Fast Boundary Element Methods in Industrial Applications

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Die Multipol–Randelementmethode in industriellen Anwendungen

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Outline:

1. Mixed boundary value problem of potential theory and linear elastostatics
2. Galerkin boundary integral equation formulation
3. Realization of the boundary integral operators by the Fast Multipole Method
4. Numerical examples and industrial applications

Mixed boundary value problem

Laplace operator:

$$
\begin{aligned}
-\Delta u(x) &= 0 && \text{for } x \in \Omega, \\
u(x) &= g_D(x) && \text{for } x \in \Gamma_D, \\
t(x) := (T_x u)(x) = (\partial_n u)(x) &= g_N(x) && \text{for } x \in \Gamma_N.
\end{aligned}
$$

linear elastostatics:

$$
\begin{aligned}
-\text{div}\,(\sigma(u)) &= 0 && \text{for } x \in \Omega \subset \mathbb{R}^3, \\
u_i(x) &= g_{D,i}(x) && \text{for } x \in \Gamma_{D,i}, i = 1, \ldots, 3, \\
t_i(x) := (T_x u)_i(x) = (\sigma(u) n(x))_i &= g_{N,i}(x) && \text{for } x \in \Gamma_{N,i}, i = 1, \ldots, 3.
\end{aligned}
$$

The stress tensor $\sigma(u)$ is related to the strain tensor $e(u)$ by **Hooke's law**

$$
\sigma(u) = \frac{E\nu}{(1+\nu)(1-2\nu)} \left(\text{tr}\; e(u) I + \frac{E}{(1+\nu)} e(u) \right).
$$

E is the Young modulus and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio. The strain tensor is defined by

$$
e(u) = \frac{1}{2}(\nabla u^\top + \nabla u).
$$

Boundary integral formulation

Representation formula:

$$u(x) = \int_\Gamma [U^*(x,y)]^\top t(y) ds_y - \int_\Gamma [T_y^* U^*(x,y)]^\top u(y) ds_y \qquad \text{for } x \in \Omega.$$

Calderon projector for the Cauchy data $u(x)$ and $t(x)$ on the boundary Γ:

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} \qquad \text{on } \Gamma$$

Boundary integral operators:

$$(Vt)(x) = \int_\Gamma [U^*(x,y)]^\top t(y) ds_y, \qquad (Ku)(x) = -\!\!\!\!\!\!\int_\Gamma [T_y^* U^*(x,y)]^\top u(y) ds_y,$$

$$(K't)(x) = -\!\!\!\!\!\!\int_\Gamma [T_x U^*(x,y)]^\top t(y) ds_y, \quad (Du)(x) = -T_x \int_\Gamma [T_y^* U^*(x,y)]^\top u(y) ds_y.$$

Fundamental solution:

$$U^*(x-y) = \frac{1}{4\pi|x-y|}, \qquad U^*_{kl}(x-y) = \frac{1+\nu}{8\pi E(1-\nu)} \left[(3-4\nu) \frac{\delta_{kl}}{|x-y|} + \frac{(x_k - y_k)(x_l - y_l)}{|x-y|^3} \right].$$

Symmetric variational boundary integral formulation

Symmetric boundary integral formulation (Sirtori '79, Costabel '87):

$$(V\widetilde{t})(x) - (K\widetilde{u})(x) = (\frac{1}{2}I + K)\widetilde{g}_D(x) - (V\widetilde{g}_N)(x) \quad \text{for } x \in \Gamma_D,$$
$$(K'\widetilde{t})(x) + (D\widetilde{u})(x) = (\frac{1}{2}I - K')\widetilde{g}_N(x) - (D\widetilde{g}_D)(x) \quad \text{for } x \in \Gamma_N.$$

Galerkin discretization with piecewise constant (φ_l) and piecewise linear (ψ_i) ansatz and test functions leads to a **system of linear equations**:

$$\begin{pmatrix} V_h & -K_h \\ K_h' & D_h \end{pmatrix} \begin{pmatrix} \widetilde{\underline{t}}_h \\ \widetilde{\underline{u}}_h \end{pmatrix} = \begin{pmatrix} \underline{f}_N \\ \underline{f}_D \end{pmatrix}.$$

Single Galerkin blocks for $k, l = 1, \ldots, m$ and $i, j = 1, \ldots, \widetilde{m}$

$$\begin{aligned} V_h[l,k] &= \langle V\varphi_k, \varphi_l \rangle_{L_2(\Gamma_D)}, & K_h[l,i] &= \langle K\psi_i, \varphi_l \rangle_{L_2(\Gamma_D)}, \\ K_h'[j,k] &= \langle K'\varphi_k, \psi_j \rangle_{L_2(\Gamma_N)}, & D_h[j,i] &= \langle D\psi_i, \psi_j \rangle_{L_2(\Gamma_N)}. \end{aligned}$$

Symmetric realization of the single layer potential

Observation: The fundamental solution $(U^*_{kl})_{l,k=1..3}$ of linear elastostatics can be written as

$$U^*_{kl}(x-y) = \frac{1+\nu}{2E(1-\nu)}\frac{1}{4\pi}\left[(3-4\nu)\frac{\delta_{kl}}{|x-y|} - \frac{1}{2}x_l\frac{\partial}{\partial x_k}\frac{1}{|x-y|} - \frac{1}{2}y_l\frac{\partial}{\partial y_k}\frac{1}{|x-y|} - \frac{1}{2}x_k\frac{\partial}{\partial x_l}\frac{1}{|x-y|} - \frac{1}{2}y_k\frac{\partial}{\partial y_l}\frac{1}{|x-y|}\right].$$

Lemma 1. *For $x \neq y$, the* **single layer potential** V^E *can be written as*

$$\left(V^E t\right)_k(x) = \frac{(1+\nu)}{2E(1-\nu)}\left[(3-4\nu)\left(V^\Delta t_k\right)(x) - \frac{1}{2}\sum_{l=1}^{3}\left(x_l\frac{\partial}{\partial x_k} + x_k\frac{\partial}{\partial x_l}\right)\left(V^\Delta t_l\right)(x) - \frac{1}{2}\int_\Gamma \sum_{l=1}^{3} y_l t_l(y)\frac{\partial}{\partial y_k}\frac{1}{4\pi|x-y|}ds_y - \frac{1}{2}\int_\Gamma y_k \sum_{l=1}^{3} t_l(y)\frac{\partial}{\partial y_l}\frac{1}{4\pi|x-y|}ds_y\right].$$

This form guarantees the **symmetry** *of the farfield part of the Galerkin matrix.*

