{O}^*$ we define the closed symmetric relation

$$
A := \left\{ \{f_0, f_0'\} \in A_0 : [f_0' - \bar{\xi} f_0, \gamma(\xi)x] = 0 \ \text{ for all } \ x \in \mathcal{G} \right\} \tag{3.7}
$$

in $\mathcal{H}$. Note that the definition of A does not depend on the choice of $\xi \in \mathcal{O} \cup \mathcal{O}^*$ and that $\operatorname{ran}(A - \bar{\lambda}) = (\operatorname{ran} \gamma(\lambda))^{[\perp]}$ holds for all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. Hence $\mathcal{N}_{\lambda, A^+} = \overline{\operatorname{ran} \gamma(\lambda)}$ or, if $\operatorname{ran} \gamma(\lambda)$ is closed, then $\mathcal{N}_{\lambda, A^+} = \operatorname{ran} \gamma(\lambda)$. Since τ is assumed to be strict it follows from Lemma 3.5 that γ is injective. Furthermore, the fact that the operator $I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}$, $\lambda \in \mathcal{O} \cup \mathcal{O}^*$, is an isomorphism of $\mathcal{N}_{\lambda_0, A^+}$ onto $\mathcal{N}_{\lambda, A^+}$ implies that $\gamma(\lambda)$, regarded as a mapping from $\mathcal{G}$ into $\mathcal{N}_{\lambda, A^+}$ is injective and has dense range. Note also that the minimality condition (3.2) together with (3.5) implies that A is an operator.

We fix a point $\mu \in \mathcal{O} \cup \mathcal{O}^*$. Then $A^+ = A_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\mu, A^+}$ holds and the linear relation

$$
T := A_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\mu, T}, \qquad \widehat{\mathcal{N}}_{\mu, T} = \left\{ \{\gamma(\mu)x, \mu\gamma(\mu)x\} : x \in \mathcal{G} \right\},
$$

is dense in A^+. The elements $\hat{f} \in T$ will be written in the form

$$\hat{f} = \{f_0, f_0'\} + \{\gamma(\mu)x, \mu\gamma(\mu)x\}, \quad \{f_0, f_0'\} \in A_0,\ x \in \mathcal{G}.$$

Let $\Gamma_0, \Gamma_1 : T \to \mathcal{G}$ be the linear mappings defined by

$$\Gamma_0 \hat{f} := x \quad \text{and} \quad \Gamma_1 \hat{f} := \gamma(\mu)^+(f_0' - \bar{\mu}f_0) + \tau(\mu)x.$$

Then obviously $\operatorname{ran} \Gamma_0 = \mathcal{G}$ and $A_0 = \ker \Gamma_0$ is selfadjoint. Moreover, for $\hat{f} \in T$ and

$$\hat{g} = \{g_0, g_0'\} + \{\gamma(\mu)y, \mu\gamma(\mu)y\} \in T, \quad \{g_0, g_0'\} \in A_0,\ y \in \mathcal{G},$$

we compute

$$\begin{aligned}
-i\left[\hat{f}, \hat{g}\right] &= [\gamma(\mu)x, g_0' - \bar{\mu}g_0] - [f_0' - \bar{\mu}f_0, \gamma(\mu)y] - (\mu - \bar{\mu})[\gamma(\mu)x, \gamma(\mu)y] \\
&= \left(x, \gamma(\mu)^+(g_0' - \bar{\mu}g_0)\right) - \left(\gamma(\mu)^+(f_0' - \bar{\mu}f_0), y\right) - \left((\tau(\mu) - \tau(\mu)^*)x, y\right) \\
&= -i\left[\Gamma\hat{f}, \Gamma\hat{g}\right],
\end{aligned}$$

where we have used $A_0 = A_0^+$ and $\tau(\mu) - \tau(\mu)^* = (\mu - \bar{\mu})\gamma(\mu)^+\gamma(\mu)$. Therefore $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple for A^+.

Let us check that the Weyl function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ coincides with τ on $\mathcal{O} \cup \mathcal{O}^*$. Note first that by the definition of Γ_0 and Γ_1 it is clear that $\tau(\mu)\Gamma_0\hat{f}_\mu = \Gamma_1\hat{f}_\mu$ holds for $\hat{f}_\mu = \{\gamma(\mu)x, \mu\gamma(\mu)x\} \in \widehat{\mathcal{N}}_{\mu,T}$. Now let $\eta \in \mathcal{O} \cup \mathcal{O}^*$ and $\hat{f}_\eta \in \widehat{\mathcal{N}}_{\eta,T}$. Since $T = A_0 \hat{+} \widehat{\mathcal{N}}_{\mu,T}$ there exist $\{f_0, f_0'\} \in A_0$ and $x \in \mathcal{G}$ such that

$$\hat{f}_\eta = \{f_\eta, \eta f_\eta\} = \{f_0, f_0'\} + \{\gamma(\mu)x, \mu\gamma(\mu)x\}. \tag{3.8}$$

It follows from (3.6) and $\gamma(\eta) = (I + (\eta - \mu)(A_0 - \eta)^{-1})\gamma(\mu)$ that

$$\begin{aligned}
\tau(\eta) &= \tau(\mu)^* + (\eta - \bar{\mu})\gamma(\mu)^+\gamma(\eta) \\
&= \tau(\mu) + \gamma(\mu)^+\left((\bar{\mu} - \mu)\gamma(\mu) + (\eta - \bar{\mu})\gamma(\eta)\right) \\
&= \tau(\mu) + \gamma(\mu)^+(\eta - \mu)\left(I + (\eta - \bar{\mu})(A_0 - \eta)^{-1}\right)\gamma(\mu).
\end{aligned}$$

Hence we have

$$\tau(\eta)\Gamma_0\hat{f}_\eta = \tau(\mu)x + \gamma(\mu)^+(\eta - \mu)\left(I + (\eta - \bar{\mu})(A_0 - \eta)^{-1}\right)\gamma(\mu)x \tag{3.9}$$

and from (3.8) it follows that

$$f_0' - \eta f_0 = (\eta - \mu)\gamma(\mu)x \quad \text{and} \quad f_0' - \bar{\mu}f_0 = (\eta - \mu)\gamma(\mu)x + (\eta - \bar{\mu})f_0$$

hold. The first identity yields $f_0 = (\eta - \mu)(A_0 - \eta)^{-1}\gamma(\mu)x$ and therefore (3.9) becomes

$$\tau(\eta)\Gamma_0\hat{f}_\eta = \tau(\mu)x + \gamma(\mu)^+(f_0' - \bar{\mu}f_0) = \Gamma_1\hat{f}_\eta,$$

i.e., τ coincides with the Weyl function of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ on $\mathcal{O} \cup \mathcal{O}^*$.

It remains to show that the triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for A^+ if and only if $\operatorname{ran}\gamma = \overline{\operatorname{ran}\gamma}$. Clearly, if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple, then the range of the γ-field is closed and hence $\operatorname{ran}\gamma = \operatorname{ran}\gamma(\lambda_0)$ is closed. Conversely, if $\operatorname{ran}\gamma$ is closed it is sufficient to check that $(\Gamma_0, \Gamma_1)^\top$ is surjective, cf. Section 2. Observe first that $\{0\} = \ker\gamma(\mu) = (\operatorname{ran}\gamma(\mu)^+)^\perp$ and that $\operatorname{ran}\gamma(\lambda)$ is closed for every $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. Hence $\operatorname{ran}\gamma(\mu)^+ = \mathcal{G}$ and for given elements $x, y \in \mathcal{G}$ there exist $\{f_0, f_0'\} \in A_0$ such that $\gamma(\mu)^+(f_0' - \bar{\mu}f_0) = y - \tau(\mu)x$. Now it easy to see that $\hat{f} = \{f_0, f_0'\} + \{\gamma(\mu)x, \mu\gamma(\mu)x\}$ satisfies $\Gamma_0\hat{f} = x$ and $\Gamma_1\hat{f} = y$. $\qquad\square$

Remark 3.7 *If τ is a strict $\mathcal{L}(\mathcal{G})$-valued function which admits a representation as in (3.1)-(3.2) and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple as in Theorem 3.6 with $T = \operatorname{dom}\Gamma$, then the span of the subspaces of $\mathcal{N}_{\lambda,T}$ is dense in $\mathcal{H}$, i.e., $\mathcal{H} = \operatorname{clsp}\{\mathcal{N}_{\lambda,T} : \lambda \in \mathcal{O} \cup \mathcal{O}^*\}$, and the closed symmetric operator $A = \ker\Gamma$ has no eigenvalues.*

If τ is a matrix-valued function, that is, $\dim\mathcal{G} < \infty$, then of course the range of the mapping $\gamma \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ in (3.1) is closed. Hence Theorem 3.6 implies the following corollary.

Corollary 3.8 *Let τ be a strict $\mathcal{L}(\mathcal{G})$-valued function represented in the form (3.1)-(3.2) and assume, in addition, that $\dim\mathcal{G}$ is finite. Then there exists a closed symmetric operator A in the Krein space $\mathcal{H}$ and an ordinary boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that τ is the corresponding Weyl function on $\mathcal{O} \cup \mathcal{O}^*$.*

3.2 Realization of non-strict operator functions

Let again $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a piecewise meromorphic operator function which is represented in the form (3.1)-(3.2). We are now interested in the case where τ is not strict, i.e., the space $\widehat{\mathcal{G}}$ in (3.3) is not trivial. Roughly speaking the next lemma states that τ can always be written as a selfadjoint constant and a smaller strict operator function. For special classes of matrix-valued functions Lemma 3.9 can be found in [6].

