On finite rank perturbations of definitizable operators

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Abstract

It was shown by P. Jonas and H. Langer that a selfadjoint definitizable operator A in a Krein space remains definitizable after a finite rank perturbation in resolvent sense if the perturbed operator B is selfadjoint and the resolvent set $\rho(B)$ is nonempty. It is the aim of this note to prove a more general variant of this perturbation result where the assumption on $\rho(B)$ is dropped. As an application a class of singular ordinary differential operators with indefinite weight functions is studied.

Key words: definitizable operator, finite rank perturbation, Krein space, differential operator, indefinite weight function

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1 Introduction

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space, i.e., $\mathcal{K}$ can be written as the direct $[\cdot,\cdot]$-orthogonal sum $\mathcal{K}_+[\dot{+}]\mathcal{K}_-$ of Hilbert spaces $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$, and let A be an operator in $\mathcal{K}$ which coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. In general such selfadjoint operators may have unpleasant spectral properties, e.g., the spectrum may cover the whole complex plane. In this paper we consider the special class of definitizable operators. A selfadjoint operator A in $\mathcal{K}$ is called *definitizable* if the resolvent set of A is nonempty and there exists a polynomial $p \neq 0$ such that $p(A)$ is a nonnegative operator in the Krein space $\mathcal{K}$, cf., [20,21]. Definitizable operators arise in various applications and have been studied extensively in the last decades, see, e.g., [3–7,14–17,19–24]. In connection with spectral problems for Sturm-Liouville operators with indefinite weights definitizable operators were studied in [1,3,4,6,15,16]. In these applications the particular operator of interest can be regarded as a perturbation of a definitizable operator $A_+ \times A_-$ in $\mathcal{K}$, where A_+ and A_- are selfadjoint operators in $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively. Therefore general perturbation results for definitizable operators are very useful and of great importance.

A classical well-known result on finite rank perturbations of definitizable operators was proved by P. Jonas and H. Langer in [13]. Assume that A is a definitizable selfadjoint operator in the Krein space $\mathcal{K}$, let B be a selfadjoint operator in $\mathcal{K}$ with nonempty resolvent set $\rho(B)$ and suppose that

$$\dim \operatorname{ran} \Big((B-\lambda)^{-1} - (A-\lambda)^{-1} \Big) < \infty$$

holds for some, and hence for all, $\lambda \in \rho(A) \cap \rho(B)$. Then it was shown in [13, Theorem 1] that also the perturbed operator B is definitizable. However, in applications it is often difficult to verify the condition on $\rho(B)$, e.g., for ordinary differential operators with indefinite weights, cf., [6], so that there is a strong desire to have a perturbation result of the above type available without any assumptions on the resolvent set of B. It is the aim of Theorem 2.2 in the present note to fill this gap. Instead of a finite rank perturbation in resolvent sense we suppose that the symmetric operator $S = A \cap B$ is of finite defect, i.e., the (graphs) of A and B "differ" by finitely many dimensions. Under this assumption we prove the following equivalence for two selfadjoint operators A and B in a Krein space: A is definitizable if and only if B is definitizable.

In Section 3 this new variant of the perturbation result from [13] is applied to ordinary differential operators with an indefinite weight function. We consider singular differential expressions of order $2n$ on $\mathbb{R}$ and generalize some of the results in [6, §2.2].

2 Finite rank perturbations of definitizable operators

2.1 Definitizable operators in Krein spaces

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be a linear operator in $\mathcal{K}$. The symbols $\operatorname{dom} A$, $\ker A$, and $\operatorname{ran} A$ stand for the domain, kernel and range of A, respectively. Suppose that A is a selfadjoint operator in $\mathcal{K}$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. Then A is said to be *definitizable* if its resolvent set $\rho(A)$ is nonempty and there exists a real polynomial p, $p \neq 0$, such that

$$[p(A)x, x] \geq 0 \quad \text{for all} \quad x \in \operatorname{dom} p(A).$$

It was shown by H. Langer that a definitizable operator A has a spectral function which is defined on all real intervals with boundary points which are not critical points of A, see [20,21]. Moreover, for a definitizable operator A the nonreal spectrum $\sigma(A) \cap (\mathbb{C}\backslash\mathbb{R})$ consists of at most finitely many pairs of eigenvalues which are symmetric with respect to the real line. Note that a selfadjoint operator A with $\rho(A) \neq \emptyset$ and the property that the hermitian form $[A\cdot,\cdot]$ defined on $\operatorname{dom} A$ has finitely many negative squares is definitizable, cf., [21].