Double layer potential

Theorem 1 (Kupradze, 1979). *The* **double layer potential** K^E *of linear elastostatics can be written as*

$$\left(K^E u\right)(x) = \frac{1}{4\pi}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y - \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\left(\mathcal{M}u\right)(y)ds_y + 2\mu\left(V^E\left(\mathcal{M}u\right)\right)(x),$$

with

$$\mu = \frac{E}{2(1+\nu)}, \qquad \mathcal{M} = \begin{pmatrix} 0 & n_2\partial_1 - n_1\partial_2 & n_3\partial_1 - n_1\partial_3 \\ n_1\partial_2 - n_2\partial_1 & 0 & n_3\partial_2 - n_2\partial_3 \\ n_3\partial_1 - n_3\partial_1 & n_2\partial_3 - n_3\partial_2 & 0 \end{pmatrix};$$

$$\left(K^E u\right)(x) = \left(K^\Delta u\right)(x) - \left(V^\Delta\left(\mathcal{M}u\right)\right)(x) + 2\mu\left(V^E\left(\mathcal{M}u\right)\right)(x)$$

The bilinear form of **adjoint double layer potential** is realized accordingly.

Hypersingular operator

Theorem 2 (Houde Han (1994), Kupradze (1979)). *The* **bilinear form** *of the* **hypersingular operator** D^E *in linear elastostatics can be written in the form*

$$\begin{aligned}
\langle D^E u, v\rangle_{L_2(\Gamma)} &= \int_\Gamma\int_\Gamma \frac{\mu}{4\pi}\frac{1}{|x-y|}\left(\sum_{k=1}^{3}\left(\mathcal{M}_{k+2,k+1}v\right)(x)\cdot\left(\mathcal{M}_{k+2,k+1}u\right)(y)\right)ds_y ds_x \\
&+\int_\Gamma\int_\Gamma \left(\mathcal{M}v\right)^\top(x)\left(\frac{\mu}{2\pi}\frac{I}{|x-y|}-4\mu^2U^*(x,y)\right)\left(\mathcal{M}u\right)(y)ds_y ds_x \\
&+\mu\int_\Gamma\int_\Gamma \sum_{i,j,k=1}^{3}\left(\mathcal{M}_{k,j}v_i\right)(x)\frac{1}{4\pi}\frac{1}{|x-y|}\left(\mathcal{M}_{k,i}u_j\right)(y)ds_y ds_x.
\end{aligned}$$

All the boundary integral operators in linear elastostatics respectively their bilinear forms are **reduced to** those of **the Laplacian**.
Together with integration by parts for bilinear form of the hypersingular operator of the Laplacian, it is **sufficient** to deal with the **single and double layer potential** of the Laplacian.

Fast Multipole Method for potential theory

$$(V^{\Delta}t)(x) = \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|x-y|} t(y)ds_y \qquad \text{for } x \in \Gamma$$

In the farfield computation by **numerical integration**:

$$\frac{1}{4\pi}\sum_{k=1}^{N}\int_{\tau_k} t(y)\frac{1}{|x-y|}ds_y \approx \frac{1}{4\pi}\sum_{k=1}^{N}\sum_{i=1}^{N_g}\underbrace{\Delta_k\omega_{k,i}t(y_{k,i})}_{=q_{k,i}}\frac{1}{|x-y_{k,i}|}.$$

multipole expansion for $|x| > |y|$

$$\sum_{j=1}^{M}\frac{q_j}{|x-y_j|} \approx \Phi_p(x) = \sum_{n=0}^{p}\sum_{m=-n}^{n}\sum_{j=1}^{M} q_j |y_j|^n Y_n^{-m}(\hat{y}_j)\frac{Y_n^m(\hat{x})}{|x|^{n+1}}$$

and **local expansion** for $|x| < |y|$

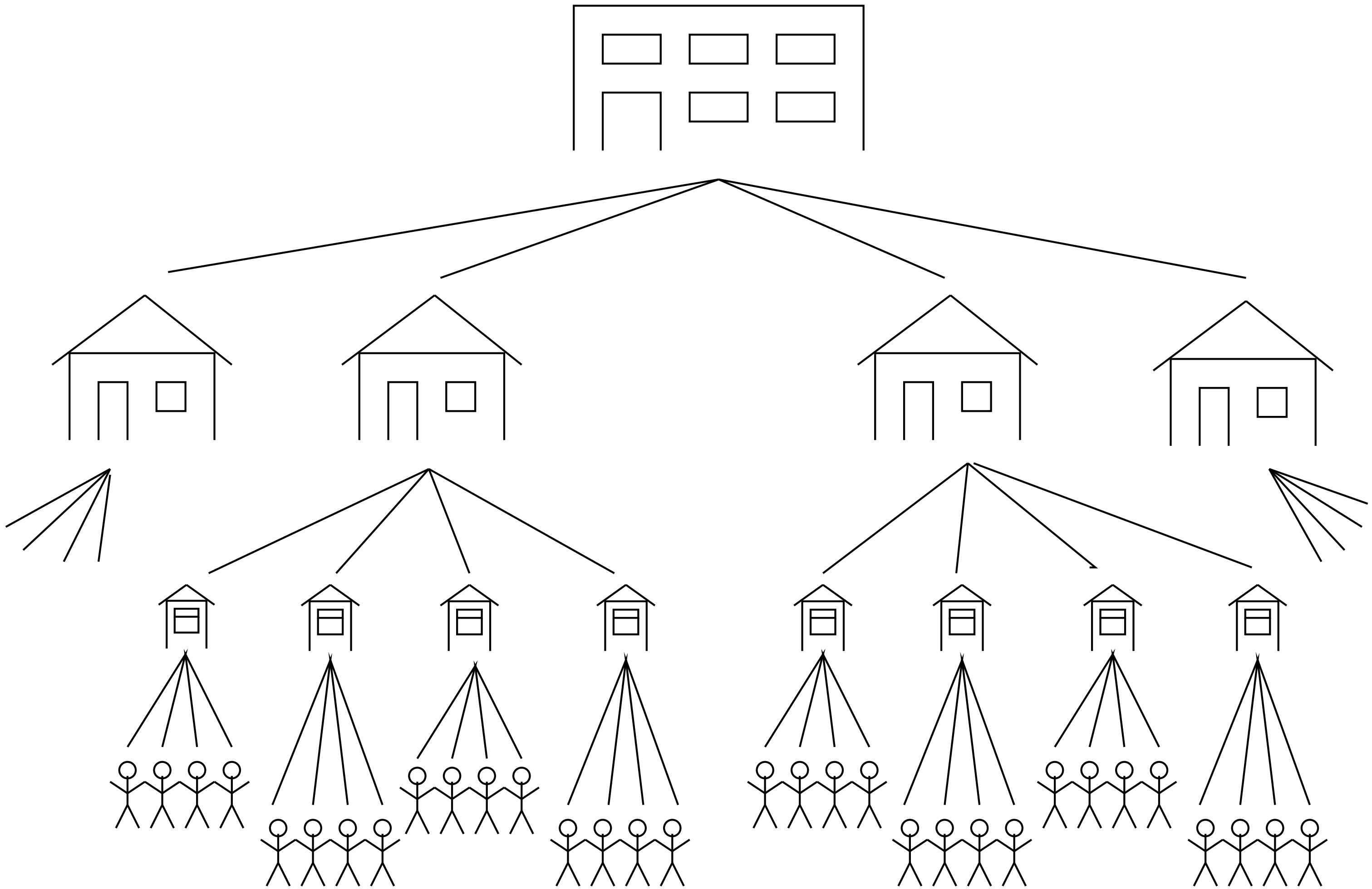
$$\sum_{j=1}^{M}\frac{q_j}{|x-y_j|} \approx \Phi_p(x) = \sum_{n=0}^{p}\sum_{m=-n}^{n}\sum_{j=1}^{M} q_j \frac{Y_n^{-m}(\hat{y}_j)}{|y_j|^{n+1}} Y_n^m(\hat{x})|x|^n$$