Lemma 3.9 *Let τ be a piecewise meromorphic $\mathcal{L}(\mathcal{G})$-valued function represented in the form (3.1)-(3.2), let $\widehat{\mathcal{G}}$ be as in (3.3) and set $\mathcal{G}' := \mathcal{G} \ominus \widehat{\mathcal{G}}$. Denote*

the corresponding orthogonal projections and canonical embeddings by $\widehat{\pi}$, π', $\widehat{\iota}$ and ι', respectively, and fix some $\mu_0 \in \mathfrak{h}(\tau)$. Then

$$\tau(\lambda) = \begin{pmatrix} \pi'\tau(\lambda)\iota' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \pi'\tau(\mu_0)\widehat{\iota} \\ \widehat{\pi}\tau(\mu_0)\iota' & \widehat{\pi}\tau(\mu_0)\widehat{\iota} \end{pmatrix} : \begin{pmatrix} \mathcal{G}' \\ \widehat{\mathcal{G}} \end{pmatrix} \to \begin{pmatrix} \mathcal{G}' \\ \widehat{\mathcal{G}} \end{pmatrix} \qquad (3.10)$$

for all $\lambda \in \mathfrak{h}(\tau)$ and the $\mathcal{L}(\mathcal{G}')$-valued function $\lambda \mapsto \pi'\tau(\lambda)\iota'$ is strict.

Proof. It follows from the definition of $\widehat{\mathcal{G}}$ in (3.3) that for $\widehat{x} \in \widehat{\mathcal{G}}$ and all $\lambda \in \mathfrak{h}(\tau)$ the relation $\tau(\lambda)\widehat{\iota}\widehat{x} = \tau(\bar{\mu}_0)\widehat{\iota}\widehat{x} = \tau(\mu_0)\widehat{\iota}\widehat{x}$ holds. Therefore

$$\tau(\lambda) = \begin{pmatrix} \cdot & \pi'\tau(\mu_0)\widehat{\iota} \\ \cdot & \widehat{\pi}\tau(\mu_0)\widehat{\iota} \end{pmatrix} : \begin{pmatrix} \mathcal{G}' \\ \widehat{\mathcal{G}} \end{pmatrix} \to \begin{pmatrix} \mathcal{G}' \\ \widehat{\mathcal{G}} \end{pmatrix}, \quad \lambda \in \mathfrak{h}(\tau),$$

and the symmetry property $\tau(\bar{\lambda}) = \tau(\lambda)^*$ implies

$$\widehat{\pi}\tau(\lambda)\iota' = (\pi'\tau(\bar{\lambda}0\widehat{\iota})^* = (\pi'\tau(\bar{\mu}_0)\widehat{\iota})^* = \widehat{\pi}\tau(\mu_0)\iota'$$

which yields the representation (3.10). Let us show that $\lambda \mapsto \pi'\tau(\lambda)\iota'$ is a strict function. Assume that $x' \in \mathcal{G}'$ belongs to

$$\bigcap_{\lambda \in \mathfrak{h}(\tau)} \ker \frac{\pi'\tau(\lambda)\iota' - \pi'\tau(\bar{\mu}_0)\iota'}{\lambda - \bar{\mu}_0}.$$

Then $\pi'\tau(\lambda)\iota'x' = \pi'\tau(\bar{\mu}_0)\iota'x'$ and also $\widehat{\pi}\tau(\lambda)\iota'x' = \widehat{\pi}\tau(\bar{\mu}_0)\iota'x'$ by (3.10) for all $\lambda \in \mathfrak{h}(\tau)$, and this implies $\iota'x' \in \widehat{\mathcal{G}}$. This is possible only for $x' = 0$, i.e., the function $\lambda \mapsto \pi'\tau(\lambda)\iota'$ is strict. $\qquad\square$

Next we construct a nondensely defined closed symmetric operator B in a Krein space and an ordinary boundary triple for B^+ such that the corresponding Weyl function is a selfadjoint constant.

Lemma 3.10 *Let $\widehat{\mathcal{G}}$ be a Hilbert space, let $\Theta = \Theta^* \in \mathcal{L}(\widehat{\mathcal{G}})$ and fix some $\vartheta \in \mathbb{C}$. Then $\widetilde{\mathcal{H}} = (\widehat{\mathcal{G}}^2, (J\cdot,\cdot))$, where $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, is a Krein space and there exists a closed symmetric operator B in $\widetilde{\mathcal{H}}$ and an ordinary boundary triple $\{\widehat{\mathcal{G}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, $B_0 = \ker\widehat{\Gamma}_0$, for B^+ such that the corresponding Weyl function is the selfadjoint constant Θ and $\sigma(B_0) = \{\vartheta, \bar{\vartheta}\}$.*

Proof. We equip $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$ with the indefinite inner product $[\cdot, \cdot] := (J\cdot, \cdot)$, where $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and $(\cdot, \cdot)$ is the Hilbert scalar product on $\widehat{\mathcal{G}}^2$. Then

$$B_0 := \begin{pmatrix} \vartheta & I \\ 0 & \bar{\vartheta} \end{pmatrix} \in \mathcal{L}(\widehat{\mathcal{G}}^2)$$

is selfadjoint in the Krein space $\widetilde{\mathcal{H}} = (\widehat{\mathcal{G}}^2, [\cdot, \cdot])$ and for every $\lambda \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$ we have

$$(B_0 - \lambda)^{-1} = \begin{pmatrix} (\vartheta - \lambda)^{-1} & (\lambda - \vartheta)^{-1}(\bar{\vartheta} - \lambda)^{-1} \\ 0 & (\bar{\vartheta} - \lambda)^{-1} \end{pmatrix} \in \mathcal{L}(\widetilde{\mathcal{H}}).$$

Let $\lambda_0 \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$, $\widehat{\gamma}_{\lambda_0} : \widehat{\mathcal{G}} \to \widetilde{\mathcal{H}}$, $x \mapsto (x, 0)^\top$, and define for $\lambda \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$

$$\widehat{\gamma}(\lambda) : \widehat{\mathcal{G}} \to \widetilde{\mathcal{H}}, \qquad x \mapsto \left(I + (\lambda - \lambda_0)(B_0 - \lambda)^{-1}\right)\widehat{\gamma}_{\lambda_0} x = \left(\frac{\vartheta - \lambda_0}{\vartheta - \lambda} x, 0\right)^\top.$$

Then obviously $\operatorname{ran}\widehat{\gamma}(\lambda) = \widehat{\mathcal{G}} \times \{0\}$. From

$$\widehat{\gamma}(\eta)^+ : \widetilde{\mathcal{H}} \to \widehat{\mathcal{G}}, \quad (x, y)^\top \mapsto \frac{\bar{\vartheta} - \bar{\lambda}_0}{\bar{\vartheta} - \bar{\eta}} y, \quad \eta \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}, \tag{3.11}$$

we obtain $\widehat{\gamma}(\eta)^+\widehat{\gamma}(\lambda) = 0$ for all $\lambda, \eta \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$. Consider the closed symmetric operator

$$B := B_0 \upharpoonright \left(\widehat{\mathcal{G}} \times \{0\}\right) \tag{3.12}$$

in $\widetilde{\mathcal{H}}$. Then we have $\mathcal{N}_{\lambda, B^+} = \widehat{\mathcal{G}} \times \{0\} = \operatorname{ran}\widehat{\gamma}(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$, the defect of B coincides with $\dim\widehat{\mathcal{G}}$ and $\mathcal{N}_{\lambda, B^+}[\bot]\mathcal{N}_{\eta, B^+}$ holds for all $\lambda, \eta \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$. For a fixed $\mu \in \mathbb{C}\backslash\{\vartheta, \bar{\vartheta}\}$ we write the elements $\hat{g} \in B^+ = B_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\mu, B^+}$ in the form

$$\hat{g} = \{g_0, B_0 g_0\} + \{\widehat{\gamma}(\mu)x, \mu\widehat{\gamma}(\mu)x\}, \qquad g_0 \in \widetilde{\mathcal{H}}, \quad x \in \widehat{\mathcal{G}}.$$

Then it follows as in the proof of Theorem 3.6 that $\{\widehat{\mathcal{G}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$, where

$$\widehat{\Gamma}_0\hat{g} := x \quad \text{and} \quad \widehat{\Gamma}_1\hat{g} := \widehat{\gamma}(\mu)^+(B_0 - \bar{\mu})g_0 + \Theta x, \tag{3.13}$$

is a boundary triple for B^+ and the corresponding Weyl function is the self-adjoint constant $\Theta \in \mathcal{L}(\widehat{\mathcal{G}})$. $\qquad\square$

Remark 3.11 *Note that the negative and the positive index of the Krein space $\widetilde{\mathcal{H}} = (\widehat{\mathcal{G}}^2, (J\cdot, \cdot))$ in Proposition 3.10 coincides with $\dim\widehat{\mathcal{G}}$, that is,*

$$\dim\left(\ker(J - I)\right) = \dim\left(\ker(J + I)\right) = \dim\widehat{\mathcal{G}}.$$

Proof of Theorem 3.1. Let $\tau : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a (in general non-strict) piecewise meromorphic function which is represented in the form (3.1)-(3.2) for a fixed $\lambda_0 \in \mathcal{O} \cup \mathcal{O}^*$ and all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. Let $\widehat{\mathcal{G}}$ be as in (3.3), set $\mathcal{G}' = \mathcal{G} \ominus \widehat{\mathcal{G}}$ and decompose τ as in (3.10).

Then by Lemma 3.9 the piecewise meromorphic function

$$\tau_s := \pi'\tau\iota' : \mathcal{D} \cup \mathcal{D}^* \to \mathcal{L}(\mathcal{G}')$$

is strict. Setting $\gamma' := \gamma\iota' \in \mathcal{L}(\mathcal{G}', \mathcal{H})$ it follows directly from (3.1) that

$$\tau_s(\lambda) = \operatorname{Re}\tau_s(\lambda_0) + \gamma'^+\Big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1}\Big)\gamma'$$

holds for a fixed $\lambda_0 \in \mathcal{O} \cup \mathcal{O}^*$ and all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$. Furthermore, (3.2) together with the fact $\widehat{\mathcal{G}} = \ker\gamma$, cf. Lemma 3.5, implies that the minimality condition

$$\mathcal{H} = \operatorname{clsp}\Big\{\big(1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}\big)\gamma'x' : \lambda \in \mathcal{O} \cup \mathcal{O}^*, \, x' \in \mathcal{G}'\Big\}$$

is satisfied. Therefore we can apply Theorem 3.6 to the function τ_s, i.e., τ_s coincides on $\mathcal{O} \cup \mathcal{O}^*$ with the Weyl function corresponding to some closed symmetric operator $A \subset A_0$ in the Krein space $\mathcal{H}$ and a generalized boundary triple $\{\mathcal{G}', \Gamma_0', \Gamma_1'\}$ for the adjoint A^+. Note that $A_0 = \ker\Gamma_0'$ and that $\operatorname{dom}\Gamma'$, $\Gamma' = (\Gamma_0', \Gamma_1')^\top$, is dense in A^+.

