Discretisation of the double curl equation by discrete differential forms and collocation techniques

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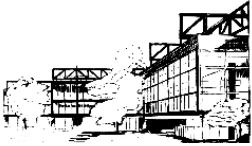
Outline



- Representation of electromagnetics by using differential forms
  - Maxwell‘s equations, constitutive laws, isomorphisms
- Generic curl-curl type problems
  - Double forms, fundamental solution, representation formula
- Boundary integral equation
  - De Rham map, (co-)homology
- Summary

Representation of Electromagnetic Fields and Potentials by Differential Forms

DF		Name, physical unit	
$\underline{\varphi}$	0-Form	Electric scalar potential	in V
$\underline{A}$	1-Form	Magnetic vector potential	in Vs
$\underline{E}$	1-Form	Electric field	in V
$\underline{H}$	1-Form	Magnetic field	in A
$\underline{D}$	2-Form	Electric flux density	in As
$\underline{B}$	2-Form	Magnetic flux density	in Vs
$\underline{j}$	2-Form	Electric current density	in A
$\underline{\rho}$	3-Form	Electric charge density	in As

Maxwell's Equations in Terms of Vector Fields and Differential Forms

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \mathrm{d}\,\underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\operatorname{div} \vec{B} = 0 \qquad \mathrm{d}\,\underline{B} = 0$$

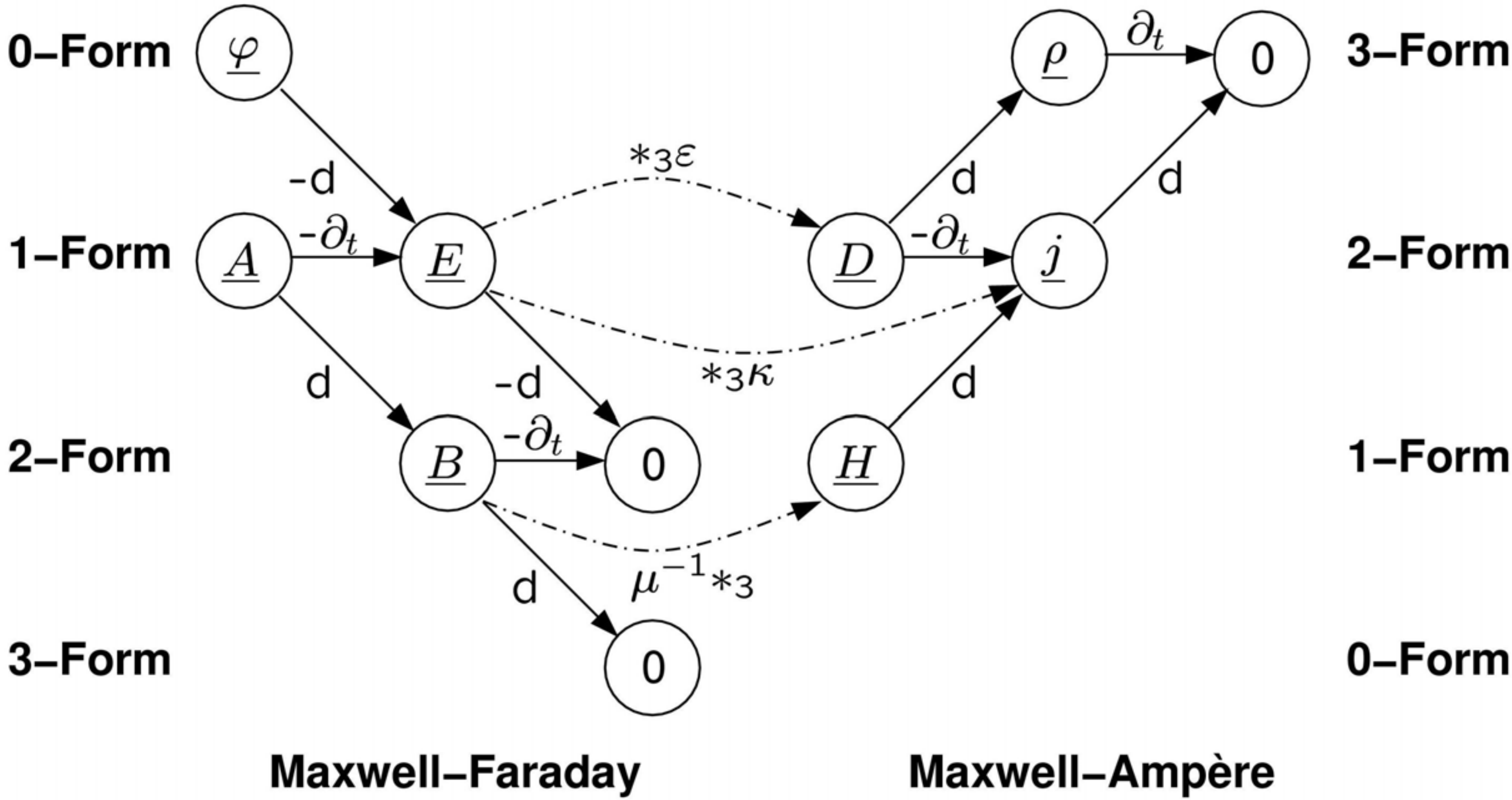
$$\operatorname{curl} \vec{H} = \vec{\jmath} + \frac{\partial \vec{D}}{\partial t} \qquad \mathrm{d}\,\underline{H} = \underline{j} + \frac{\partial \underline{D}}{\partial t}$$