with **spherical harmonics** for $m \geq 0$

$$Y_n^{\pm m}(\hat{x}) = \sqrt{\frac{(n-m)!}{(n+m)!}}(-1)^m\frac{d^m}{d\hat{x}_3^m}P_n(\hat{x}_3)(\hat{x}_1 \pm i\hat{x}_2)^m.$$

Post model

Dividing in **nearfield** and **farfield** by a hierarchical structure.



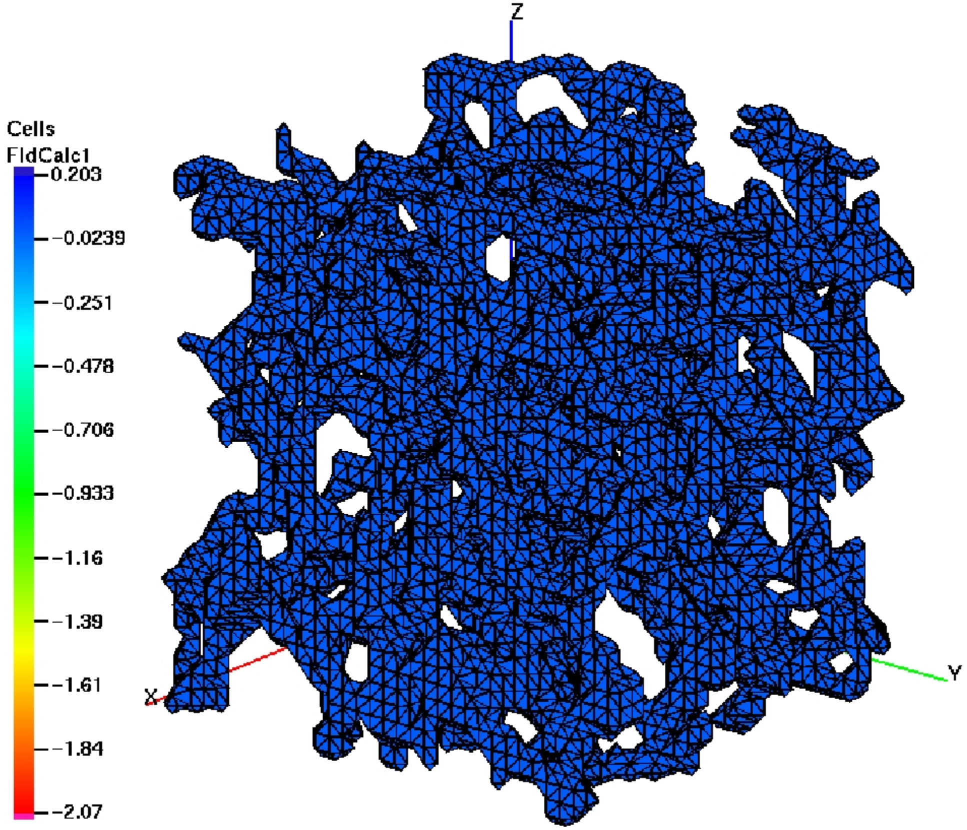
Properties of the FMM approximation

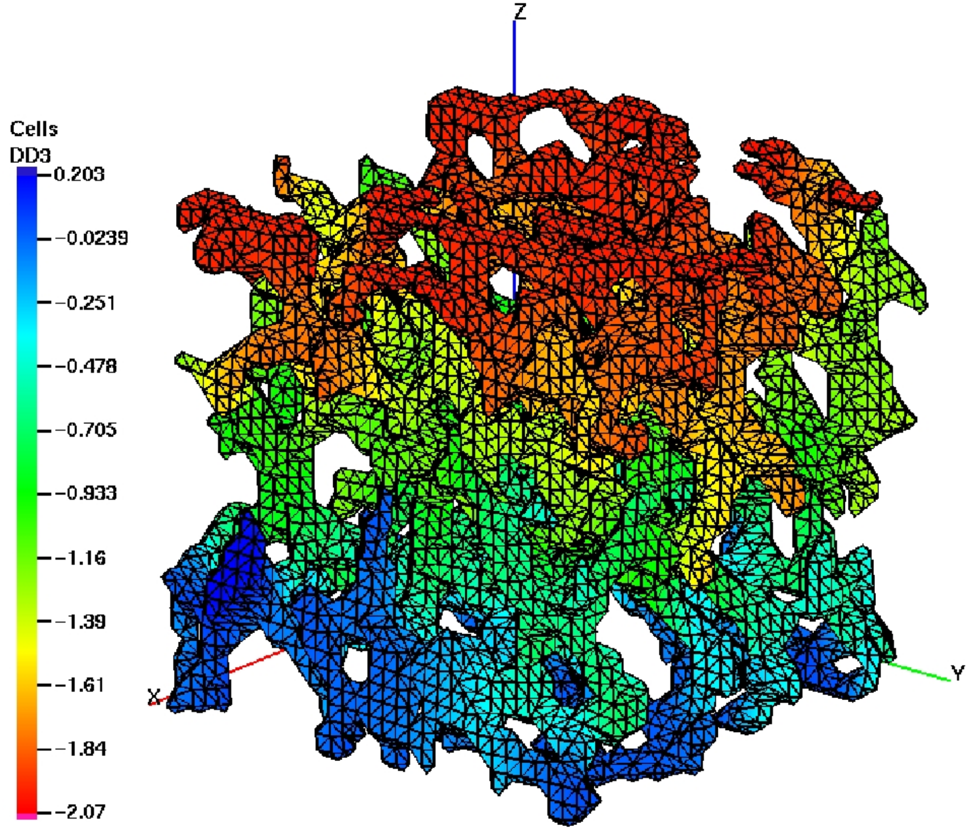
With an appropriate choice of the parameters of the multipole approximation:

- positive definiteness of the approximated matrices $\widetilde{V}_h$ and $\widetilde{D}_h$
- same (optimal) convergence rate as in the standard approach
- All the boundary integral operators in linear elastostatics are **reduced to** those of **the Laplacian**: Single and double layer potential are sufficient.
- **regularization** of boundary integral operators $\Rightarrow$ **less effort** on integration
- **symmetric implementation** of boundary integral operators
- Fast boundary element method with $\mathcal{O}(N \log^2 N)$ demand of time and memory, applicable to **complex problems of industrial interest**
- **preconditioning** by boundary integral operators and using hierarchical strategies

Example: foam

(H. Andrä, ITWM Kaiserslautern)

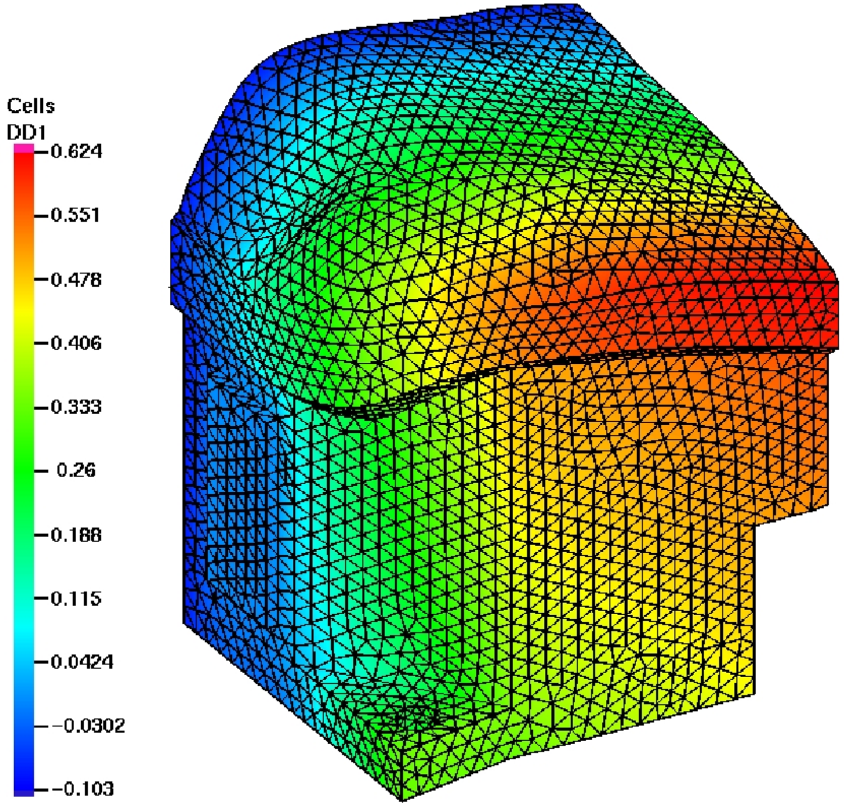


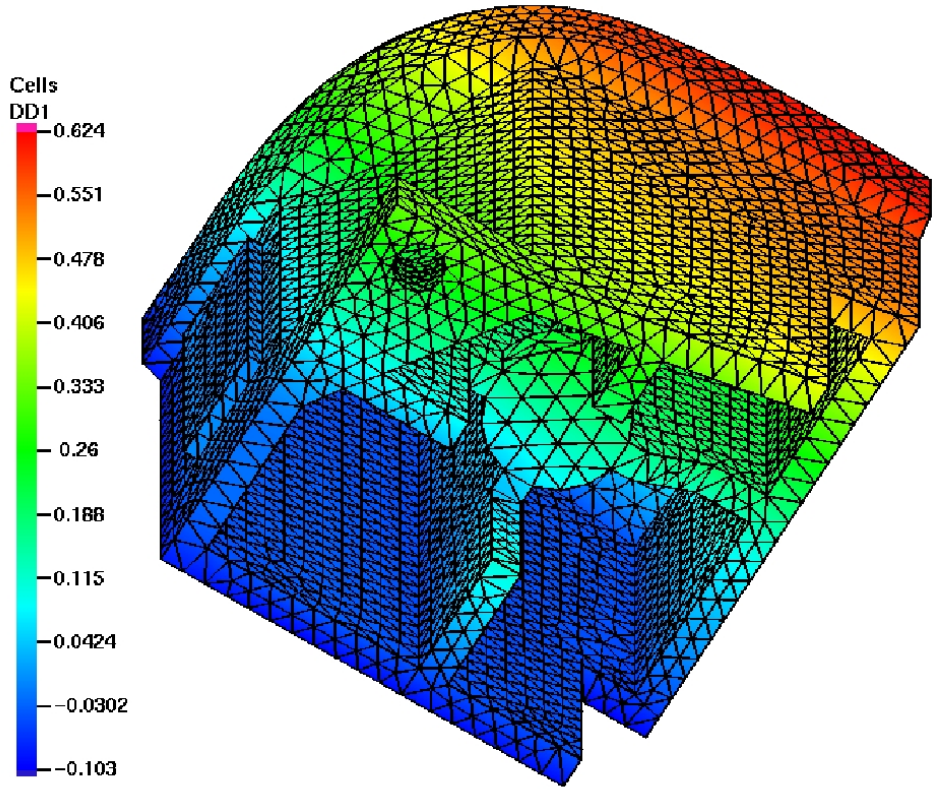


N	generation	solving	It
28952	0.7 h	7.3 h	246

Example: part of a machine

(W. Volk, M. Wagner, S. Wittig, BMW)