According to Lemma 3.10 there exists a Krein space $\widetilde{\mathcal{H}}$, a closed symmetric operator B in $\widetilde{\mathcal{H}}$ and an ordinary boundary triple $\{\widehat{\mathcal{G}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ such that the corresponding Weyl function is the selfadjoint constant

$$\widehat{\pi}\tau(\mu_0)\widehat{\iota} \in \mathcal{L}(\widehat{\mathcal{G}}).$$

Moreover, the spectrum of the selfadjoint relation $B_0 = \ker\widehat{\Gamma}_0$ consists of a pair of eigenvalues $\{\vartheta, \bar{\vartheta}\}$ and it is no restriction to assume that $\vartheta, \bar{\vartheta} \notin \mathcal{O} \cup \mathcal{O}^*$ holds.

In the following we consider the closed symmetric operator $S := A \times B$ in the Krein space $\mathcal{K} := \mathcal{H} \times \widetilde{\mathcal{H}}$ and its adjoint $S^+ = A^+ \times B^+$. Note that $\operatorname{dom}\Gamma' \times B^+$ is dense in S^+. The elements in $\operatorname{dom}\Gamma' \times B^+$ will be denoted in the form $\{\hat{f}, \hat{g}\}$, $\hat{f} \in \operatorname{dom}\Gamma'$, $\hat{g} \in B^+$. We claim that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0\{\hat{f}, \hat{g}\} := \begin{pmatrix} \Gamma_0'\hat{f} \\ \widehat{\Gamma}_0\hat{g} \end{pmatrix} \quad \text{and} \quad \Gamma_1\{\hat{f}, \hat{g}\} := \begin{pmatrix} \Gamma_1'\hat{f} + \pi'\tau(\mu_0)\widehat{\iota}\,\widehat{\Gamma}_0\hat{g} \\ \widehat{\Gamma}_1\hat{g} + \widehat{\pi}\tau(\mu_0)\iota'\,\Gamma_0'\hat{f} \end{pmatrix},$$

$\{\hat{f}, \hat{g}\} \in \operatorname{dom}\Gamma' \times B^+$, is a generalized boundary triple for S^+ such that the corresponding Weyl function coincides with τ on $\mathcal{O} \cup \mathcal{O}^*$. In fact, since $\{\mathcal{G}', \Gamma_0', \Gamma_1'\}$ and $\{\widehat{\mathcal{G}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ are generalized and ordinary boundary triples for

A^+ and B^+, respectively, it follows that for $\{\hat{f}, \hat{g}\}, \{\hat{h}, \hat{k}\} \in \operatorname{dom} \Gamma' \times B^+$

$$\left[\Gamma\{\hat{f}, \hat{g}\}, \Gamma\{\hat{h}, \hat{k}\}\right]_{(\mathcal{G}' \oplus \widehat{\mathcal{G}})^2}$$
$$= i\left(\begin{pmatrix} \Gamma'_0 \hat{f} \\ \widehat{\Gamma}_0 \hat{g} \end{pmatrix}, \begin{pmatrix} \Gamma'_1 \hat{h} + \pi'\tau(\mu_0)\hat{\iota}\,\widehat{\Gamma}_0\hat{k} \\ \widehat{\Gamma}_1 \hat{k} + \widehat{\pi}\tau(\mu_0)\iota'\,\Gamma'_0\hat{h} \end{pmatrix}\right) - i\left(\begin{pmatrix} \Gamma'_1 \hat{f} + \pi'\tau(\mu_0)\hat{\iota}\,\widehat{\Gamma}_0\hat{g} \\ \widehat{\Gamma}_1 \hat{g} + \widehat{\pi}\tau(\mu_0)\iota'\,\Gamma'_0\hat{f} \end{pmatrix}, \begin{pmatrix} \Gamma'_0 \hat{h} \\ \widehat{\Gamma}_0 \hat{k} \end{pmatrix}\right)$$
$$= \left[\Gamma'\hat{f}, \Gamma'\hat{h}\right]_{\mathcal{G}'^2} + \left[\widehat{\Gamma}\hat{g}, \widehat{\Gamma}\hat{k}\right]_{\widehat{\mathcal{G}}^2} = \left[\hat{f}, \hat{h}\right]_{\mathcal{H}^2} + \left[\hat{g}, \hat{k}\right]_{\widetilde{\mathcal{H}}^2} = \left[\{\hat{f}, \hat{g}\}, \{\hat{h}, \hat{k}\}\right]_{(\mathcal{H} \times \widetilde{\mathcal{H}})^2}$$

holds. Here we also have used $(\pi'\tau(\mu_0)\hat{\iota})^* = \widehat{\pi}\tau(\mu_0)\iota'$. Moreover, since $A_0 = \ker \Gamma'_0$ and $B_0 = \ker \widehat{\Gamma}_0$ are selfadjoint in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, it is clear that $\ker \Gamma_0 = A_0 \times B_0$ is a selfadjoint relation in $\mathcal{K} = \mathcal{H} \times \widetilde{\mathcal{H}}$. As $\operatorname{ran} \Gamma'_0 = \mathcal{G}'$ and $\operatorname{ran} \widehat{\Gamma}_0 = \widehat{\mathcal{G}}$ we also have that $\operatorname{ran} \Gamma_0$ coincides with $\mathcal{G} = \mathcal{G}' \oplus \widehat{\mathcal{G}}$. Hence $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a generalized boundary triple for $S^+ = A^+ \times B^+$. It remains to show that the corresponding Weyl function coincides with τ. For this, note that

$$\widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma} = \widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma' \times B^+} = \widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma'} \times \widehat{\mathcal{N}}_{\lambda, B^+}, \qquad \lambda \in \mathcal{O} \cup \mathcal{O}^*,$$

and let $\{\hat{f}_\lambda, \hat{g}_\lambda\} \in \operatorname{dom}\Gamma' \times B^+$, where $\hat{f}_\lambda \in \widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma'}$ and $\hat{g}_\lambda \in \widehat{\mathcal{N}}_{\lambda, B^+}$. Since

$$\tau_s(\lambda)\Gamma'_0\hat{f}_\lambda = \Gamma'_1\hat{f}_\lambda \quad \text{and} \quad \widehat{\pi}\tau(\mu_0)\hat{\iota}\,\widehat{\Gamma}_0\hat{g}_\lambda = \widehat{\Gamma}_1\hat{g}_\lambda, \quad \lambda \in \mathcal{O} \cup \mathcal{O}^*,$$

we conclude

$$\tau(\lambda)\Gamma_0\{\hat{f}_\lambda, \hat{g}_\lambda\} = \begin{pmatrix} \tau_s(\lambda) & \pi'\tau(\mu_0)\hat{\iota} \\ \widehat{\pi}\tau(\mu_0)\iota' & \widehat{\pi}\tau(\mu_0)\hat{\iota} \end{pmatrix} \begin{pmatrix} \Gamma'_0\hat{f}_\lambda \\ \widehat{\Gamma}_0\hat{g}_\lambda \end{pmatrix}$$
$$= \begin{pmatrix} \Gamma'_1\hat{f}_\lambda + \pi'\tau(\mu_0)\hat{\iota}\,\widehat{\Gamma}_0\hat{g}_\lambda \\ \widehat{\pi}\tau(\mu_0)\iota'\,\Gamma'_0\hat{f}_\lambda + \widehat{\Gamma}_1\hat{g}_\lambda \end{pmatrix} = \Gamma_1\{\hat{f}_\lambda, \hat{g}_\lambda\}$$

for all $\lambda \in \mathcal{O} \cup \mathcal{O}^*$, that is, τ coincides with the Weyl function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ on $\mathcal{O} \cup \mathcal{O}^*$. $\qquad \square$

Remark 3.12 *Let τ be as in (3.1)-(3.2) and let $\mathcal{K} = \mathcal{H} \times \widetilde{\mathcal{H}}$, $S = A \times B$ and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be as in the proof of Theorem 3.1. If τ is non-strict, then $\widehat{\mathcal{G}} \neq \{0\}$ and in contrast to Theorem 3.6 and Remark 3.7 here the defect subspaces $\mathcal{N}_{\lambda, \operatorname{dom}\Gamma}$, $\lambda \in \mathcal{O} \cup \mathcal{O}^*$, are not dense in $\mathcal{K}$. Indeed, it follows from the construction in the proof of Lemma 3.10 that*

$$\operatorname{clsp}\left\{\mathcal{N}_{\lambda, B^+} : \lambda \in \mathcal{O} \cup \mathcal{O}^*\right\} = \widehat{\mathcal{G}} \times \{0\} \neq \widetilde{\mathcal{H}} = \widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$$

holds. Therefore

$$\operatorname{clsp}\left\{\mathcal{N}_{\lambda, \operatorname{dom}\Gamma} : \lambda \in \mathcal{O} \times \mathcal{O}^*\right\} = \mathcal{H} \times \widehat{\mathcal{G}} \times \{0\} \neq \mathcal{K}.$$

This implies that the analytic properties of τ are in general not completely reflected by the spectral properties of the selfadjoint operator or relation $S_0 = \ker\Gamma_0$ in $\mathcal{K}$, but this disadvantage arises only at the points $\vartheta, \bar{\vartheta}$ which can be chosen arbitrary, e.g. in $\mathbb{C}\backslash(\mathcal{D}\cup\mathcal{D}^*)$. In Section 4 we shall see that the non-minimality does not affect solvability properties of a certain class of elliptic boundary value problems investigated here. Note also, that ϑ is the only eigenvalue of the symmetric operator $S = A\times B$, since $\sigma_p(A) = \emptyset$ by Remark 3.7 and $\sigma_p(B) = \{\vartheta\}$; cf. (3.12).