$$\operatorname{div} \vec{D} = \rho \qquad \mathrm{d}\,\underline{D} = \underline{\rho}$$

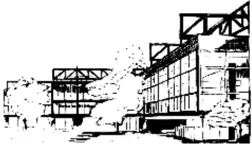
Expressing the Material Laws

Vector field representation	Differential form representation
$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ $\vec{D} = \varepsilon \vec{E}$	$\underline{D} = *\varepsilon_0 \underline{E} + \underline{P}$ $\underline{D} = *\varepsilon \underline{E}$
$\vec{B} = \mu_0 (\vec{H} + \vec{M})$ $\vec{B} = \mu \vec{H}$	$\underline{B} = *\mu_0 (\underline{H} + \underline{M})$ $\underline{B} = *\mu \underline{H}$
$\vec{j} = \kappa \vec{E} + \vec{j}_\mathrm{s}$	$\underline{j} = *\kappa \underline{E} + \underline{j}_\mathrm{s}$

Tonti Diagram



Translating Scalar and Vector Fields into Forms



Translating scalar and vector fields into differential forms by means of metric induced isomorphisms

p	Field	Associated DF
0	f	${}^0f = f$
1	$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$	${}^1\mathbf{a} = a_x \,\mathrm{d}x + a_y \,\mathrm{d}y + a_z \,\mathrm{d}z$
2	$\mathbf{b} = b_x \mathbf{e}_x + b_y \mathbf{e}_y + b_z \mathbf{e}_z$	${}^2\mathbf{b} = b_x \mathrm{d}y \wedge \mathrm{d}z + b_y \mathrm{d}z \wedge \mathrm{d}x + b_z \mathrm{d}x \wedge \mathrm{d}y$
3	g	${}^3g = g \,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

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- Generic second order curl-curl type equation

$$\delta \mathrm{d}\boldsymbol{\alpha} = *\boldsymbol{\eta} \quad \Leftrightarrow \quad \mathrm{d}*\mathrm{d}\,\boldsymbol{\alpha} = (-1)^{p+1}\boldsymbol{\eta} \qquad \text{in} \quad V \subset E_3,$$

$$\boldsymbol{\alpha} \in \mathcal{F}^p(V), \quad \boldsymbol{\eta} \in \mathcal{F}^{3-p}(V), \quad p = 0, 1, 2$$

d = exterior derivative
δ = co-derivative, $\delta\omega = (-1)^{\deg \omega} * \mathrm{d} * \omega$
$*$ = Hodge operator of the Euclidean metric