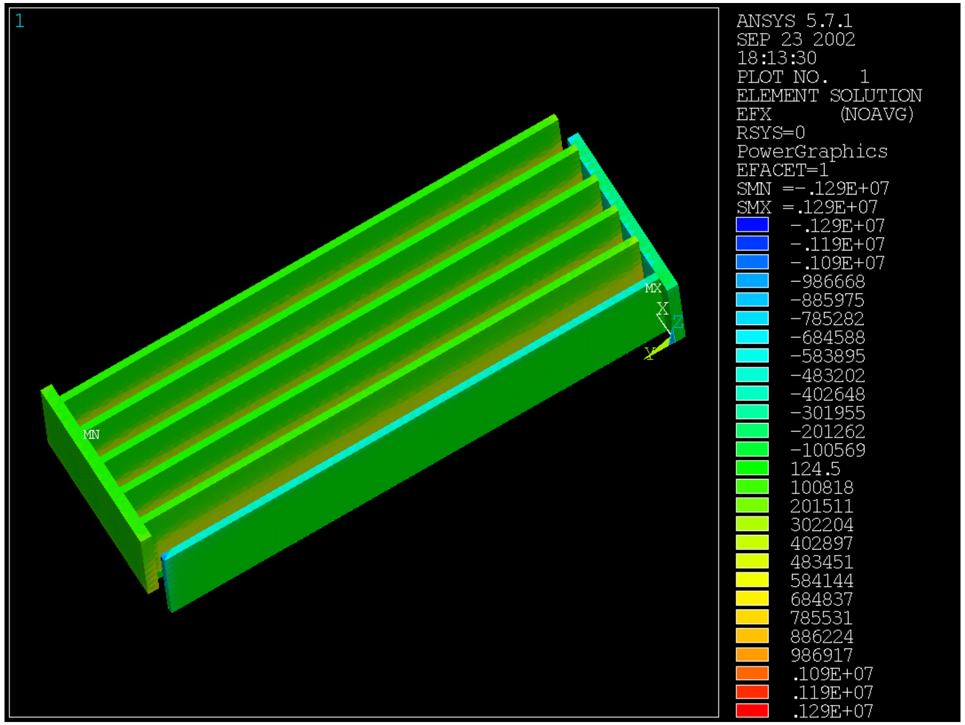


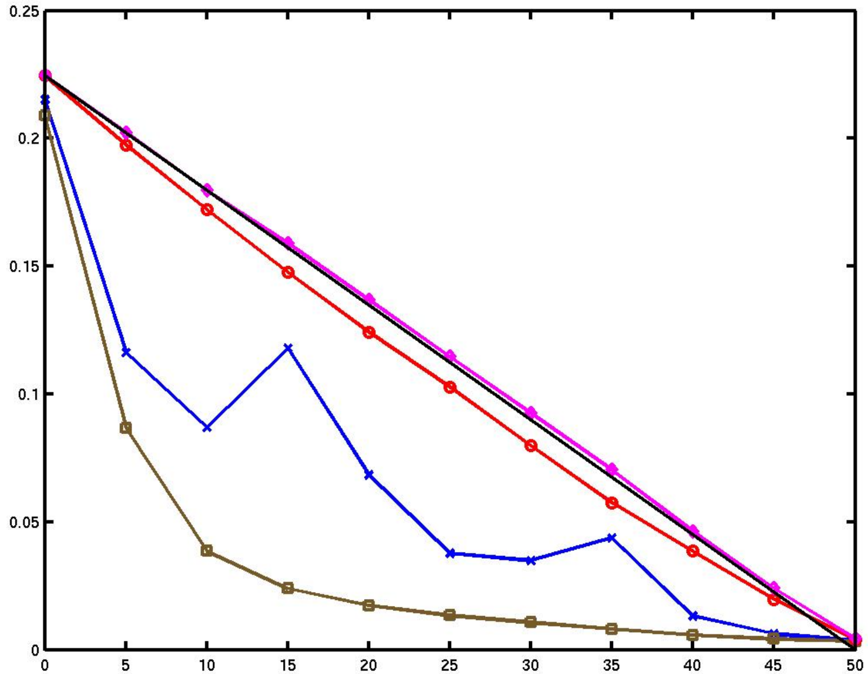


N	generation	solving	It
13144	28 min	2.35 h	342

Example: Capacitance

(M. Kaltenbacher, Universität Erlangen)