3.3 Some special classes of operator functions

Many classes of $\mathbb{R}$-symmetric operator functions satisfy the general assumptions in the beginning of Section 3, cf. Remark 3.3. In this subsection we briefly recall some necessary definitions and we formulate some corollaries of Theorem 3.1.

The first corollary concerns the case of a locally holomorphic operator function. We refer to [2,28,48] for the existence of the representation (3.1)-(3.2).

Corollary 3.13 *Let $\tau : \mathcal{D}\cup\mathcal{D}^* \to \mathcal{L}(\mathcal{G})$ be a piecewise holomorphic function which satisfies $\tau(\bar{\lambda}) = \tau(\lambda)^*$, $\lambda \in \mathcal{D}\cup\mathcal{D}^*$, and let $\mathcal{O}$ be a simply connected open set with $\overline{\mathcal{O}}\subset\mathcal{D}$. Then there exists a Krein space $\mathcal{K}$, a closed symmetric operator S in $\mathcal{K}$ and a generalized boundary triple $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ for S^+ such that the corresponding Weyl function coincides with τ on $\mathcal{O}\cup\mathcal{O}^*$. If, in addition, $\dim\mathcal{G} < \infty$ holds, then $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is an ordinary boundary triple.*

The classes of generalized Nevanlinna functions were introduced and studied by M.G. Krein and H. Langer, see, e.g., [49–51]. Recall that an $\mathcal{L}(\mathcal{G})$-valued function τ belongs to the *generalized Nevanlinna class* $N_\kappa(\mathcal{L}(\mathcal{G}))$, $\kappa\in\mathbb{N}_0$, if τ is piecewise meromorphic in $\mathbb{C}\backslash\mathbb{R}$ and $\mathbb{R}$-symmetric, i.e., $\tau(\bar{\lambda}) = \tau(\lambda)^*$ for all λ belonging to the set of points of holomorphy $\mathfrak{h}(\tau)$ of τ, and the kernel

$$K_\tau(\lambda,\mu) := \frac{\tau(\lambda) - \tau(\mu)^*}{\lambda - \bar{\mu}}, \qquad \lambda,\mu\in\mathbb{C}^+\cap\mathfrak{h}(\tau),$$

has κ negative squares, that is, for all $n\in\mathbb{N}$, $\lambda_1,\ldots,\lambda_n\in\mathbb{C}^+\cap\mathfrak{h}(\tau)$ and all $x_1,\ldots,x_n\in\mathcal{G}$ the selfadjoint matrix

$$\Big((K_\tau(\lambda_i,\lambda_j)x_i,x_j) \Big)_{i,j=1}^n$$

has at most κ negative eigenvalues, and κ is minimal with this property. The functions in the class $N_0(\mathcal{L}(\mathcal{G}))$ are called *Nevanlinna functions*. A function $\tau\in N_0(\mathcal{L}(\mathcal{G}))$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and $\operatorname{Im}\tau(\lambda)$ is nonnegative for all $\lambda\in\mathbb{C}^+$. It is well-known that Nevanlinna functions can equivalently be characterized

by integral representations. More precisely, τ is a $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function if and only if there exist selfadjoint operators $\alpha, \beta \in \mathcal{L}(\mathcal{G})$, $\beta \geq 0$, and a nondecreasing selfadjoint operator function $t \mapsto \Sigma(t) \in \mathcal{L}(\mathcal{G})$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma(t) \in \mathcal{L}(\mathcal{G})$ and

$$\tau(\lambda) = \alpha + \lambda\beta + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t) \tag{3.14}$$

holds for all $\lambda \in \mathfrak{h}(\tau)$. It is worth to note that a Nevanlinna function τ is strict if and only if $\operatorname{Im} \tau(\lambda)$ is uniformly positive for some (and hence for all) $\lambda \in \mathbb{C}^+$.

It was shown in [45,50] that every function $\tau \in N_\kappa(\mathcal{L}(\mathcal{G}))$ can be represented in the form (3.1)-(3.2) with $\mathcal{D} = \mathbb{C}^+$, $\mathcal{O} = \mathfrak{h}(\tau) \cap \mathbb{C}^+$ and $\mathcal{H}$ is a Pontryagin space with negative index κ. For generalized Nevanlinna functions our main result reads as follows, cf. Remark 3.11 and [7, Theorem 3.2] for the special case of $\mathcal{L}(\mathbb{C}^n)$-valued Nevanlinna functions.

Corollary 3.14 *Let $\tau \in N_\kappa(\mathcal{L}(\mathcal{G}))$, $\kappa \in \mathbb{N}_0$, and let $\widehat{\mathcal{G}}$ be as in (3.3). Then there exists a Krein space $\mathcal{K}$ with negative index $\kappa + \dim \widehat{\mathcal{G}}$, a closed symmetric operator S in $\mathcal{K}$ and a generalized boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+ such that the corresponding Weyl function coincides with τ on $\mathfrak{h}(\tau)$. If, in addition, $\dim \mathcal{G} < \infty$, then $\mathcal{K}$ is a Pontryagin space with negative index $\kappa + \dim \widehat{\mathcal{G}}$ and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.*

Next we briefly recall the definitions of definitizable and locally definitizable operator functions introduced by P. Jonas in [46–48]. An $\mathbb{R}$-symmetric piecewise meromorphic $\mathcal{L}(\mathcal{G})$-valued function τ in $\mathbb{C}\backslash\mathbb{R}$ is called *definitizable* if there exists an $\mathbb{R}$-symmetric scalar rational function r such that $r\tau$ is the sum of a Nevanlinna function $G \in N_0(\mathcal{L}(\mathcal{G}))$ and an $\mathcal{L}(\mathcal{G})$-valued rational function P with the poles of P belonging to $\mathfrak{h}(\tau)$,

$$r(\lambda)\tau(\lambda) = G(\lambda) + P(\lambda), \qquad \lambda \in \mathfrak{h}(r\tau).$$

The classes $N_\kappa(\mathcal{L}(\mathcal{G}))$, $\kappa \in \mathbb{N}_0$, are contained in the set of definitizable functions, see [46,47]. Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\mathbb{R}$, such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ are simply connected. A $\mathcal{L}(\mathcal{G})$-valued function τ is said to be *definitizable in* Ω if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, the restriction of τ to Ω' can be written as the sum of a definitizable function τ_d and an $\mathbb{R}$-symmetric $\mathcal{L}(\mathcal{G})$-valued function τ_h holomorphic in Ω', $\tau(\lambda) = \tau_d(\lambda) + \tau_h(\lambda)$ for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$.

Operator representations of the form (3.1)-(3.2) for definitizable and locally definitizable functions can be found in [47,48]. If τ is definitizable in Ω and Ω' is a domain as Ω, $\overline{\Omega'} \subset \Omega$, one can choose $\mathcal{D} = \Omega \cap \mathbb{C}^+$ and $\mathcal{O} = \Omega' \cap \mathfrak{h}(\tau) \cap \mathbb{C}^+$. This yields the following corollary.

Corollary 3.15 *Let τ be a $\mathcal{L}(\mathcal{G})$-valued function definitizable in Ω and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{K}$, a closed symmetric operator S in $\mathcal{K}$ and a generalized boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+ such that the corresponding Weyl function coincides with τ on $\Omega' \cap \mathfrak{h}(\tau) \cap \mathbb{C} \backslash \mathbb{R}$. If, in addition, $\dim \mathcal{G} < \infty$ holds, then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.*

4 Elliptic PDEs with λ-dependent boundary conditions

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n > 1$, with C^∞-boundary $\partial\Omega$ and consider the second order differential expression

$$\ell = -\sum_{j,k=1}^{n} \partial_j \, a_{jk} \, \partial_k \, + a \tag{4.1}$$

on Ω with coefficients $a_{jk}, a \in C^\infty(\overline{\Omega})$ such that $a_{jk} = \overline{a_{kj}}$ for all $j, k = 1, \ldots, n$ and a is real-valued. In addition, it is assumed that the ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C \sum_{k=1}^{n} \xi_k^2, \qquad \xi = (\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n, \; x \in \overline{\Omega},$$

holds for some constant $C > 0$. In this section we investigate the following λ-dependent elliptic boundary value problem: For a given function $g \in L^2(\Omega)$ and $\lambda \in \mathfrak{h}(\tau)$ find $f \in L^2(\Omega)$ such that

$$(\ell - \lambda)f = g \qquad \text{and} \qquad \tau(\lambda)f|_{\partial\Omega} = \frac{\partial f_D}{\partial \nu_\ell}\Big|_{\partial\Omega} \tag{4.2}$$

holds. Here τ is assumed to be a piecewise meromorphic $\mathcal{L}(L^2(\partial\Omega))$-valued function and f_D denotes the component of f in the domain of the Dirichlet operator. The precise formulation of the problem will be given in Section 4.2.

4.1 *Preliminaries and ordinary boundary triples for elliptic PDEs*

The Sobolev space of kth order on Ω is denoted by $H^k(\Omega)$ and the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$. Sobolev spaces on the boundary are denoted by $H^s(\partial\Omega)$, $s \in \mathbb{R}$. Let $(\cdot, \cdot)_{-1/2 \times 1/2}$ and $(\cdot, \cdot)_{-3/2 \times 3/2}$ be the extensions of the $L^2(\partial\Omega)$ inner product to $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$, respectively, and let $\iota_\pm : H^{\pm 1/2}(\partial\Omega) \to L^2(\partial\Omega)$ be isomorphisms such that $(x, y)_{-1/2 \times 1/2} = (\iota_- x, \iota_+ y)$ holds for all $x \in H^{-1/2}(\partial\Omega)$ and $y \in H^{1/2}(\partial\Omega)$.