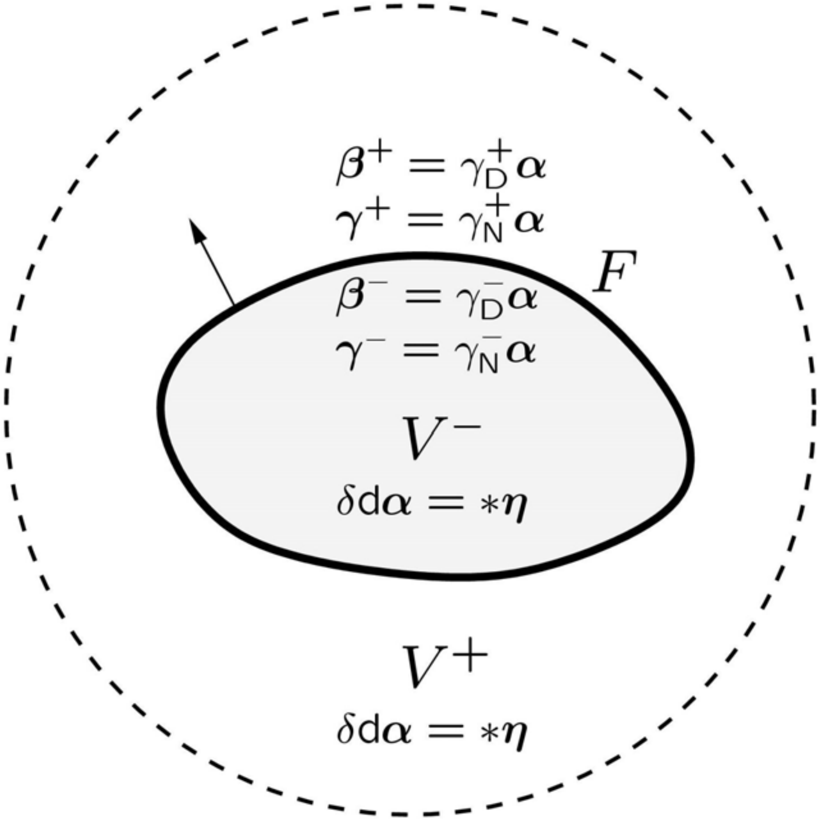
- Representation of boundary conditions by trace operators

Dirichlet trace $\quad \gamma_\mathrm{D} : \mathcal{F}^p(V) \to \mathcal{F}^p(F) \quad : \boldsymbol{\alpha} \mapsto \boldsymbol{\beta} = \mathrm{t}\boldsymbol{\alpha}$

Neumann trace $\quad \gamma_\mathrm{N} : \mathcal{F}^p(V) \to \mathcal{F}^{2-p}(F) : \boldsymbol{\alpha} \mapsto \boldsymbol{\gamma} = \mathrm{t}*\mathrm{d}\,\boldsymbol{\alpha}$

t = standard trace operator

p	$\boldsymbol{\alpha}$	$\boldsymbol{\eta}$	$\delta \mathrm{d}\boldsymbol{\alpha} = *\boldsymbol{\eta}$	$\gamma_{\mathrm{D}}\boldsymbol{\alpha} = \boldsymbol{\beta}$ $\gamma_{\mathrm{N}}\boldsymbol{\alpha} = \boldsymbol{\gamma}$
0	${}^0\varphi$	$\frac{1}{\varepsilon_0}{}^3\rho$	$\Delta\varphi = -\frac{1}{\varepsilon_0}\rho$	$\gamma_{\mathrm{D}}\left({}^0\varphi\right) = {}^0\left(\varphi\big\vert_F\right)$ $\gamma_{\mathrm{N}}\left({}^0\varphi\right) = -\frac{1}{\varepsilon_0}{}^2\left(D_{\mathrm{n}}\big\vert_F\right)$
1	${}^1\vec{A}$	$\mu_0\,{}^2\vec{J}$	$\mathrm{curl}\,\mathrm{curl}\,\vec{A} = \mu_0\vec{J}$	$\gamma_{\mathrm{D}}\left({}^1\vec{A}\right) = {}^1\left(\vec{A}_{\mathrm{t}}\big\vert_F\right)$ $\gamma_{\mathrm{N}}\left({}^1\vec{A}\right) = \mu_0\,{}^1\left(\vec{H}_{\mathrm{t}}\big\vert_F\right)$