Definitizability of selfadjoint operators in Krein spaces can also be characterized in a different form, see Theorem 2.1 below. Recall that for a selfadjoint operator A in $\mathcal{K}$ a point λ from the approximative point spectrum is said to be a *spectral point of positive type* (*negative type*) *of* A if for each sequence $(x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(A - \lambda)x_n\| \to 0$ for $n \to \infty$,

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad \Big(\limsup_{n\to\infty}\, [x_n, x_n] < 0, \text{ respectively}\Big)$$

holds, cf. [12,22]. The selfadjointness of A implies that the spectral points of positive and negative type are real. An open set $\Delta \subset \mathbb{R}$ is said to be of *positive type* (*negative type*) with respect to A if $\Delta \cap \sigma(A)$ consists of spectral points of positive type (negative type, respectively). We say that an open set $\Delta \subset \mathbb{R}$ is of *definite type* with respect to A if Δ is either of positive or negative type with respect to A.

The next theorem follows from [11, Section 2.1] and [12] where the concept of local definitizability of selfadjoint operators in Krein spaces is investigated in detail. We shall use the equivalent characterization of definitizable operators from Theorem 2.1 in the proof of Theorem 3.1. The one-point compactifications of the real line and the complex plane are denoted by $\overline{\mathbb{R}}$ and $\overline{\mathbb{C}}$, respectively.

Theorem 2.1 *Let A be a selfadjoint operator in the Krein space $\mathcal{K}$. Then A is definitizable if and only if the following holds.*

(i) *Every point $\mu \in \overline{\mathbb{R}}$ has an open connected neighborhood $\mathcal{U}_\mu$ in $\overline{\mathbb{R}}$ such that both intervals $\mathcal{U}_\mu \backslash \{\mu\}$ are of definite type with respect to A;*

(ii) *$\sigma(A) \cap (\mathbb{C} \backslash \mathbb{R})$ consists of at most finitely many isolated points which are poles of the resolvent of A;*

(iii) *There exist $m \geq 1$, $M > 0$ and an open neighborhood $\mathcal{O}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$ such that*

$$\|(A-\lambda)^{-1}\| \leq M(1+|\lambda|)^{2m-2}\,|\mathrm{Im}\,\lambda|^{-m} \quad \textit{for all} \quad \lambda \in \mathcal{O} \backslash \overline{\mathbb{R}}.$$

2.2 Finite rank perturbations

In this section a classical result from [13] on finite rank perturbations of definitizable operators is generalized, see Theorem 2.2 below. Roughly speaking we drop the assumption from [13, Theorem 1] that the perturbed operator has a nonempty resolvent set. In order to formulate our variant of the perturbation result we remind the reader that a (possibly nondensely defined) operator S in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called *symmetric* if $[Sx, x]$ is real for all $x \in \mathrm{dom}\, S$. Recall also that a closed symmetric operator S in $\mathcal{K}$ is said to be of *defect* $m \in \mathbb{N}_0$ if there exists a selfadjoint extension A of S in $\mathcal{K}$ such that $\dim(\mathrm{graph}(A)/\mathrm{graph}(S)) = m$. Note that m is independent of the choice of the selfadjoint extension A of S.

Theorem 2.2 *Let A and B be selfadjoint operators in the Krein space $\mathcal{K}$ and assume that $A \cap B$ is of finite defect. Then A is definitizable if and only if B is definitizable.*

Proof. Assume that A is definitizable and let $S := A \cap B$, i.e.,

$$\begin{aligned} \mathrm{dom}\, S &= \big\{ f \in \mathrm{dom}\, A \cap \mathrm{dom}\, B : Af = Bf \big\}, \\ Sf &= Af = Bf, \qquad f \in \mathrm{dom}\, S. \end{aligned} \tag{2.1}$$

We will prove in the following that $\rho(B)$ is nonempty. Then the assumption that the defect of S is finite implies that

$$\dim\Big(\mathrm{dom}\,(A-\lambda)^{-1}/\mathrm{dom}\,(S-\lambda)^{-1}\Big) = \dim\Big(\mathrm{dom}\,(B-\lambda)^{-1}/\mathrm{dom}\,(S-\lambda)^{-1}\Big)$$

is finite for all $\lambda \in \rho(A) \cap \rho(B)$ and $(A-\lambda)^{-1}f = (B-\lambda)^{-1}f$, $f \in \mathrm{dom}\,(S-\lambda)^{-1}$, yields

$$\dim \mathrm{ran}\,\Big((B-\lambda)^{-1} - (A-\lambda)^{-1}\Big) < \infty \quad \text{for all} \quad \lambda \in \rho(A) \cap \rho(B). \tag{2.2}$$

Therefore the statement of Theorem 2.2 follows from [13, Theorem 1].