Recall that the *Dirichlet operator*

$$T_D f_D = \ell f_D, \qquad \operatorname{dom} T_D = H^2(\Omega) \cap H_0^1(\Omega),$$

associated to the elliptic differential expression ℓ in (4.1) is selfadjoint in $L^2(\Omega)$ and the resolvent of T_D is compact, cf. [34, VI. Theorem 1.4] and [52,55,61]. Furthermore, the *minimal operator*

$$T f = \ell f, \qquad \operatorname{dom} T = H_0^2(\Omega),$$

is a densely defined closed symmetric operator in $L^2(\Omega)$ and the adjoint operator $T^* f = \ell f$ is defined on the maximal domain

$$\operatorname{dom} T^* = \mathcal{D}_{\max} = \left\{ f \in L^2(\Omega) : \ell f \in L^2(\Omega) \right\}.$$

Let us fix some $\eta \in \mathbb{R} \cap \rho(T_D)$. Then for each function $f \in \mathcal{D}_{\max}$ there is a unique decomposition $f = f_D + f_\eta$, where $f_D \in \operatorname{dom} T_D$ and $f_\eta \in \mathcal{N}_{\eta, T^*} = \ker(T^* - \eta)$. In fact, as $T_D - \eta$ is surjective for a given $f \in \mathcal{D}_{\max}$ there exists $f_D \in \operatorname{dom} T_D$ such that $(T^* - \eta) f = (T_D - \eta) f_D$ holds. It follows that $f_\eta := f - f_D \in \mathcal{N}_{\eta, T^*}$ and hence $f = f_D + f_\eta$ is the desired decomposition. The uniqueness follows from $\ker(T_D - \eta) = \{0\}$.

Let $\mathfrak{n} = (\mathfrak{n}_1, \ldots, \mathfrak{n}_n)^\top$ be the unit outward normal of Ω. It is well-known that the map

$$C^\infty(\overline{\Omega}) \ni f \mapsto \left\{ f|_{\partial\Omega}, \frac{\partial f}{\partial\nu_\ell}\Big|_{\partial\Omega} \right\}, \quad \text{where} \quad \frac{\partial f}{\partial\nu_\ell} := \sum_{j,k=1}^n a_{jk} \mathfrak{n}_j \partial_k f,$$

can be extended to a linear operator from $\mathcal{D}_{\max}$ into $H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$ and that for $f \in \mathcal{D}_{\max}$ and $g \in H^2(\Omega)$ Green's identity

$$(T^* f, g) - (f, T^* g) = \left(f|_{\partial\Omega}, \frac{\partial g}{\partial\nu_\ell}\Big|_{\partial\Omega} \right)_{-\frac{1}{2} \times \frac{1}{2}} - \left(\frac{\partial f}{\partial\nu_\ell}\Big|_{\partial\Omega}, g|_{\partial\Omega} \right)_{-\frac{3}{2} \times \frac{3}{2}} \tag{4.3}$$

holds, see [41,55,61].

The λ-dependent boundary condition in (4.2) will be rewritten with the help of an ordinary boundary triple for the maximal realization of ℓ in $L^2(\Omega)$. The ordinary boundary triple in the next proposition can also be found in [16,39,43,44]. For the convenience of the reader we include a short proof based on the general observations in [41,42].

Proposition 4.1 *The triple* $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$, *where*

$$\Upsilon_0 \hat{f} := \iota_- f_\eta|_{\partial\Omega} \qquad and \qquad \Upsilon_1 \hat{f} := -\iota_+ \frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega},$$

$\hat{f} = \{f, T^*f\}$, $f = f_D + f_\eta \in \mathcal{D}_{\max}$, is an ordinary boundary triple for the maximal operator $T^*f = \ell f$, $\mathrm{dom}\, T^* = \mathcal{D}_{\max}$, such that $T_D = \ker \Upsilon_0$. The corresponding γ-field and Weyl function are given by

$$\gamma(\lambda)y = (I + (\lambda - \eta)(T_D - \lambda)^{-1})f_\eta(y), \qquad \lambda \in \rho(T_D),$$

and

$$M(\lambda)y = (\eta - \lambda)\iota_+ \frac{\partial(T_D - \lambda)^{-1}f_\eta(y)}{\partial\nu_\ell}\Big|_{\partial\Omega}, \qquad \lambda \in \rho(T_D),$$

respectively, where $f_\eta(y)$ is the unique function in $\ker(T^* - \eta)$ satisfying $\iota_- f_\eta(y)|_{\partial\Omega} = y$.

Proof. Let $f, g \in \mathcal{D}_{\max}$ be decomposed in the form $f = f_D + f_\eta$ and $g = g_D + g_\eta$. As T_D is selfadjoint and $\eta \in \mathbb{R}$ we find

$$(T^*f, g) - (f, T^*g) = (T_D f_D, g_\eta) - (f_D, T^*g_\eta) + (T^*f_\eta, g_D) - (f_\eta, T_D g_D)$$

and then $f_D|_{\partial\Omega} = g_D|_{\partial\Omega} = 0$ together with Green's identity (4.3) implies

$$(T^*f, g) - (f, T^*g) = -\left(\frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega}, g_\eta|_{\partial\Omega}\right)_{\frac{1}{2} \times -\frac{1}{2}} + \left(f_\eta|_{\partial\Omega}, \frac{\partial g_D}{\partial\nu_\ell}\Big|_{\partial\Omega}\right)_{-\frac{1}{2} \times \frac{1}{2}}$$
$$= (\Upsilon_1\hat{f}, \Upsilon_0\hat{g}) - (\Upsilon_0\hat{f}, \Upsilon_1\hat{g}).$$

Hence (2.4) in Definition 2.2 holds, cf. (2.3). Furthermore, by the classical trace theorem the map $H^2(\Omega) \cap H_0^1(\Omega) \ni f_D \mapsto \frac{\partial f_D}{\partial\nu_\ell}|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ is onto and the same holds for the map $\ker(T^* - \eta) \ni f_\eta \mapsto f_\eta|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$, which is an isomorphism according to [42, Theorem 2.1]. Hence $(\Upsilon_0, \Upsilon_1)^\top$ maps T^* onto $L^2(\partial\Omega) \times L^2(\partial\Omega)$ and therefore $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ is an ordinary boundary triple for T^* with $T_D = \ker \Upsilon_0$.

It remains to show that the corresponding γ-field and Weyl function have the asserted form. For this let $y \in L^2(\partial\Omega)$, choose the unique function $f_\eta(y)$ in $\ker(T^* - \eta)$ such that $y = \iota_- f_\eta(y)|_{\partial\Omega}$ holds and set

$$f_\lambda := (\lambda - \eta)(T_D - \lambda)^{-1}f_\eta(y) + f_\eta(y) \tag{4.4}$$

for $\lambda \in \rho(T_D)$. It is easy to see that $(T^* - \lambda)f_\lambda = 0$ holds and since $(T_D - \lambda)^{-1}f_\eta(y) \in \mathrm{dom}\, T_D$ and $f_\eta(y) \in \ker(T^* - \eta)$ we obtain

$$\Gamma_0\hat{f}_\lambda = \Gamma_0\{f_\lambda, \lambda f_\lambda\} = \iota_- f_\eta(y)|_{\partial\Omega} = y,$$

i.e. $\gamma(\lambda)y = f_\lambda = (I + (\lambda - \eta)(T_D - \lambda)^{-1})f_\eta(y)$. Finally, by the definition of the Weyl function and (4.4) we have

$$M(\lambda)y = \Gamma_1\hat{f}_\lambda = (\eta - \lambda)\iota_+ \frac{\partial(T_D - \lambda)^{-1}f_\eta(y)}{\partial\nu_\ell}\Big|_{\partial\Omega}.$$

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$\square$

4.2 Elliptic boundary value problems with eigenvalue depending boundary conditions

Let $\mathcal{D} \subset \mathbb{C}^+$ be a simply connected open set and let τ be a piecewise mero-morphic $\mathcal{L}(L^2(\partial\Omega))$-valued function on $\mathcal{D} \cup \mathcal{D}^*$ which admits a representation of the form (3.1)-(3.2) via the resolvent of some selfadjoint relation on an open subset $\mathcal{O} \cup \mathcal{O}^*$ of $\mathcal{D} \cup \mathcal{D}^*$. Note that τ is holomorphic on $\mathcal{O} \cup \mathcal{O}^*$. We study the following λ-dependent elliptic boundary value problem: For a given function $g \in L^2(\Omega)$ and $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ find $f \in \mathcal{D}_{\max}$ such that

$$(\ell - \lambda)f = g \qquad \text{and} \qquad \tau(\lambda)\iota_- f|_{\partial\Omega} = \iota_+ \frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega} \qquad (4.5)$$

holds. According to Theorem 3.1 there exists a Krein space $\mathcal{K}$, a closed sym-metric operator S in $\mathcal{K}$ and a generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ for $S^+ = \overline{\operatorname{dom}\Gamma}$ such that the corresponding Weyl function coincides with τ on $\mathcal{O} \cup \mathcal{O}^*$. In particular, the set $\mathcal{O} \cup \mathcal{O}^*$ is a subset of the resolvent set of the selfadjoint relation $S_0 = \ker\Gamma_0$ in $\mathcal{K}$. With the help of the operator S, the generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ and the ordinary boundary triple $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ for the elliptic operator from Proposition 4.1 we construct a linearization of the boundary value problem (4.5) in the next theorem.

Theorem 4.2 *Let $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ be the ordinary boundary triple for the maximal differential operator T^* associated to ℓ from Proposition 4.1 with cor-responding γ-field γ and Weyl function M, and assume that $(M(\mu) + \tau(\mu))^{-1} \in \mathcal{L}(L^2(\partial\Omega))$ holds for some $\mu \in \mathcal{O}$.*

Then the operator

$$\widetilde{A}\begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} \ell f \\ k' \end{pmatrix},$$

$$\operatorname{dom}\widetilde{A} = \left\{ \begin{pmatrix} f \\ k \end{pmatrix} \in \mathcal{D}_{\max} \times \mathcal{K} : \begin{pmatrix} \Upsilon_0\hat{f} - \Gamma_0\hat{k} = 0 \\ \Upsilon_1\hat{f} + \Gamma_1\hat{k} = 0 \end{pmatrix} \text{ for } \begin{matrix} \hat{f} = \{f, T^*f\}, \\ \hat{k} = \{k, k'\} \in \operatorname{dom}\Gamma \end{matrix} \right\},$$

is a selfadjoint extension of the minimal differential operator T in the Krein space $L^2(\Omega) \times \mathcal{K}$, the set

$$\mathcal{U} := \left\{ \lambda \in \mathcal{O} \cup \mathcal{O}^* : (M(\lambda) + \tau(\lambda))^{-1} \in \mathcal{L}(L^2(\partial\Omega)) \right\}$$

is a subset of $\rho(\widetilde{A}) \cap \rho(T_D) \cap \mathfrak{h}(\tau)$ and for every $\lambda \in \mathcal{U}$ the unique solution of the boundary value problem (4.5) is given by

$$f = P_{L^2}\big(\widetilde{A} - \lambda\big)^{-1} \restriction_{L^2} g = (T_D - \lambda)^{-1}g - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\bar{\lambda})^* g.$$
$$(4.6)$$

Proof. The proof of Theorem 4.2 is divided into two parts. In the first part it will be shown that $\widetilde{A}$ is a selfadjoint operator in the Krein space $L^2(\Omega) \times \mathcal{K}$ and in the second part it is verified that the unique solution of (4.5) is given by the function f in the theorem.