$$V = V^- \cup V^+$$
$$F = \partial V^-$$

V^- = interior domain, Lipschitz curvilinear polyhedron
V^+ = exterior domain
F = piecewise smooth boundary

$$[\gamma \cdot]_F = \gamma^+ \cdot - \gamma^- \cdot$$

$\gamma^- \cdot$ = interior trace
$\gamma^+ \cdot$ = exterior trace
$[\gamma \cdot]_F$ = jump of some trace across F

- Necessary conditions for existence of solutions

$$\text{Apply d}: \quad \mathrm{d}\boldsymbol{\eta} = 0 \qquad\qquad p = 1: \quad \operatorname{div}\vec{\jmath} = 0$$

$$\text{Apply } \mathrm{t}^-: \quad \mathrm{d}\gamma = (-1)^{p+1}\,\mathrm{t}^-\boldsymbol{\eta} \qquad\qquad \oint_{\Gamma=\partial\Omega} \vec{H}\cdot\mathrm{d}\vec{\Gamma} = \int_{\Omega\subset F} \hat{\mathbf{n}}\cdot\vec{\jmath}\,\mathrm{d}\Omega$$

- Subsequently we will assume $\mathrm{t}^-\boldsymbol{\eta} = 0$, therefore γ must be **closed**,

$$\mathrm{d}\gamma = 0$$

- Definition of the space of closed p-forms

$$\mathcal{F}^p(\mathrm{d}_0, X) = \left\{\boldsymbol{\omega} \,\middle|\, \boldsymbol{\omega} \in \mathcal{F}^p(X), \mathrm{d}\boldsymbol{\omega} = 0\right\}$$

$$\rightarrow \quad \boldsymbol{\eta} \in \mathcal{F}^{3-p}(\mathrm{d}_0, V^-), \qquad \gamma \in \mathcal{F}^{2-p}(\mathrm{d}_0, F)$$

Double Forms (1)

- **Double forms** are forms in one space with coefficients that are forms in another space, or DF-valued DFs.

- The double forms to be used here are associated with $E_3' \times E_3$.

  E_3' = observation space $\quad$ $\mathbf{x}'$ = observation co-ordinates
  E_3 = source space $\quad$ $\mathbf{x}$ = source co-ordinates

- A double form $\boldsymbol{K}_p(\mathbf{x}, \mathbf{x}')$ can be used as a transformation kernel

$$\mathcal{F}^p(E_3) \to \mathcal{F}^p(E_3') : \quad \boldsymbol{\omega}(\mathbf{x}) \mapsto \boldsymbol{\omega}'(\mathbf{x}') = \int_{E_3} \boldsymbol{K}_p(\mathbf{x}, \mathbf{x}') \wedge *\boldsymbol{\omega}(\mathbf{x})$$

see: G. de Rham, *Differentiable Manifolds*, Springer, 1984, pp. 30-33.

- The **identity kernel** $\boldsymbol{\delta}_p(\mathrm{x}, \mathrm{x}')$ has the property

$$\boldsymbol{\omega}'(\mathrm{x}') = \int_{E_3} \boldsymbol{\delta}_p(\mathrm{x}, \mathrm{x}') \wedge *\boldsymbol{\omega}(\mathrm{x}) = \boldsymbol{\omega}(\mathrm{x}')$$

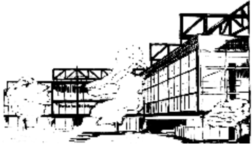
$$\boldsymbol{\delta}_p(\mathrm{x}, \mathrm{x}') = \delta(\mathrm{x}, \mathrm{x}')\boldsymbol{I}_p$$