Let $p \neq 0$ be a definitizing real polynomial for the selfadjoint operator A, that is, $p(A)$ is a nonnegative operator in $\mathcal{K}$ and with the exception of at most finitely many points the set $\mathbb{C}\backslash\mathbb{R}$ belongs to $\rho(A)$. It is clear that $p(A)$ is symmetric in the Krein space $\mathcal{K}$ and it follows from $\sigma(p(A)) = p(\sigma(A))$ (see, e.g., [10, §VII.9, Theorem 10]) that $\rho(p(A)) \cap (\mathbb{C}\backslash\mathbb{R})$ is nonempty. Therefore $p(A)$ is a selfadjoint operator in $\mathcal{K}$ and as $p(A)$ is nonnegative we have

$$\mathbb{C}\backslash\mathbb{R} \subset \rho\Big(p(A)\Big). \tag{2.3}$$

Observe that $\operatorname{dom} S$ in (2.1) is in general not a dense subspace in $\mathcal{K}$ and therefore the adjoint of S has to be defined in the sense of linear relations, i.e., S^+ is the linear subspace

$$S^+ := \Big\{\{f, f'\} \in \mathcal{K}^2 : [Sg, f] = [g, f'] \text{ for all } g \in \operatorname{dom} S\Big\}$$

of $\mathcal{K}\times\mathcal{K}$, cf., e.g., [8]. Here and in the following the elements of a linear relation are written in curly brackets. Operators are regarded as linear relations via their graphs. Note that the definition of S^+ extends the usual definition of the adjoint of a densely defined operator and, moreover, S^+ is (the graph of) an operator if and only if $\operatorname{dom} S$ is dense in $\mathcal{K}$.

We claim that for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the linear relation $p(S^+)$ (see, e.g., [9,26]), can be decomposed in the form

$$p(S^+) = p(A) \dotplus \Big\{\{h, \lambda h\} : h \in \ker(p(S^+) - \lambda)\Big\}, \tag{2.4}$$

where $\dotplus$ denotes the direct sum of subspaces. In fact, $S \subset A$ and $A = A^+$ implies $A \subset S^+$, and hence also $p(A) \subset p(S^+)$. Therefore the inclusion

$$p(A) \dotplus \Big\{\{h, \lambda h\} : h \in \ker(p(S^+) - \lambda)\Big\} \subset p(S^+)$$

holds and the sum is direct since by (2.3) we have $\ker(p(A) - \lambda) = \{0\}$ for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. In order to verify the reverse inclusion let $\{f, f'\} \in p(S^+)$. By (2.3) we have $\operatorname{ran}(p(A) - \lambda) = \mathcal{K}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and hence there exists $\{g, g'\} \in p(A)$ such that $f' - \lambda f = g' - \lambda g$. This, together with $\{f, f' - \lambda f\} \in p(S^+) - \lambda$ and $\{g, g' - \lambda g\} \in (p(A) - \lambda) \subset (p(S^+) - \lambda)$ implies

$$\{f - g, 0\} = \{f, f' - \lambda f\} - \{g, g' - \lambda g\} \in p(S^+) - \lambda,$$

i.e., $f - g \in \ker(p(S^+) - \lambda)$. Thus $\{f, f'\} = \{g, g'\} + \{f - g, \lambda(f - g)\}$ is decomposed as in (2.4).

Next it will be shown that $p(S^+)$ is a finite dimensional extension of $p(A)$. According to (2.4) it is sufficient to check that $\ker(p(S^+) - \lambda_0)$ is finite dimensional for some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. Observe first that the polynomial

$$q(\mu) := p(\mu) - \lambda_0$$

has no real zeros since p is a real polynomial and $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. Hence there exist $m \in \mathbb{N}$, $k_1, \ldots, k_m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in \mathbb{C}\backslash\mathbb{R}$ and $\alpha \in \mathbb{C}\backslash\{0\}$ such that

$$q(\mu) = \alpha \prod_{i=1}^{m} (\mu - \beta_i)^{k_i}.$$

Furthermore we can assume that $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ was chosen in such a way that none of the nonreal eigenvalues of the definitizable operator A is a zero of q. According to [26, Theorem 3.4]

$$\ker q(S^+) = \ker\Big(p(S^+) - \lambda_0\Big) = \sum_{i=1}^{m} \ker(S^+ - \beta_i)^{k_i} \tag{2.5}$$

holds. As the defect of S is finite, S^+ is a finite dimensional extension of A and from the fact that each β_i belongs to $\rho(A)$ we conclude from

$$S^+ = A \dotplus \Big\{\{g, \beta_i g\} : g \in \ker(S^+ - \beta_i)\Big\}$$

that the dimension of $\ker(S^+ - \beta_i)$, $i = 1, \ldots, m$, is also finite. In a similar way as for operators one then verifies

$$\dim\Big(\ker(S^+ - \beta_i)^{k_i}\Big) < \infty$$

and thus (2.4) and (2.5) imply

$$n := \dim\Big(p(S^+)/p(A)\Big) = \dim\Big(\ker(p(S^+) - \lambda_0)\Big) < \infty. \tag{2.6}$$