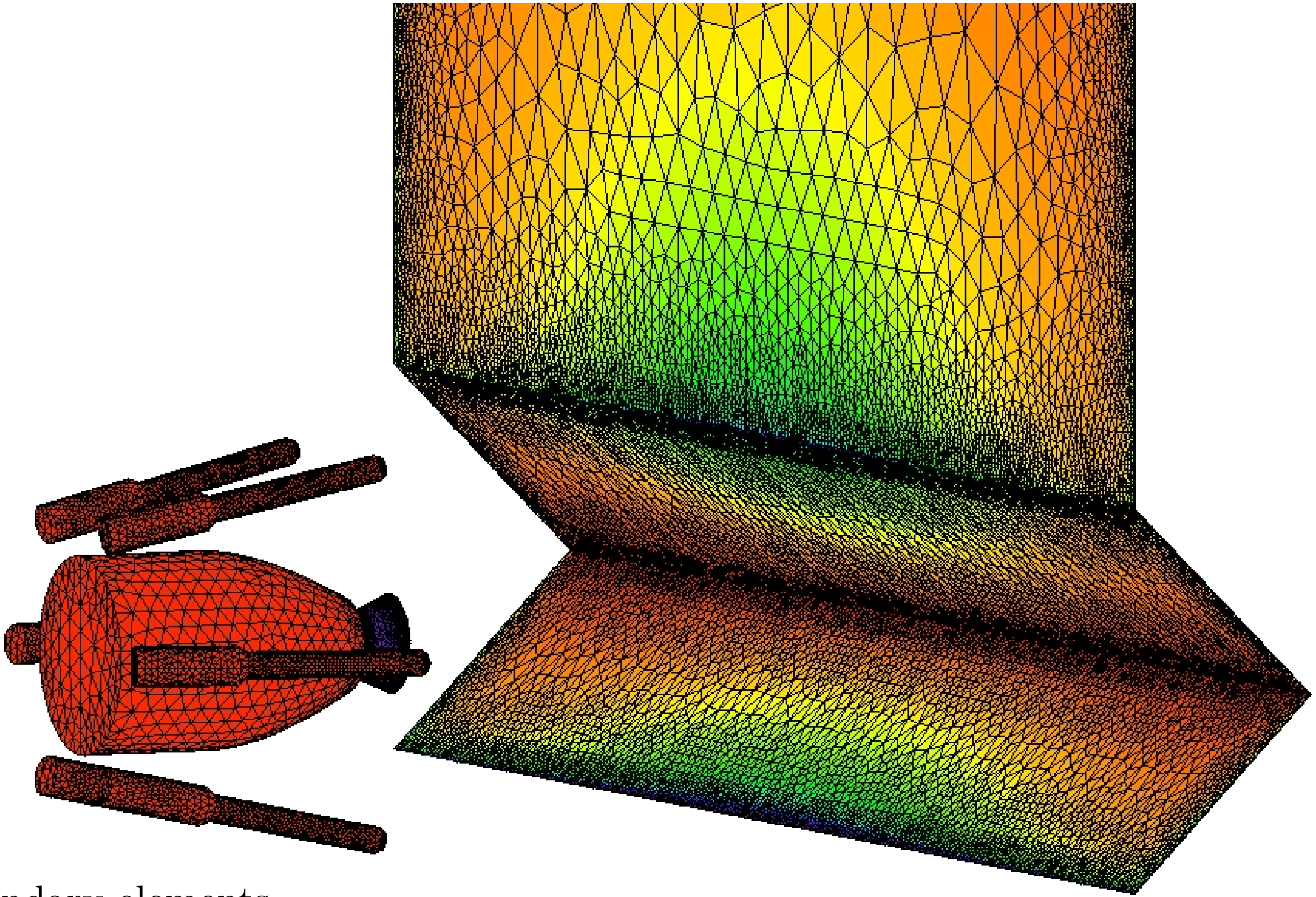


minimal distance distance between the fingers: 10^{-8}.

size of elements	number of elements
$5 \cdot 10^{-5}$	376
$2 \cdot 10^{-5}$	1112
$5 \cdot 10^{-6}$	9436
$2.5 \cdot 10^{-6}$	37744

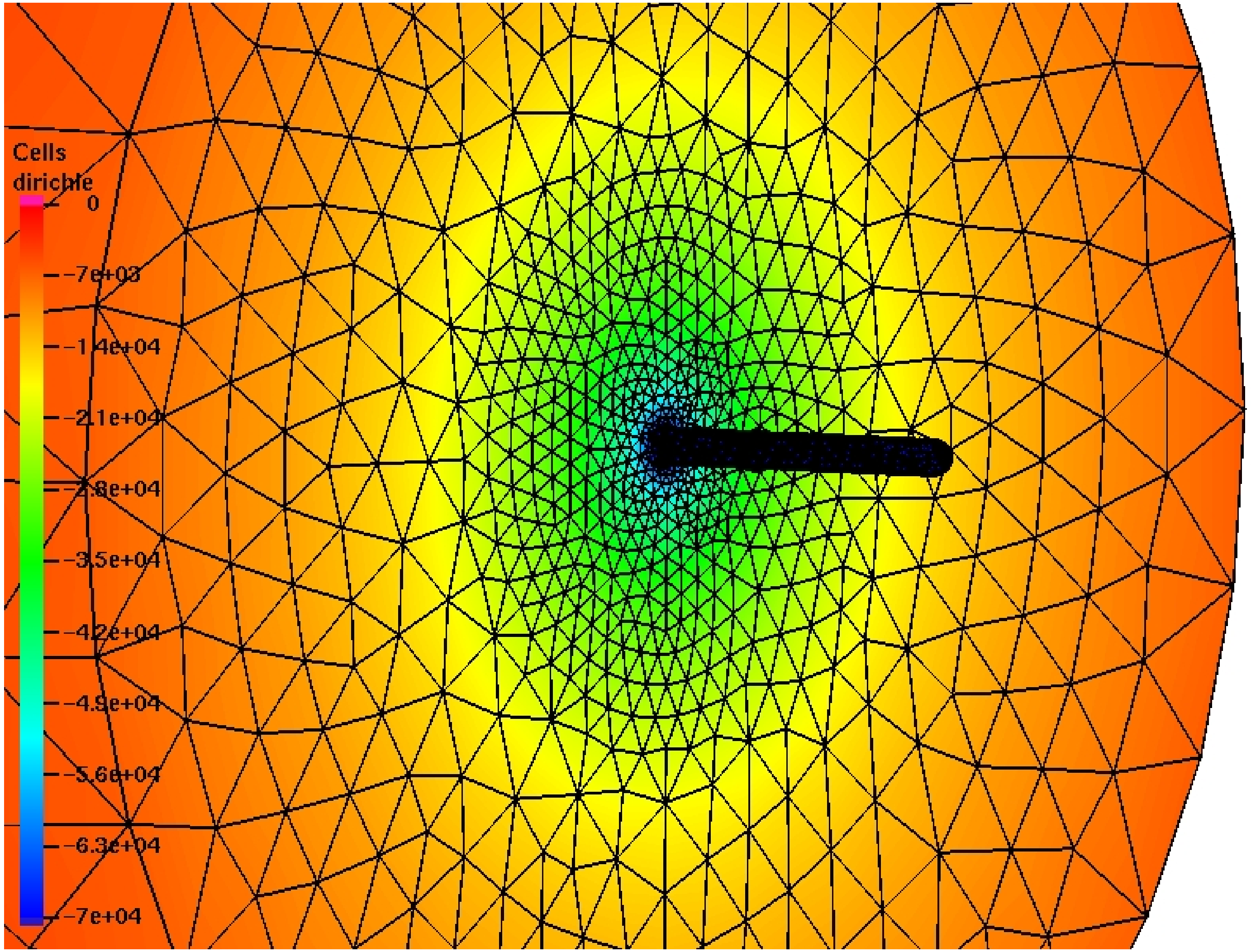
Example: spraying

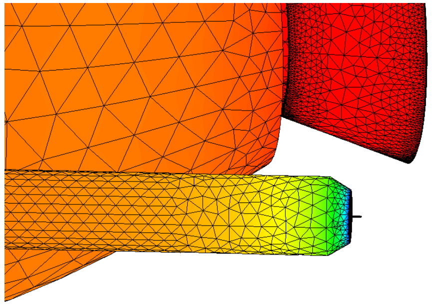
(R. Sonnenschein, Daimler Chrysler, Dornier)



112146 boundary elements.