Step 1. Let us check first that $\widetilde{A}$ is an operator. In fact, if

$$\begin{pmatrix} f \\ k \end{pmatrix} \in \operatorname{dom} \widetilde{A} \quad \text{and} \quad f = k = 0,$$

then obviously $T^* f = 0$ and hence $\hat{f} = 0$. This yields $\Upsilon_0 \hat{f} = 0 = \Gamma_0 \hat{k}$ and $\Upsilon_1 \hat{f} = 0 = \Gamma_1 \hat{k}$. Therefore $\hat{k} = \{0, k'\} \in S$ and as S is an operator $k' = 0$ follows. The fact that $\widetilde{A}$ is symmetric in the Krein space $L^2(\partial\Omega) \times \mathcal{K}$ follows from the special form of $\operatorname{dom} \widetilde{A}$ and the identities (2.4) and (2.2) for the ordinary boundary triple $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ and the generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$. Indeed, for $\begin{pmatrix} f \\ k \end{pmatrix}, \begin{pmatrix} g \\ h \end{pmatrix} \in \operatorname{dom} \widetilde{A}$ we have $\Upsilon_0 \hat{f} = \Gamma_0 \hat{k}$, $\Upsilon_0 \hat{g} = \Gamma_0 \hat{h}$, $\Upsilon_1 \hat{f} = -\Gamma_1 \hat{k}$, $\Upsilon_1 \hat{g} = -\Gamma_1 \hat{h}$ and hence

$$\left[\widetilde{A}\begin{pmatrix} f \\ k \end{pmatrix}, \begin{pmatrix} g \\ h \end{pmatrix}\right] - \left[\begin{pmatrix} f \\ k \end{pmatrix}, \widetilde{A}\begin{pmatrix} g \\ h \end{pmatrix}\right]$$
$$= (\Upsilon_1 \hat{f}, \Upsilon_0 \hat{g}) - (\Upsilon_0 \hat{f}, \Upsilon_1 \hat{g}) + (\Gamma_1 \hat{k}, \Gamma_0 \hat{h}) - (\Gamma_0 \hat{k}, \Gamma_1 \hat{h}) = 0.$$

In order to prove that $\widetilde{A}$ is selfadjoint in $L^2(\Omega) \times \mathcal{K}$ it is sufficient to verify that the operators $\widetilde{A} - \mu$ and $\widetilde{A} - \bar{\mu}$ are surjective for some $\mu \in \mathcal{U}$. We show only $\operatorname{ran}(\widetilde{A} - \mu) = L^2(\Omega) \times \mathcal{K}$, the same reasoning applies for $\widetilde{A} - \bar{\mu}$. By assumption $\mu \in \mathcal{O}$ is such that $(M(\mu) + \tau(\mu))^{-1} \in \mathcal{L}(L^2(\partial\Omega))$ and moreover, μ belongs to $\rho(T_D) \cap \rho(S_0)$ as $\sigma(T_D) \subset \mathbb{R}$ and τ is holomorphic on $\mathcal{O} \cup \mathcal{O}^*$. Let $g \in L^2(\Omega)$, $h \in \mathcal{K}$ and define $\hat{f} = \{f, \mu f + g\}$ and $\hat{k} = \{k, \mu k + h\}$ by

$$f := (T_D - \mu)^{-1}g - \gamma(\mu)\big(M(\mu) + \tau(\mu)\big)^{-1}(\gamma(\bar{\mu})^* g + \gamma_\tau(\bar{\mu})^+ h) \in L^2(\Omega) \quad (4.7)$$

and

$$k := (S_0 - \mu)^{-1}h - \gamma_\tau(\mu)\big(M(\mu) + \tau(\mu)\big)^{-1}(\gamma(\bar{\mu})^* g + \gamma_\tau(\bar{\mu})^+ h) \in \mathcal{K}.$$

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Here γ is the γ-field of the ordinary boundary triple $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ and γ_τ is the γ-field corresponding to the generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$. Note that $\hat{f} \in T^*$ since $\gamma(\mu)(M(\mu) + \tau(\mu))^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h) \in \mathcal{N}_{\mu,T^*}$ and

$$\left\{(T_D - \mu)^{-1}g, (I + \mu(T_D - \mu)^{-1})g\right\} \in T_D. \tag{4.8}$$

An analogous argument shows $\hat{k} \in \mathrm{dom}\,\Gamma \subset S^+$. We claim that $\{\hat{f}, \hat{k}\}$ satisfies the boundary conditions $\Upsilon_0\hat{f} = \Gamma_0\hat{k}$ and $\Upsilon_1\hat{f} = -\Gamma_1\hat{k}$, so that $\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right)$ belongs to $\mathrm{dom}\,\widetilde{A}$. In fact, as $T_D = \ker\Upsilon_0$ it follows from (4.7), (4.8) and (2.7) that

$$\Upsilon_0\hat{f} = -\Big(M(\mu) + \tau(\mu)\Big)^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h),$$
$$\Upsilon_1\hat{f} = \gamma(\bar{\mu})^*g - M(\mu)\Big(M(\mu) + \tau(\mu)\Big)^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h),$$

and analogously,

$$\Gamma_0\hat{k} = -\Big(M(\mu) + \tau(\mu)\Big)^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h),$$
$$\Gamma_1\hat{k} = \gamma_\tau(\bar{\mu})^+h - \tau(\mu)\Big(M(\mu) + \tau(\mu)\Big)^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h).$$

Hence we have $\Upsilon_0\hat{f} = \Gamma_0\hat{k}$ and

$$\Upsilon_1\hat{f} = \gamma(\bar{\mu})^*g - (\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h)$$
$$+ \tau(\mu)\Big(M(\mu) + \tau(\mu)\Big)^{-1}(\gamma(\bar{\mu})^*g + \gamma_\tau(\bar{\mu})^+h) = -\Gamma_1\hat{k},$$

i.e., $\{\hat{f}, \hat{k}\} \in \widetilde{A}$ and it follows that

$$(\widetilde{A} - \mu)\begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} \mu f + g \\ \mu k + h \end{pmatrix} - \mu\begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

holds. As the elements $g \in L^2(\Omega)$ and $h \in \mathcal{K}$ were chosen arbitrary we conclude $\mathrm{ran}\,(\widetilde{A} - \mu) = L^2(\Omega) \times \mathcal{K}$.

Step 2. Next it will be verified that for $\lambda \in \mathcal{U}$ the unique solution of (4.5) is given by

$$f = P_{L^2}(\widetilde{A} - \lambda)^{-1}\begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{4.9}$$

We note first that the set $\mathcal{U}$ is a subset of $\rho(\widetilde{A})$. In fact, for every $\lambda \in \mathcal{U}$ the same argument as in Step 1 of the proof shows that $\widetilde{A} - \lambda$ and $\widetilde{A} - \bar{\lambda}$ are surjective and hence $\ker(\widetilde{A} - \bar{\lambda}) = \{0\} = \ker(\widetilde{A} - \lambda)$, i.e. $\lambda, \bar{\lambda} \in \rho(\widetilde{A})$. For f in

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(4.9) we have

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix} = \begin{pmatrix} f \\ k \end{pmatrix}, \quad \text{where} \quad k := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix},$$

and from $\widetilde{A} \subset T^* \times \operatorname{dom} \Gamma$ and

$$\widetilde{A} \begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} g + \lambda f \\ \lambda k \end{pmatrix}$$

we conclude that $T^* f = g + \lambda f$ and $k \in \mathcal{N}_{\lambda, S^+} = \ker(S^+ - \lambda)$ holds. As τ is the Weyl function corresponding to the generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ and S^+ it follows that $\hat{k} = \{k, \lambda k\} \in \widehat{\mathcal{N}}_{\lambda, S^+} \cap \operatorname{dom} \Gamma$ satisfies $\tau(\lambda)\Gamma_0 \hat{k} = \Gamma_1 \hat{k}$. Therefore, making use of the specific form of $\operatorname{dom} \widetilde{A}$ and the ordinary boundary triple in Proposition 4.1 we obtain

$$\tau(\lambda)\iota_- f|_{\partial\Omega} = \tau(\lambda)\Upsilon_0 \hat{f} = \tau(\lambda)\Gamma_0 \hat{k} = \Gamma_1 \hat{k} = -\Upsilon_1 \hat{f} = \iota_+ \frac{\partial f_D}{\partial \nu_\ell}\Big|_{\partial\Omega}.$$

Hence (4.9) is a solution of the boundary value problem (4.5). The fact that the compression of the resolvent of $\widetilde{A}$ onto $L^2(\Omega)$ has the asserted form follows from Step 1 of the proof by setting $\hat{f} = \{f, \lambda f + g\}$ and $\hat{k} = \{k, \lambda k\}$. In this case (4.7) reduces to

$$f = (T_D - \lambda)^{-1} g - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1} \gamma(\bar{\lambda})^* g$$

and coincides with $P_{L^2}(\widetilde{A} - \lambda)^{-1}|_{L^2} g$ by (4.9).