Hence, $p(S^+)$ is a finite dimensional extension of $p(A)$. From (2.6) we conclude that the closed symmetric operator $(p(S^+))^+$ in $\mathcal{K}$ has finite defect n and $(p(S^+))^+ \subset p(A)$ implies that $(p(S^+))^+$ is nonnegative.

Since p is a real polynomial it follows that $p(B)$ is a symmetric operator in $\mathcal{K}$. From $B = B^+$ and $S \subset B$ we obtain $B \subset S^+$, hence $p(B)$ is a restriction of $p(S^+)$ and an extension of $(p(S^+))^+$,

$$(p(S^+))^+ \subset p(B) \subset p(S^+).$$

As $(p(S^+))^+$ has finite defect and $p(B)$ is a symmetric operator, it follows that $p(B)$ admits selfadjoint extensions in $\mathcal{K}$ which are operators. Then it follows in the same way as in the proof of [6, Proposition 1.1] that such a selfadjoint (operator) extension T of $p(B)$ has a nonempty resolvent set. In fact, by (2.3) we have $\operatorname{ran}(p(A)-\lambda) = \mathcal{K}$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$, hence the ranges of $p(S^+)^+ - \lambda$ are closed and the same holds for the ranges of the finite dimensional extensions

$p(B)-\lambda$ and $T-\lambda$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Suppose now $\rho(T) = \emptyset$. Then it follows that in at least one of the halfplanes there are infinitely many points belonging to $\sigma_p(T)$. Let $f_1, \dots, f_{n+1}$ be eigenvectors corresponding to $n+1$ different eigenvalues of T in that halfplane. Choose vectors $g_1, \dots, g_{n+1}$ in the dense subspace $\operatorname{dom} T$ such that $[Tf_i, g_j] = \delta_{ij}$, $i, j = 1, \dots, n+1$, holds, cf. [6]. Then the Krein space

$$\mathcal{L} := \Big(\operatorname{span}\{f_1, \dots, f_{n+1}, g_1, \dots, g_{n+1}\}, [T\cdot, \cdot]\Big)$$

contains an $n+1$-dimensional neutral subspace. Hence $\mathcal{L}$ contains also an $n+1$-dimensional negative subspace, which contradicts the fact that T is an n-dimensional extension of the nonnegative operator $p(S^+)^+$. Therefore $\rho(T) \neq \emptyset$.

Since $[T\cdot, \cdot]$ has finitely many negative squares and $\rho(T) \neq \emptyset$ it follows that T is a definitizable operator, cf. [21]. In particular, the set $\mathbb{C}\backslash\mathbb{R}$ with the possible exception of at most finitely many points belongs to $\rho(T)$. Therefore, up to a finite set each $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is a point of regular type of the finite dimensional restriction $p(B)$ of T, that is, $\ker(p(B) - \lambda) = \{0\}$ and $\operatorname{ran}(p(B) - \lambda)$ is closed. This together with $\sigma_p(p(B)) = p(\sigma_p(B))$ and the fact that the range of $B - \lambda$ is closed for all $\lambda \in \rho(A)$ implies that there exists a pair $\{\mu, \bar{\mu}\}$, $\mu \in \mathbb{C}\backslash\mathbb{R}$, of points of regular type of B, i.e., $\operatorname{ran}(B - \mu)$ and $\operatorname{ran}(B - \bar{\mu})$ are closed and $\ker(B - \mu) = \ker(B - \bar{\mu}) = \{0\}$. But this is possible only if $\operatorname{ran}(B - \mu) = \operatorname{ran}(B - \bar{\mu}) = \mathcal{K}$, therefore $\{\mu, \bar{\mu}\} \in \rho(B)$. Thus $\rho(B) \neq \emptyset$ and the statement of Theorem 2.2 follows from (2.2) and [13, Theorem 1]. We note for the sake completeness that $\rho(B) \neq \emptyset$ implies $\rho(p(B)) \neq \emptyset$ and hence $p(B)$ and T coincide. □