$\delta(\mathrm{x}, \mathrm{x}')$ = three-dimensional Dirac delta distribution

- In Cartesian co-ordinates

$$\begin{aligned}
\boldsymbol{I}_0 &= 1 \\
\boldsymbol{I}_1 &= \mathrm{d}x\,\mathrm{d}x' + \mathrm{d}y\,\mathrm{d}y' + \mathrm{d}z\,\mathrm{d}z' \\
\boldsymbol{I}_2 &= (\mathrm{d}x\wedge\mathrm{d}y)(\mathrm{d}x\wedge\mathrm{d}y)' + (\mathrm{d}y\wedge\mathrm{d}z)(\mathrm{d}y\wedge\mathrm{d}z)' + (\mathrm{d}z\wedge\mathrm{d}x)(\mathrm{d}z\wedge\mathrm{d}x)' \\
\boldsymbol{I}_3 &= (\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z)(\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z)'
\end{aligned}$$

Fundamental Solution



- Starting point: Fundamental solution of the scalar Laplace equation

$$g(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}$$

- Consider the Laplace-Beltrami operator

$$\boldsymbol{\Delta} : \mathcal{F}^p(V) \to \mathcal{F}^p(V), \qquad \boldsymbol{\Delta} = \delta \circ \mathrm{d} + \mathrm{d} \circ \delta$$

- Fundamental solution

$$\boldsymbol{G}_p(\mathbf{x}, \mathbf{x}') = g(\mathbf{x}, \mathbf{x}')\boldsymbol{I}_p \qquad \to \qquad \boldsymbol{\Delta G}_p(\mathbf{x}, \mathbf{x}') = \boldsymbol{\delta}_p(\mathbf{x}, \mathbf{x}')$$

- Useful properties of $\boldsymbol{G}_p(\mathbf{x}, \mathbf{x}')$

$$\left.\begin{aligned} \mathrm{d}'\boldsymbol{G}_p(\mathbf{x}, \mathbf{x}') &= \delta\, \boldsymbol{G}_{p+1}(\mathbf{x}, \mathbf{x}') \\ \mathrm{d}\, \boldsymbol{G}_p(\mathbf{x}, \mathbf{x}') &= \delta' \boldsymbol{G}_{p+1}(\mathbf{x}, \mathbf{x}') \end{aligned}\right\} \quad \mathbf{x} \neq \mathbf{x}', \quad p = 0, 1, 2$$

Representation Formula

- From Green's theorem for $\mathbf{x}' \in V^-$

$$\boldsymbol{\alpha}' = \underbrace{\int_F (\gamma_{\mathsf{D}}^- \boldsymbol{G}_p) \wedge \boldsymbol{\gamma}}_{\Psi_{\mathsf{SL},p}(\boldsymbol{\gamma})} - \underbrace{(-1)^p \int_F (\gamma_{\mathsf{N}}^- \boldsymbol{G}_p) \wedge \boldsymbol{\beta}}_{\Psi_{\mathsf{DL},p}(\boldsymbol{\beta})} + \underbrace{\int_{V^-} \boldsymbol{G}_p \wedge \boldsymbol{\eta}}_{\Psi_{\mathsf{Newton},p}(\boldsymbol{\eta})}$$

$$+ \mathsf{d}' \left(\Psi_{\mathsf{SL},p-1}(\boldsymbol{\varphi}) + \int_{V^-} \boldsymbol{G}_{p-1} \wedge *\delta\boldsymbol{\alpha} \right)$$

$$\widetilde{\gamma_{\mathsf{N}}}^- : \mathcal{F}^p(V^-) \to \mathcal{F}^{3-p}(F) : \boldsymbol{\alpha} \mapsto \boldsymbol{\varphi} = \mathbf{t}^- * \boldsymbol{\alpha} \qquad \widetilde{\gamma_{\mathsf{N}}}^- ({}^1\vec{A}) = {}^2\big(A_{\mathsf{n}}\big|_F\big)$$