Needles and adaptivity

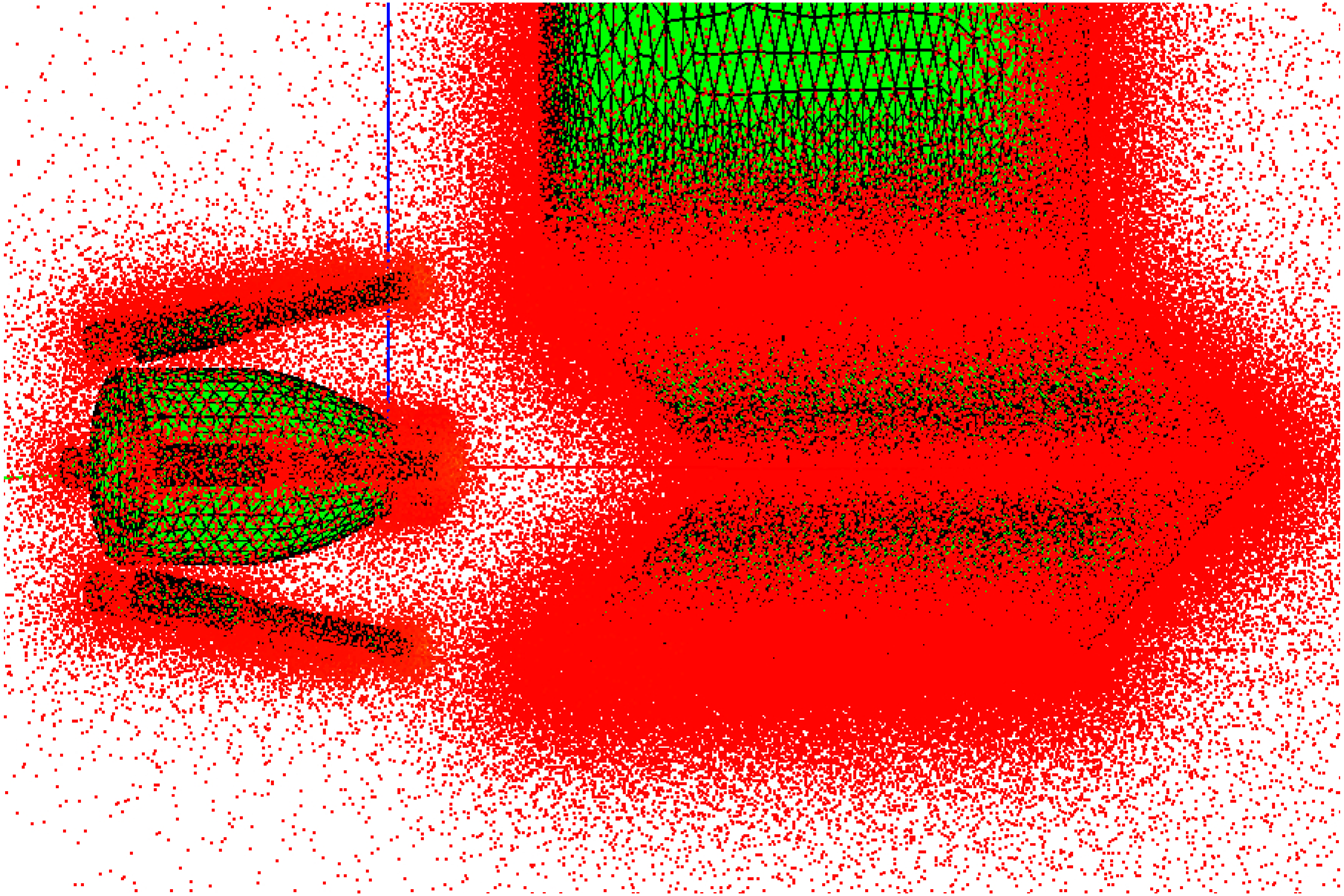




mesh ratio $\approx$ 1454,5.

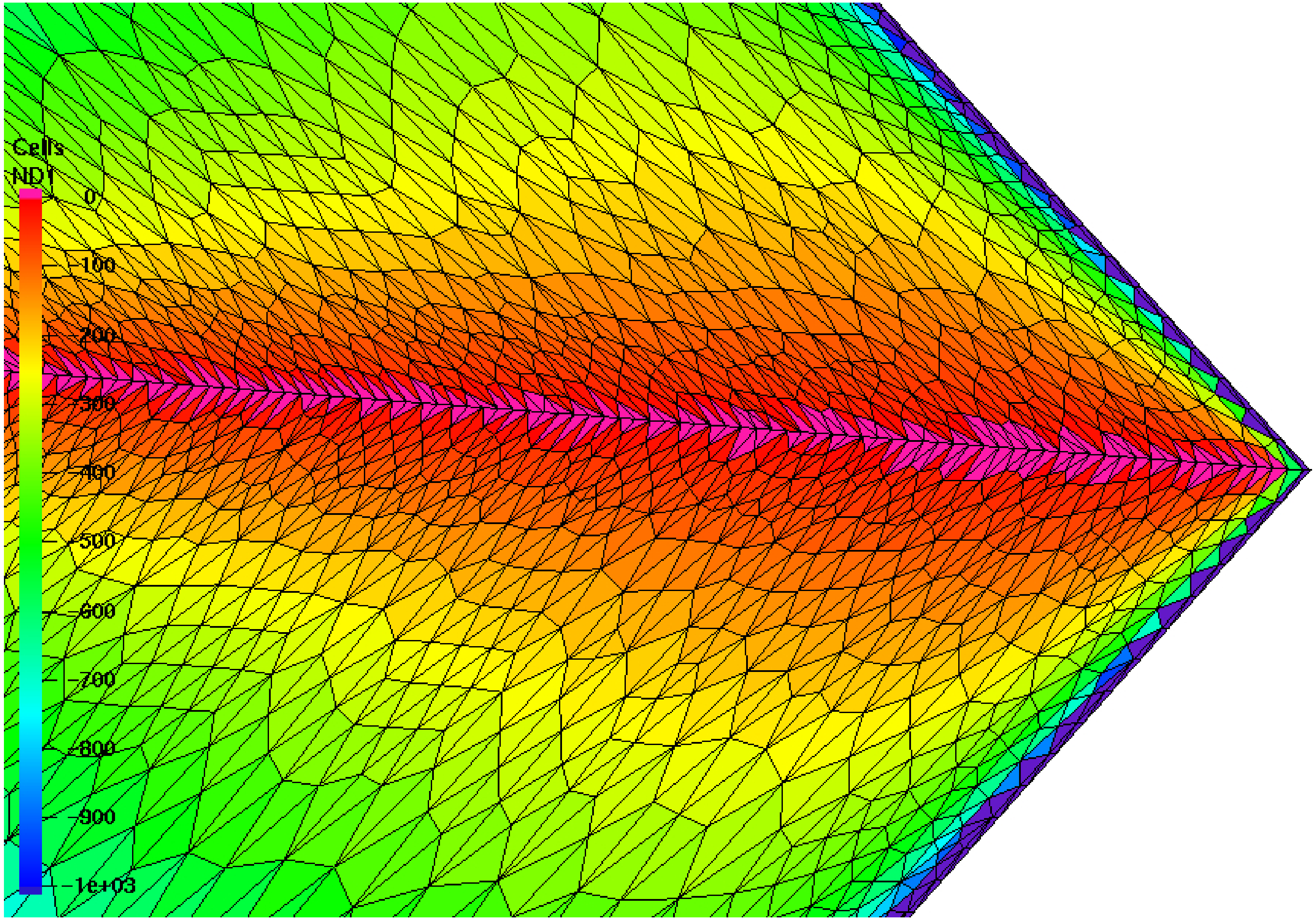
Field evaluation

in 570930 points or better interactive on demand. $\Longrightarrow$ Fast Multipole Methode