Remark 2.3 *The notion of definitizability of selfadjoint operators can be generalized to selfadjoint relations, see, e.g., [9]. We note that Theorem 2.2 is not true if B (or A) is allowed to be a selfadjoint relation since then the assumption that $A \cap B$ has finite defect in general does not imply that B has a nonempty resolvent set. However, if $\rho(B)$ is assumed to be nonempty then definitizability of the selfadjoint relation A implies definitizability of the selfadjoint relation B, see, e.g., [2].*

Remark 2.4 *Locally definitizable selfadjoint operators and relations were comprehensively studied by P. Jonas, see, e.g., [12], and it was shown in [2] that the notion of local definitizability is also stable under finite rank perturbations in resolvent sense if the perturbed operator or relation is selfadjoint and has a nonempty resolvent set. Nevertheless, Theorem 2.2 does not hold for locally definitizable operators. This is due to fact that the assumption of finite defect of $A \cap B$ does not imply that B (or A) has a nonempty resolvent set, see [2, §3.3] for a simple counterexample.*

The following corollary is generalization of [6, Proposition 1.1].

Corollary 2.5 *Let S be a closed symmetric operator of finite defect in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension of S in $\mathcal{K}$ which is definitizable. Then the following holds:*

(i) *every selfadjoint extension of S in $\mathcal{K}$ which is an operator has a nonempty resolvent set and is definitizable;*
(ii) *if S is densely defined, then every selfadjoint extension of S in $\mathcal{K}$ has a nonempty resolvent set and is definitizable.*

3 An application: ordinary differential operators with indefinite weight functions

We consider the formal differential expression of order $2n$ on $\mathbb{R}$ given by

$$\ell(f) = \frac{1}{r}\Big((-1)^n(p_0 f^{(n)})^{(n)} + (-1)^{n-1}(p_1 f^{(n-1)})^{(n-1)} + \cdots + p_n f\Big), \qquad (3.1)$$

where $r, p_0^{-1}, p_1, \ldots, p_n \in L^1_{\text{loc}}(\mathbb{R})$ are assumed to be real functions such that $r \neq 0$ and $p_0 > 0$ a.e. on $\mathbb{R}$. With the help of the quasi-derivatives

$$f^{[0]} := f, \qquad f^{[k]} := \frac{d^k f}{dx^k}, \quad k = 1, 2, \ldots, n-1,$$
$$f^{[n]} := p_0 \frac{d^n f}{dx^n}, \quad f^{[n+k]} := p_k \frac{d^{n-k} f}{dx^{n-k}} - \frac{d}{dx} f^{[n+k-1]}, \quad k = 1, 2, \ldots, n,$$

cf. [18,25], the formal expression (3.1) can be written as

$$\ell(f) = \frac{1}{r} f^{[2n]}. \qquad (3.2)$$

Following the lines of [1,6] we show that under suitable assumptions definitizable selfadjoint operators in a Krein space can be associated to the differential expression ℓ.

For the weight function r the following condition (I) is supposed to hold (cf., [1] and [6, Proposition 2.5]):

(I) There exist $a, b \in \mathbb{R}$, $a < b$, such that the restrictions $r_+ := r \restriction (b, \infty)$ and $r_- := r \restriction (-\infty, a)$ satisfy $r_+ > 0$ a.e. on (b, ∞) and $r_- < 0$ a.e. on $(-\infty, a)$.

In the following we agree to choose $a, b \in \mathbb{R}$ in such a way that the sets $\{x \in (a, b) \,|\, r(x) > 0\}$ and $\{x \in (a, b) \,|\, r(x) < 0\}$ have positive Lebesgue measure. This is no restriction. We note that the case $r_+ < 0$ and $r_- > 0$ can be treated analogously. We do not consider the case that r_+ and r_- have the same signs. Under suitable assumptions these cases are contained in the considerations in [6], cf., Remark 3.3 below.

Let $L^2_{|r|}(\mathbb{R})$ be the Hilbert space of all equivalence classes of measurable functions f defined on $\mathbb{R}$ for which $\int_{\mathbb{R}} |f(x)|^2|r(x)|dx$ is finite. We equip $L^2_{|r|}(\mathbb{R})$ with the indefinite inner product

$$[f,g] := \int_{\mathbb{R}} f(x)\overline{g(x)}r(x)dx, \quad f,g \in L^2_{|r|}(\mathbb{R}), \tag{3.3}$$