Finally, we check that for $\lambda \in \mathcal{U}$ the solution f of (4.5) in (4.9) is unique. Assume that $f_1 \in \mathcal{D}_{\max}$ is also a solution of (4.5). Then $f - f_1 \in \mathcal{N}_{\lambda, T^*}$ and as M is the Weyl function of $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ we have

$$M(\lambda)\Upsilon_0(\hat{f} - \hat{f}_1) = \Upsilon_1(\hat{f} - \hat{f}_1), \qquad \hat{f} = \{f, T^* f\}, \ \hat{f}_1 = \{f_1, T^* f_1\}.$$

On the other hand, since f and f_1 both satisfy the boundary condition in (4.5) it is clear that $\tau(\lambda)\Upsilon_0(\hat{f} - \hat{f}_1) = -\Upsilon_1(\hat{f} - \hat{f}_1)$ holds and this implies

$$(M(\lambda) + \tau(\lambda))\Upsilon_0(\hat{f} - \hat{f}_1) = 0.$$

Since $\lambda \in \mathcal{U}$ we conclude $\Upsilon_0(\hat{f} - \hat{f}_1) = 0$, i.e., $\hat{f} - \hat{f}_1 \in T_D = \ker \Upsilon_0$. ¿From $\lambda \in \rho(T_D)$ we then obtain $\hat{f} = \hat{f}_1$ and hence the solution f in (4.9) is unique. This completes the proof of Theorem 4.2. $\qquad\square$

Remark 4.3 *The method applied in the proof of Theorem 4.2 differs from the coupling techniques in [6, Theorem 4.3] and [21, § 5.2], where only ordinary boundary triples were used. The principal difficulty here is to ensure selfadjointness of $\widetilde{A}$, a fact that follows immediately via the abstract boundary condition in [6,21].*

In the special case that τ in (4.5) is a (in general non-strict) $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function the condition $0 \in \rho(M(\mu) + \tau(\mu))$ in Theorem 4.2 is automatically satisfied for every nonreal μ, because the imaginary part of the Weyl function M of the ordinary boundary triple $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ for T^* is uniformly positive (uniformly negative) for $\lambda \in \mathbb{C}^+$ ($\lambda \in \mathbb{C}^-$, respectively). This proves the following corollary.

Corollary 4.4 *Assume that the function τ in the boundary condition in (4.5) belongs to the class $N_0(\mathcal{L}(L^2(\partial\Omega)))$ and let $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ be the ordinary boundary triple for T^* from Proposition 4.1 with corresponding γ-field γ and Weyl function M. Then the operator $\widetilde{A}$ in Theorem 4.2 is a selfadjoint extension of T in $L^2(\Omega) \times \mathcal{K}$ and for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the unique solution of the boundary value problem (4.5) is given by (4.6).*

Observe that for $g = 0$ in (4.5) and $\lambda \in \mathcal{U}$ the unique solution of the homogeneous boundary value problem

$$(\ell - \lambda)f = 0 \qquad \text{and} \qquad \tau(\lambda)\iota_- f|_{\partial\Omega} = \iota_+ \frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega} \qquad (4.10)$$

is given by $f = P_{L^2}(\widetilde{A} - \lambda)^{-1}|_{L^2}0 = 0$, cf. Theorem 4.2. The following proposition shows, roughly speaking, that the nontrivial solutions of the homogeneous problem (4.10) are given by the eigenvalues and eigenvectors of the operator $\widetilde{A}$.

Proposition 4.5 *Let the assumptions be as in Theorem 4.2 and let $\widetilde{A}$ be the selfadjoint operator in $L^2(\Omega) \times \mathcal{K}$ from the same theorem. Then the following holds.*

(i) *If $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ is an eigenvalue of $\widetilde{A}$ and $\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) \in \ker(\widetilde{A} - \lambda)$ is a corresponding eigenvector, then $f \in \mathcal{D}_{\max}$ is a nontrivial solution of (4.10).*
(ii) *If $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ and $f \in \mathcal{D}_{\max}$ is a nontrivial solution of (4.10), then λ is an eigenvalue of $\widetilde{A}$ and $\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) \in \ker(\widetilde{A} - \lambda)$ for some $k \in \mathcal{K}$.*

Proof. (i) Suppose that $\left(\begin{smallmatrix} f \\ k \end{smallmatrix}\right) \in \operatorname{dom}\widetilde{A}$ is an eigenvector corresponding to the eigenvalue $\lambda \in \mathcal{O} \cup \mathcal{O}^*$ of $\widetilde{A}$. Then we have $\ell f = \lambda f$ and since $\hat{k} = \{k, \lambda k\} \in \widehat{\mathcal{N}}_{\lambda,S^+} \cap \operatorname{dom}\Gamma$ it follows from the specific form of $\operatorname{dom}\widetilde{A}$ and the fact that τ

is the Weyl function of the generalized boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ that

$$\tau(\lambda)\iota_- f|_{\partial\Omega} = \tau(\lambda)\Upsilon_0\hat{f} = \tau(\lambda)\Gamma_0\hat{k} = \Gamma_1\hat{k} = -\Upsilon_1\hat{f} = \iota_+ \frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega}$$

holds. Therefore $f \in \mathcal{D}_{\max}$ is a solution of the homogeneous boundary value problem (4.10). It remains to show $f \neq 0$. Assume the contrary. Then $\hat{f} = \{f, T^*f\} = 0$ and it follows from $0 = \Upsilon_0\hat{f} = \Gamma_0\hat{k}$ that $\hat{k} = \{k, \lambda k\}$ belongs to $S_0 = \ker\Gamma_0$. Since $(\mathcal{O} \cup \mathcal{O}^*) \subset \rho(S_0)$ (cf. the beginning of Section 4.2, Theorem 3.1 and Remark 3.12) we conclude $k = 0$, a contradiction to $\binom{f}{k}$ being an eigenvector.

(ii) Let $f \in \mathcal{D}_{\max}$ be a nontrivial solution of (4.10). Then the boundary condition $\tau(\lambda)\Upsilon_0\hat{f} = -\Upsilon_1\hat{f}$, $\hat{f} = \{f, \lambda f\}$, is fulfilled and as $\lambda \in (\mathcal{O} \cup \mathcal{O}^*) \subset \rho(S_0)$, $S_0 = \ker\Gamma_0$, we can decompose $\operatorname{dom}\Gamma$ in the form $\operatorname{dom}\Gamma = S_0 \,\widehat{+}\, \widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma}$, cf. (2.5). Since $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ is a generalized boundary triple for $S^+ = \overline{\operatorname{dom}\Gamma}$ the map $\Gamma_0 : \operatorname{dom}\Gamma \to L^2(\partial\Omega)$ is onto and hence there exists $\hat{k} = \{k, \lambda k\} \in \widehat{\mathcal{N}}_{\lambda, \operatorname{dom}\Gamma} = \widehat{\mathcal{N}}_{\lambda, S^+} \cap \operatorname{dom}\Gamma$ such that $\Gamma_0\hat{k} = \iota_- f|_{\partial\Omega}$ holds. Hence we have $\Gamma_0\hat{k} = \Upsilon_0\hat{f}$, $\tau(\lambda)\Gamma_0\hat{k} = \Gamma_1\hat{k}$, and therefore

$$\Upsilon_1\hat{f} = -\tau(\lambda)\Upsilon_0\hat{f} = -\tau(\lambda)\Gamma_0\hat{k} = -\Gamma_1\hat{k},$$

i.e., $\binom{f}{k} \in \operatorname{dom}\widetilde{A}$ is an eigenvector corresponding to the eigenvalue λ of $\widetilde{A}$. $\square$

4.3 An example: A rational Nevanlinna function τ

Let $\alpha_i, \beta_i \in \mathcal{L}(L^2(\partial\Omega))$, $i = 1, \ldots, m$, be bounded selfadjoint operators in $L^2(\partial\Omega)$ and assume that $\beta_i \geq 0$ holds for all $i = 1, \ldots, m$ and $0 \in \rho(\beta_1)$. We consider the boundary value problem (4.5) with a function τ of the form

$$\tau(\lambda) = \alpha_1 + \lambda\beta_1 + \sum_{i=2}^m \beta_i^{1/2}(\alpha_i - \lambda)^{-1}\beta_i^{1/2}, \qquad \lambda \in \bigcap_{i=2}^m \rho(\alpha_i). \tag{4.11}$$

Observe that τ is an $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function with the property $0 \in \rho(\operatorname{Im}\tau(\lambda))$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and hence τ is (uniformly) strict. The next theorem, in which a solution operator $\widetilde{A}$ of the boundary value problem (4.5), (4.11) is explicitly constructed, is essentially a consequence of Theorem 4.2 and an explicit realization of the function (4.11) as the Weyl function of an ordinary boundary triple in the product space

$$L^2(\partial\Omega)^m = L^2(\partial\Omega) \times \ldots \times L^2(\partial\Omega) \qquad (m \text{ copies}).$$

A special case of Theorem 4.6 below was announced in [5]. For ordinary second order differential operators in $L^2(I)$, $I \subset \mathbb{R}$, and scalar rational Nevanlinna

functions in the boundary condition a solution operator of similar form in $L^2(I) \oplus \mathbb{C}^m$ as in the next result can be found in [12], see also [13,14].

Theorem 4.6 *Let τ be a rational $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function of the form (4.11) and let γ and M be as in Proposition 4.1. Then*

$$
\widetilde{A} \begin{pmatrix} f \\ k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix} = \begin{pmatrix} \ell f \\ k_1' \\ \beta_2^{1/2}\beta_1^{-1/2}k_1 + \alpha_2 k_2 \\ \vdots \\ \beta_m^{1/2}\beta_1^{-1/2}k_1 + \alpha_m k_m \end{pmatrix}, \qquad \begin{array}{l} f = f_D + f_\eta \in \mathcal{D}_{\max}, \\[2ex] k_1, \dots, k_m, k_1' \in L^2(\partial\Omega), \end{array}
$$

$$
\operatorname{dom} \widetilde{A} = \left\{ \begin{pmatrix} f \\ k_1 \\ \vdots \\ k_m \end{pmatrix} : \begin{array}{l} \iota_- f_\eta|_{\partial\Omega} = \beta_1^{-1/2}k_1 \\[1ex] \iota_+ \frac{\partial f_D}{\partial \nu_\ell}\big|_{\partial\Omega} = \alpha_1 \beta_1^{-1/2}k_1 + \beta_1^{1/2}k_1' - \sum_{i=2}^m \beta_i^{1/2}k_i \end{array} \right\},
$$

is a selfadjoint operator in the Hilbert space $L^2(\Omega) \times L^2(\partial\Omega)^m$ and for every λ in $\rho(\widetilde{A}) \cap \rho(T_D) \cap \mathfrak{h}(\tau)$ the unique solution of the boundary value problem (4.5) is given by (4.6).