- Encompasses the usual Kirchhoff (p=0) and Stratton-Chu (p=1) representation formulas

$$\varphi'(\mathbf{x}') = \int\limits_F \left(g \frac{\partial \varphi}{\partial n} - \frac{\partial g}{\partial n} \varphi \right) \mathrm{d}\, F + \frac{1}{\varepsilon_0} \int\limits_{V^-} g\, \rho\, \mathrm{d}\, V$$

$$\vec{A}'(\mathbf{x}') = \int\limits_F g\, \hat{\mathbf{n}} \times \mu_0 \vec{H}\, \mathrm{d}\, F + \mathrm{curl}' \int\limits_F g\, \hat{\mathbf{n}} \times \vec{A}\, \mathrm{d}\, F + \mu_0 \int\limits_{V^-} g\, \vec{\jmath}\, \mathrm{d}\, V$$

$$- \mathrm{grad}' \left(\int\limits_F g\, \hat{\mathbf{n}} \cdot \vec{A}\, \mathrm{d}\, F + \int\limits_{V^-} g\, \mathrm{div}\, \vec{A}\, \mathrm{d}\, V \right)$$

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- Consider $\boldsymbol{\eta} = 0$ (no sources) and $\delta\boldsymbol{\alpha} = 0$ (Coulomb gauge) for simplicity.
- Boundary integral equation from representation formula by applying the trace operator ${\gamma_{\mathrm{D}}^{-}}'$

$$\left(\frac{\ominus^{-}}{4\pi}\mathcal{I}_p + \mathcal{K}_p\right)\boldsymbol{\beta} = \mathcal{V}_p\gamma + \mathrm{d}'\,\mathcal{V}_{p-1}\varphi$$

- How to get rid of the additional Neumann data φ?

problem type	Dirichlet	Neumann
prescribed data	$\boldsymbol{\beta}$	γ, φ
unknown data	γ, φ	$\boldsymbol{\beta}$
elimination of φ	evaluation in $\mathcal{F}^p(\mathrm{d}_0, F)^{\perp}$	$\varphi = 0$

Solution Techniques: de Rham Map

- Consider the space $\mathcal{C}_p(F)$ of p-dimensional integration domains in F, i.e. p-chains, and the space of closed p-chains, i.e. p-cycles

$$\mathcal{C}_p(\partial_0, F) = \left\{ \Gamma \in \mathcal{C}_p(F) \,\middle|\, \partial\Gamma = 0 \right\}.$$

- Introduce the map $\mathcal{P}_{\mathsf{deRham},p}$:

$$\mathcal{F}^p(F) \times \mathcal{C}_p(F) \to \mathbb{R} : \quad (\boldsymbol{\beta}, \Gamma) \mapsto \boldsymbol{\beta}|\Gamma = \int_\Gamma \boldsymbol{\beta}$$

- $\mathcal{P}_{\mathsf{deRham},p}(\mathsf{d}\,\boldsymbol{\lambda}, \Gamma) = 0 \quad \forall \quad \boldsymbol{\lambda} \in \mathcal{F}^{p-1}(F), \Gamma \in \mathcal{C}_p(\partial_0, F)$, since

$$\mathsf{d}\,\boldsymbol{\lambda}|\Gamma = \boldsymbol{\lambda}|\partial\Gamma = 0 \quad \text{by Stokes theorem}$$

- Application of $\mathcal{P}_{\mathrm{deRham},p}$ to the boundary integral equation yields the "collocation" formulation for the Dirichlet problem

For a given $\beta \in \mathcal{F}^p(F)$ *find* $\gamma \in \mathcal{F}^{2-p}(\mathrm{d}_0, F)$ *such that*