and denote the corresponding Krein space $(L^2_{|r|}(\mathbb{R}), [\cdot,\cdot])$ by $L^2_r(\mathbb{R})$. The maximal operator $S_{\max} f = \ell(f)$ associated to (3.2) is defined on the dense subspace $\mathcal{D}_{\max}$ consisting of all functions $f \in L^2_r(\mathbb{R})$ which have absolutely continuous quasi derivatives $f^{[0]}, f^{[1]}, \ldots, f^{[2n-1]}$ such that $\ell(f) \in L^2_r(\mathbb{R})$. The restriction $S^0_{\min}$ of $S_{\max}$ to functions with compact support is a densely defined symmetric operator in the Krein space $L^2_r(\mathbb{R})$. The minimal operator $S_{\min}$ is the closure of $S^0_{\min}$. It is a symmetric operator in $L^2_r(\mathbb{R})$ of defect m, $0 \le m \le 2n$, and $S^+_{\min} = S_{\max}$ holds, cf., [6,25]. In particular, the selfadjoint realizations of ℓ in $L^2_r(\mathbb{R})$ are finite dimensional extensions of $S_{\min}$ in $L^2_r(\mathbb{R})$.

Denote by ℓ_-, ℓ_{ab} and ℓ_+ the differential expressions on the intervals $(-\infty, a)$, (a, b) and (b, ∞), respectively, which are defined in the same way as ℓ, except that the functions $r, p_0, p_1, \ldots, p_n$ in (3.1) are replaced by their restrictions onto $(-\infty, a)$, (a, b) and (b, ∞), respectively. By condition (I) the inner product (3.3) is positive definite on functions with support in (b, ∞) and negative definite on functions with support in $(-\infty, a)$. Furthermore, (3.3) is indefinite on functions with support in (a, b). Therefore

$$L^2_{r_+}((b,\infty)) := \left(L^2_{|r_+|}((b,\infty)), [\cdot,\cdot]\right)$$

is a Hilbert space,

$$L^2_{r_-}((-\infty,a)) := \left(L^2_{|r_-|}((-\infty,a)), [\cdot,\cdot]\right)$$

is an anti Hilbert space, i.e., $(L^2_{|r_-|}((-\infty, a)), -[\cdot,\cdot])$ is a Hilbert space, and

$$L^2_{r_{ab}}((a,b)) := \left(L^2_{|r_{ab}|}((a,b)), [\cdot,\cdot]\right), \quad r_{ab} := r \restriction (a,b),$$

is a Krein space with infinite positive and negative index. Since a and b are regular endpoints, the minimal closed symmetric operators $S_{\min,+}$ and $S_{\min,-}$ associated to ℓ_+ and ℓ_- have defect m, $n \le m \le 2n$, cf. [25, §17.5], and the selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$ are finite dimensional extensions of $S_{\min,+}$ and $S_{\min,-}$, respectively.

Theorem 3.1 *Suppose that the weight function r satisfies condition* (I) *and assume that A_+ and A_- are selfadjoint realizations of ℓ_+ and ℓ_- in the spaces $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, such that the following holds:*

(i) A_+ *is semibounded from below and* A_- *is semibounded from above;*
(ii) *the set* $e := \sigma(A_+) \cap \sigma(A_-)$ *is finite;*
(iii) *there exist disjoint open intervals* $\mathcal{I}_1, \ldots, \mathcal{I}_n \subset \mathbb{R}$ *and some* $j_0 \in \{1, \ldots, n\}$ *such that*

$$\sigma(A_+)\backslash\{e\} \subset \bigcup_{k=1}^{j_0} \mathcal{I}_k \quad \textit{and} \quad \sigma(A_-)\backslash\{e\} \subset \bigcup_{k=j_0+1}^{n} \mathcal{I}_k.$$

Then every selfadjoint realization of the differential expression ℓ *in the Krein space* $L^2_r(\mathbb{R})$ *has a nonempty resolvent set and is a definitizable operator.*

Proof. Denote the minimal closed symmetric operator associated to ℓ_{ab} in the Krein space $L^2_{r_{ab}}((a,b))$ by $S_{\min,ab}$. The defect of $S_{\min,ab}$ is $2n$, cf., [25, §17.3]. Let A_{ab} be a selfadjoint extension of $S_{\min,ab}$ in the Krein space $L^2_{r_{ab}}((a,b))$. Then according to [6, Proposition 2.2, Proposition 1.1 and Corollary 1.4] the spectrum $\sigma(A_{ab})$ is discrete, $\rho(A_{ab})$ is nonempty, the hermitian sesquilinear form $[A_{ab}\cdot,\cdot]$ defined on $\operatorname{dom} A_{ab}$ has finitely many negative squares and A_{ab} is definitizable. Let A_+ and A_- be selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, such that (i)-(iii) hold. We claim that the direct sum

$$A = A_- \times A_{ab} \times A_+, \qquad \operatorname{dom} A = \operatorname{dom} A_- \times \operatorname{dom} A_{ab} \times \operatorname{dom} A_+, \tag{3.4}$$

is a definitizable operator in the Krein space

$$L^2_{r_-}((-\infty,a)) \times L^2_{r_{ab}}((a,b)) \times L^2_{r_+}((b,\infty)) = L^2_r(\mathbb{R}).$$