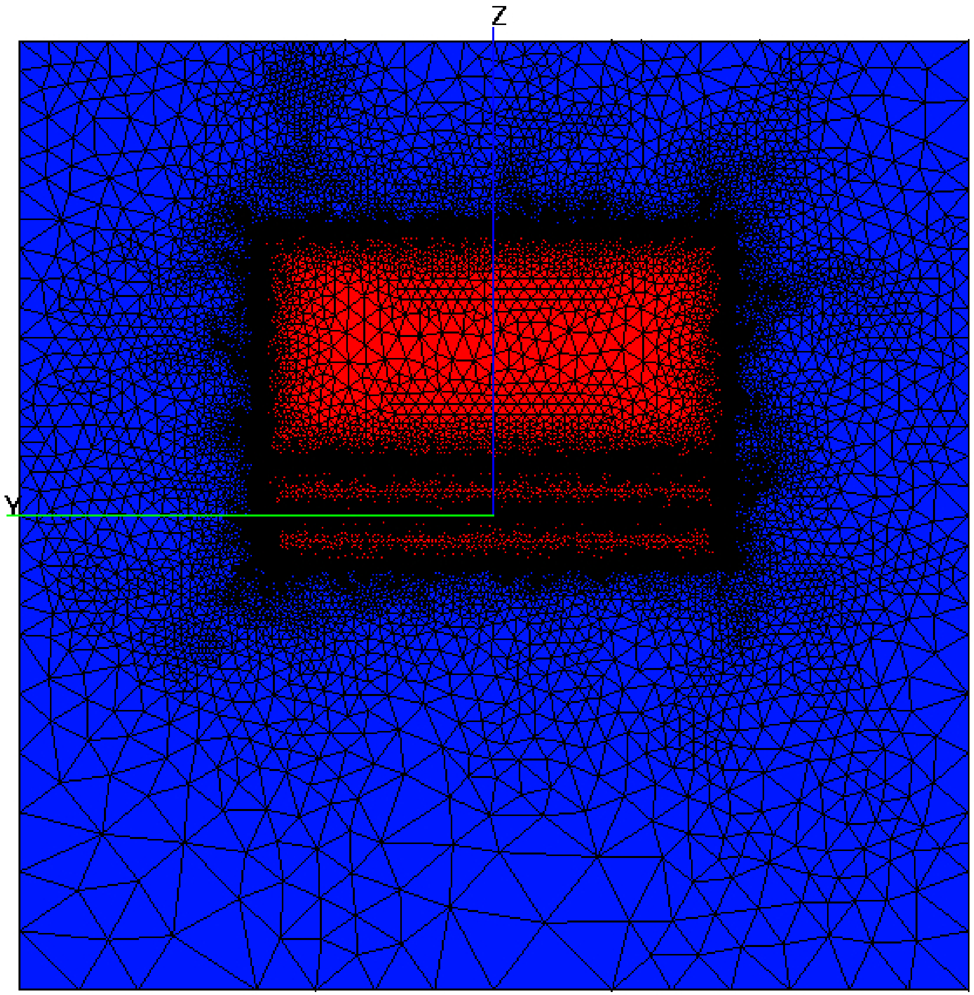
Thickness of the wall

0.8 mm, size of the wall about 1 m. data range: $-2.128 \cdot 10^5 \ldots 5.857 \cdot 10^8$

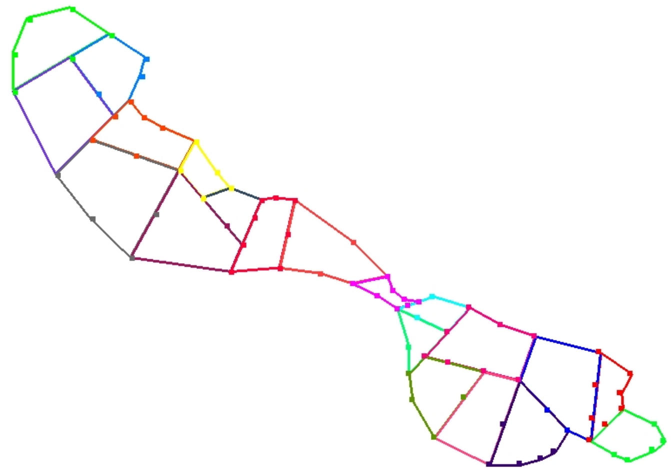


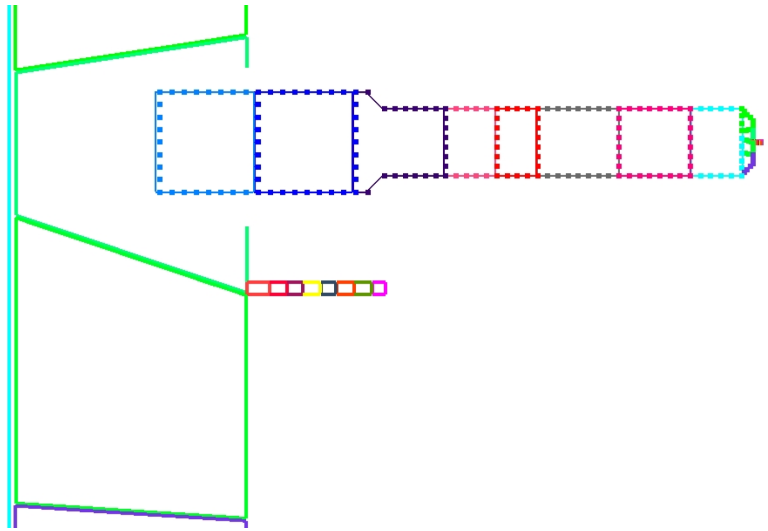
Domain Decomposition

create domain decomposition



automatic domain decomposition





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Domain Decomposition

- automatic domain decomposition
- preconditioners
- BETI
- parallel solvers