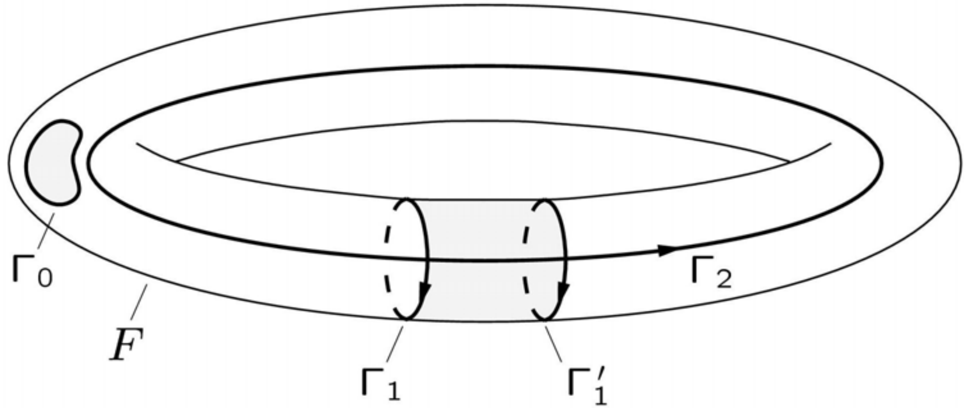
$$\mathcal{V}_p\gamma\big|\Gamma' = \left(\tfrac{\Theta^-}{4\pi}\mathcal{I}_p + \mathcal{K}_p\right)\beta\big|\Gamma' \quad \forall\Gamma' \in \mathcal{C}_p(\partial_0, F')$$

- Electromagnetic interpretation

$p = 0$	Laplace equation	The boundary integral equation is enforced in each point of the boundary.
$p = 1$	curl-curl equation	The boundary integral equation is enforced w.r.t. the magnetic flux through arbitrary cycles which are contained in the boundary.

- Each cycle is either a bounding cycle or a member of the homology group.

  Example: Torus



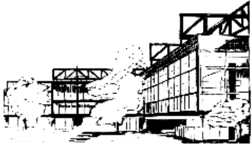
  Γ_0 bounding cycle
  $\Gamma_{1,2}$ members of $\mathcal{H}_1(F)$

  $$\mathcal{C}_p(\partial_0, F) = \partial\mathcal{C}_{p+1}(F) \oplus \mathcal{H}_p(F)$$

- De Rham theorem for DFs: Each closed form can be decomposed into an exact form and a member of the cohomolgy group

  $$\mathcal{F}^{2-p}(\mathrm{d}_0, F) = \mathrm{d}\,\mathcal{F}^{1-p}(F) \oplus \mathcal{H}^{2-p}(F)$$

- Simplest case: Trivial topology $\rightarrow$ let $\gamma = \mathrm{d}\,\omega, \quad \Gamma' = \partial'\Omega'$



- Electromagnetic interpretation

p	$\gamma \quad = \mathrm{d}\omega$	interpretation
1	$\vec{H}_{\mathrm{t}} = -\,\mathrm{grad}_S\,\varphi^*$	magnetic surface scalar potential (mandatory)
0	$D_{\mathrm{n}} = -\,\mathrm{curl}_S\,\vec{A}_{\mathrm{t}}^*$	electric surface vector potential (optional)

- The remaining problem reads

For a given $\beta \in \mathcal{F}^p(F)$ *find* $\omega \in \mathcal{F}^{1-p}(F)$ *such that*

$$\mathcal{V}_p\,\mathrm{d}\omega\big|\partial'\Omega' = \Big(\frac{\ominus^-}{4\pi}\mathcal{I}_p + \mathcal{K}_p\Big)\beta\big|\partial'\Omega' \quad \forall\Omega' \in \mathcal{C}_{p+1}(F')$$

- Go for a discretisation of $\mathcal{F}^{1-p}(F)$ and $\mathcal{C}_{p+1}(F)$:

  Needs triangulation $\mathcal{T}_h$ of the boundary F.

- The laws of electromagnetics can be stated concisely by means of differential forms (DFs)
- Some integral equations of electromagnetics have been reformulated in terms of DFs
  - The integral kernels become double forms
- Uniform treatment of electro- (p=0) and magnetostatics (p=1)
- Since DFs possess discrete counterparts, the discrete DFs, such schemes lend themselves naturally to discretisation…