This will be verified with the help of Theorem 2.1. First of all $A_\pm$ are selfadjoint operators in Hilbert or anti Hilbert spaces and thus their spectrum $\sigma(A_\pm)$ is real. Therefore $\sigma(A) \cap (\mathbb{C}\backslash\mathbb{R}) = \sigma(A_{ab}) \cap (\mathbb{C}\backslash\mathbb{R})$. As A_{ab} is definitizable, condition (ii) in Theorem 2.1 is satisfied. Similarly the definitizability of A_{ab} together with the growth properties of the resolvents $(A_\pm - \lambda)^{-1}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, implies (iii) in Theorem 2.1.

It remains to check that each point $\mu \in \overline{\mathbb{R}}$ has an open connected neighborhood $\mathcal{U}_\mu$ such that both intervals $\mathcal{U}_\mu\backslash\{\mu\}$ are of definite type with respect to A. Assume first $\mu \in \mathbb{R}$. As A_+ (A_-) is a selfadjoint operator in a Hilbert space (anti Hilbert space, respectively) $\sigma(A_+)$ ($\sigma(A_-)$) consists only of points of positive type (negative type, respectively). Now (ii) and (iii) imply that $\mathcal{U}_\mu$ can be chosen such that both intervals $\mathcal{U}_\mu\backslash\{\mu\}$ are of definite type with respect to $A_- \times A_+$. Since $\sigma(A_{ab})$ is discrete we can assume $\mathcal{U}_\mu\backslash\{\mu\} \subset \rho(A_{ab})$ and hence both intervals $\mathcal{U}_\mu\backslash\{\mu\}$ are also of definite type with respect to A. Let us now consider the case $\mu = \infty$. As the hermitian sesquilinear form $[A_{ab}\cdot,\cdot]$ has finitely many negative squares it follows that there exist $\mu_+ \in (0,\infty)$ and $\mu_- \in (-\infty,0)$ such that the interval (μ_+,∞) is of positive type with respect to A_{ab} and the interval $(-\infty, \mu_-)$ is of negative type with respect to A_{ab}. Since by (i) A_+ and A_- are semibounded from below and above, respectively, μ_+ and μ_- can be chosen such that $(\mu_+, \infty) \subset \rho(A_-)$ and $(-\infty, \mu_-) \subset \rho(A_+)$. As $\sigma(A_+)$ is of positive type and $\sigma(A_-)$ is of negative type we conclude that (μ_+, ∞) is of positive type with respect to A and $(-\infty, \mu_-)$ is of negative type with respect to A. Thus (i) in Theorem 2.1 holds and it follows that A is a definitizable operator in the Krein space $L^2_r(\mathbb{R})$.

Since $A_\pm$ are selfadjoint extensions of the operators $S_{\min,\pm}$ and A_{ab} is a selfadjoint extension of $S_{\min,ab}$ it is clear that A is a selfadjoint extension of the closed symmetric operator $S = S_{\min,-} \times S_{\min,ab} \times S_{\min,+}$ in $L^2_r(\mathbb{R})$. Furthermore, $\operatorname{dom} S$ is dense and S has finite defect m, $4n \le m \le 6n$. Hence by Corollary 2.5(ii) every selfadjoint extension of S is definitizable. Since each selfadjoint realization of ℓ in $L^2_r(\mathbb{R})$ is an extension of the minimal operator $S_{\min}$ associated to ℓ and $S \subset S_{\min}$ the assertion of Theorem 3.1 follows. □

Remark 3.2 *Conditions* (i)-(iii) *in Theorem 3.1 do not depend on the choice of the selfadjoint extensions A_+ and A_- of $S_{\min,+}$ and $S_{\min,-}$. In fact, if A'_+ and A'_- are arbitrary selfadjoint realizations of ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, then semiboundedness of $A_\pm$ implies semiboundedness of $A'_\pm$ since the resolvents of $A_\pm$ and $A'_\pm$ differ by a finite rank operator. Furthermore, condition* (ii) *and* (iii) *ensure that $A_+ \times A_-$ is definitizable and hence $A'_+ \times A'_-$ is definitizable by [13, Theorem 1]. Therefore* (ii) *and* (iii) *hold also for A'_+ and A'_-.*