Proof. The statements in Theorem 4.6 will follow by applying Theorem 4.2 to an explicit realization of the function τ in (4.11) as the Weyl function of an ordinary boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ for some closed symmetric operator in $L^2(\partial\Omega)^m$.

Denote the functions $k \in L^2(\partial\Omega)^m$ in the form $k = (k_1, \dots, k_m)^\top$, $k_i \in L^2(\partial\Omega)$, $i = 1, \dots, m$, and consider the non-densely defined operator

$$
S(k_1, \dots, k_m)^\top = \left(\sum_{i=2}^m \beta_1^{-1/2}\beta_i^{1/2}k_i, \alpha_2 k_2, \dots, \alpha_m k_m \right)^\top,
$$
$$
\operatorname{dom} S = \left\{ (k_1, \dots, k_m)^\top \in L^2(\partial\Omega)^m : k_1 = 0 \right\},
$$

in $L^2(\partial\Omega)^m$. The scalar products in $L^2(\partial\Omega)$ and $L^2(\partial\Omega)^m$ will both be denoted by $(\cdot, \cdot)$. We hope that this does not lead to any confusion. As $\alpha_i = \alpha_i^*$, $i = 1, \dots, m$, it follows that (Sk, k) is real for all $k \in \operatorname{dom} S$ and hence S is

symmetric. We claim that the adjoint of S is given by

$$S^* = \left\{ \left\{ \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix}, \begin{pmatrix} k_1' \\ \beta_2^{1/2}\beta_1^{-1/2}k_1 + \alpha_2 k_2 \\ \vdots \\ \beta_m^{1/2}\beta_1^{-1/2}k_1 + \alpha_m k_m \end{pmatrix} \right\} : k_1, \ldots, k_m, k_1' \in L^2(\partial\Omega) \right\}.$$

$$(4.12)$$

In fact, for $l \in \operatorname{dom} S$ and an element $\hat{k} = \{k, k'\}$ belonging to the right hand side of (4.12) we compute

$$(Sl, k) - (l, k') = \sum_{i=2}^{m} (\beta_1^{-1/2}\beta_i^{1/2}l_i, k_1) + \sum_{i=2}^{m} (\alpha_i l_i, k_i)$$
$$- \sum_{i=2}^{m} (l_i, \beta_i^{1/2}\beta_1^{-1/2}k_1 + \alpha_i k_i) = 0$$

and hence the right hand side of (4.12) is a subset of S^*. Furthermore, for each $l \in \operatorname{dom} S$ and $\hat{k} = \{k, k'\} \in S^*$ we have

$$0 = (Sl, k) - (l, k') = \sum_{i=2}^{m} (\beta_1^{-1/2}\beta_i^{1/2}l_i, k_1) + \sum_{i=2}^{m} (\alpha_i l_i, k_i) - \sum_{i=2}^{m} (l_i, k_i').$$

Therefore, by inserting $l = (0, \ldots, 0, l_j, 0, \ldots, 0)^\top$, $l_j \in L^2(\partial\Omega)$, $j = 2, \ldots, m$, we obtain

$$k_j' = \beta_j^{1/2}\beta_1^{-1/2}k_1 + \alpha_j k_j, \qquad j = 2, \ldots, m,$$

i.e., S^* is a subset of the right hand side of (4.12) and hence S^* is given by (4.12).

Let us check that $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 \hat{k} = \beta_1^{-1/2} k_1 \quad \text{and} \quad \Gamma_1 \hat{k} = \alpha_1 \beta_1^{-1/2} k_1 + \beta_1^{1/2} k_1' - \sum_{i=2}^{m} \beta_i^{1/2} k_i, \quad \hat{k} \in S^*,$$

is an ordinary boundary triple for S^* with τ in (4.11) as corresponding Weyl function. Since for an element $\hat{k} = \{k, k'\} \in S^*$ the entries k_1 and k_1' are arbitrary elements in $L^2(\partial\Omega)$ it follows immediately from $0 \in \rho(\beta_1)$ that the mapping $(\Gamma_0, \Gamma_1)^\top : S^* \to L^2(\partial\Omega) \times L^2(\partial\Omega)$ is onto. Next we verify the identity

(2.3). For $\hat{l} = \{l, l'\}$ and $\hat{k} = \{k, k'\} \in S^*$ a straightforward computation shows

$$(l', k) - (l, k') = (\beta_1^{1/2} l_1', \beta_1^{-1/2} k_1) - (\beta_1^{-1/2} l_1, \beta_1^{1/2} k_1')$$
$$+ \sum_{i=2}^{m} (\beta_i^{1/2} \beta_1^{-1/2} l_1 + \alpha_i l_i, k_i) - \sum_{i=2}^{m} (l_i, \beta_i^{1/2} \beta_1^{-1/2} k_1 + \alpha_i k_i)$$
$$= \left(\beta_1^{1/2} l_1' - \sum_{i=2}^{m} \beta_i^{1/2} l_i \,,\, \beta_1^{-1/2} k_1 \right) - \left(\beta_1^{-1/2} l_1 \,,\, \beta_1^{1/2} k_1' - \sum_{i=2}^{m} \beta_i^{1/2} k_i \right)$$
$$= (\Gamma_1 \hat{l}, \Gamma_0 \hat{k}) - (\Gamma_0 \hat{l}, \Gamma_1 \hat{k}),$$

where we have used $\alpha_1 = \alpha_1^*$ in the last step. Observe that the selfadjoint relation $S_0 = \ker \Gamma_0$ is given by

$$S_0 = \left\{ \left\{ (0, k_2 \ldots, k_m)^\top, (k_1', \alpha_2 k_2, \ldots, \alpha_m k_m)^\top \right\} : k_1', k_2, \ldots, k_m \in L^2(\partial\Omega) \right\}$$

and that for $\lambda \in \rho(S_0) = \bigcap_{i=2}^{m} \rho(\alpha_i)$ the resolvent of S_0 is a diagonal block operator matrix in $L^2(\partial\Omega)^m$ with entries $0, (\alpha_2 - \lambda)^{-1}, \ldots, (\alpha_m - \lambda)^{-1}$ on the diagonal. Let now $\hat{k} = \{k, \lambda k\} \in \widehat{\mathcal{N}}_{\lambda, S^*}$ and $\lambda \in \rho(S_0)$. Then we have

$$k_1' = \lambda k_1 \qquad \text{and} \qquad \beta_i^{1/2} \beta_1^{-1/2} k_1 = (\lambda - \alpha_i) k_i, \quad i = 2, \ldots, m,$$

and this implies

$$\left(\alpha_1 + \lambda \beta_1 + \sum_{i=2}^{m} \beta_i^{1/2} (\alpha_i - \lambda)^{-1} \beta_i^{1/2} \right) \Gamma_0 \hat{k}$$
$$= \alpha_1 \beta_1^{-1/2} k_1 + \lambda \beta_1^{1/2} k_1 + \sum_{i=2}^{m} \beta_i^{1/2} (\alpha_i - \lambda)^{-1} \beta_i^{1/2} \beta_1^{-1/2} k_1$$
$$= \alpha_1 \beta_1^{-1/2} k_1 + \beta_1^{1/2} k_1' - \sum_{i=2}^{m} \beta_i^{1/2} k_i = \Gamma_1 \hat{k}$$

for $\lambda \in \rho(S_0)$. Hence τ is the Weyl function of the ordinary boundary triple $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$.

Now we apply Theorem 4.2 to the present situation. It follows directly from (4.12) and the definition of the boundary triples $\{L^2(\partial\Omega), \Upsilon_0, \Upsilon_1\}$ in Proposition 4.1 and $\{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ above that the solution operator $\widetilde{A}$ in Theorem 4.2 has the asserted form. As τ is a Nevanlinna function $\mathbb{C}\backslash\mathbb{R}$ is subset of $\mathcal{U}$, cf. the consideration before Corollary 4.4, and hence for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the unique solution $f \in \mathcal{D}_{\max}$ of (4.5) is given by (4.6). It can be shown with similar arguments as in step 1 of the proof of Theorem 4.2 that this is also true on the larger set $\rho(\widetilde{A}) \cap \rho(T_D) \cap \mathfrak{h}(\tau)$. $\qquad \square$

In the next corollary we consider the special case of a linear $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function τ in the boundary condition of (4.5). Similar λ-linear elliptic boundary value problems were investigated in, e.g., [15,35,36].

Corollary 4.7 *Let α, β be bounded selfadjoint operators in $L^2(\partial\Omega)$ and assume that β is uniformly positive. Then*

$$\widetilde{A}\begin{pmatrix} f \\ k \end{pmatrix} = \begin{pmatrix} \ell f \\ \beta^{-1/2}\iota_+ \frac{\partial f_D}{\partial\nu_\ell}|_{\partial\Omega} - \beta^{-1/2}\alpha\beta^{-1/2}k \end{pmatrix}$$

$$\mathrm{dom}\,\widetilde{A} = \left\{ \begin{pmatrix} f \\ k \end{pmatrix} \in \mathcal{D}_{\max} \times L^2(\partial\Omega) : \iota_- f_\eta|_{\partial\Omega} = \beta^{-1/2}k \right\}$$

is a selfadjoint operator in $L^2(\Omega) \times L^2(\partial\Omega)$ and for $g \in L^2(\Omega)$ and $\lambda \in \rho(\widetilde{A}) \cap \rho(T_D)$ the unique solution $f \in \mathcal{D}_{\max}$ of the boundary value problem

$$(\ell - \lambda)f = g, \qquad (\alpha + \lambda\beta)\iota_- f|_{\partial\Omega} = \iota_+ \frac{\partial f_D}{\partial\nu_\ell}\Big|_{\partial\Omega},$$

is given by (4.6).

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