Remark 3.3 *The case that the weight function r is positive (negative) on $(-\infty, a)$ and (b,∞) is not considered in Theorem 3.1. We note that, e.g., the positivity of r_+, r_- and the semiboundedness of A_+ and A_- from below imply that for some $\alpha \in \mathbb{R}$ the selfadjoint operator $A - \alpha$, where $A = A_- \times A_{ab} \times A_+$ is as in* (3.4)*, has a finite number of negative squares and $\sigma(A) \cap (-\infty, \eta)$ is discrete for some $\eta \in \mathbb{R}$. Then the same is true for all selfadjoint realizations of ℓ in $L^2_r(\mathbb{R})$, cf., [6].*

Definitizability of selfadjoint realizations of indefinite Sturm-Liouville differential expressions of the form (3.1)-(3.2) was already studied in [6]. In addition, the selfadjoint differential operators arising in [6] have finitely many negative squares. The following two corollaries connect Theorem 3.1 with the results in [6, §2.2].

Corollary 3.4 *Suppose that the weight function r satisfies condition* (I) *and assume that A_+ and A_- are selfadjoint realizations of ℓ_+ and ℓ_- in the spaces $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively, such that $\sigma(A_+) \cap (-\infty, 0)$ and $\sigma(A_-) \cap (0,\infty)$ consist of finitely many eigenvalues.*

Then every selfadjoint realization B of the differential expression ℓ in the Krein *space $L^2_r(\mathbb{R})$ has a nonempty resolvent set and the form $[B\cdot,\cdot]$ has finitely many negative squares.*

Proof. The assumption that $\sigma(A_+)\cap(-\infty,0)$ and $\sigma(A_-)\cap(0,\infty)$ consist of finitely many eigenvalues implies that conditions (i)-(iii) in Theorem 3.1 hold. Hence every selfadjoint realization of ℓ in $L^2_r(\mathbb{R})$ has a nonempty resolvent set and is definitizable. Furthermore, it is not difficult to see that the selfadjoint operator $A_-\times A_+$ in $L^2_{r_-}((-\infty,a))\times L^2_{r_+}((b,\infty))$ has finitely many negative squares (cf., e.g., [4, §4.2]) and the same holds for the selfadjoint operator $A=A_-\times A_{ab}\times A_+$ in $L^2_r(\mathbb{R})$, cf. (3.4). Therefore the symmetric operator $S=S_{\min,-}\times S_{\min,ab}\times S_{\min,+}$ also has finitely many negative squares and hence every selfadjoint realization B of ℓ in $L^2_r(\mathbb{R})$ has finitely many negative squares. □

Corollary 3.5 *Suppose that the weight function r satisfies condition* (I) *and let $S_{\min,+}$ and $S_{\min,-}$ be the minimal closed symmetric operators associated to ℓ_+ and ℓ_- in $L^2_{r_+}((b,\infty))$ and $L^2_{r_-}((-\infty,a))$, respectively. Assume that there exist $b'\in(b,\infty)$ and $a'\in(-\infty,a)$ such that $[S_{\min,+}\cdot,\cdot]$ and $[S_{\min,-}\cdot,\cdot]$ are positive on the set of functions from* $\operatorname{dom} S_{\min,+}$ *and* $\operatorname{dom} S_{\min,-}$ *which have compact support in (b',∞) and $(-\infty,a')$, respectively.*

Then every selfadjoint realization B of the differential expression ℓ in the Krein space $L^2_r(\mathbb{R})$ has a nonempty resolvent set and the form $[B\cdot,\cdot]$ has finitely many negative squares.

Proof. As in the proof of [6, Proposition 2.3] one verifies that the inner product $[S_{\min,+}\cdot,\cdot]$ has a finite number of negative squares on $\operatorname{dom} S_{\min,+}$. Hence, if A_+ is an arbitrary selfadjoint extensions of $S_{\min,+}$ in $L^2_{r_+}((b,\infty))$, then also the form $[A_+\cdot,\cdot]$ defined on $\operatorname{dom} A_+$ has a finite number of negative squares, so that $\sigma(A_+)\cap(-\infty,0)$ consists of finitely many eigenvalues. Analogously it follows that for any selfadjoint extension A_- of $S_{\min,-}$ the form $-[A_-\cdot,\cdot]$ has finitely many positive squares, hence the positive spectrum of A_- in $L^2_{r_-}((-\infty,a))=(L^2_{|r_-|}((-\infty,a),-[\cdot,\cdot])$ consists of at most finitely many eigenvalues. Therefore the statement follows from Corollary 3.4. □

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