Wavelet based matrix compression for boundary integral equations on complex geometries

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Overview

- Motivation - presentation of problem

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- Wavelet basis - stiffness matrix

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- Numerical results

Preliminaries

$$A\rho = f \quad \text{on } \Gamma = \partial\Omega \subset \mathbb{R}^2$$

$$A : H^q(\Gamma) \to H^{-q}(\Gamma) \quad (A\rho)(x) = \int_\Gamma k(x,y)\rho(y)\partial\Gamma_y$$

- single layer potential: $q = -\frac{1}{2}$, $A = K$

$$K = -\frac{1}{2\pi} \int_\Gamma \log|y - x| \rho(y) \partial\Gamma_y$$

- double layer potential: $q = 0$, $A = -\frac{1}{2} + K$

$$K = -\frac{1}{2\pi} \int_\Gamma \frac{\langle n(y), y - x \rangle}{|y - x|^2} \rho(y) \partial\Gamma_y$$

Galerkin scheme

- Variational formulation: find $\rho \in H^q(\Gamma)$: $(A\rho, v)_{L^2(\Gamma)} = (f, v)_{L^2(\Gamma)} \quad \forall v \in H^q(\Gamma)$
- $V_N = \text{span}\{\phi_1, ..., \phi_N\} \subseteq H^q(\Gamma)$

$$\Rightarrow A^\Phi \rho^\Phi = f^\Phi$$

Geometry

- Γ_N - polygonial approximations of the surface Γ

  $\Rightarrow$ finest level is fixed

- $\text{diam}(\Omega) < 1$

- ansatzfunctions:

$$\phi_i(x) = \begin{cases} \frac{1}{\sqrt{\int_{\Gamma_i} \partial\Gamma}}, & \text{for } x \in \Gamma_i \ , \\ 0, & \text{else} \end{cases}$$

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$$\phi_i(x(s)) = \begin{cases} \frac{s}{\sqrt{\frac{1}{3}\int_{\Gamma_{i-1}} \partial\Gamma + \frac{1}{3}\int_{\Gamma_i} \partial\Gamma}} & \text{for } x(s) \in \Gamma_{i-1}, \\ \frac{1-s}{\sqrt{\frac{1}{3}\int_{\Gamma_{i-1}} \partial\Gamma + \frac{1}{3}\int_{\Gamma_i} \partial\Gamma}} & \text{for } x(s) \in \Gamma_i, \\ 0 & \text{else} \end{cases}$$

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Objectives

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- Dahmen-Prößdorf-Schneider/ von Petersdorff-Schwab: A^ψ is a quasi sparse matrix with $O(N\log(N))$ entries.

- idea:

$$\int_\Gamma \int_\Gamma k(x,y)\psi_k(x)\psi_{k'}(y)\partial\Gamma_x\partial\Gamma_y$$

$$= \sum_{(\alpha,\beta)\in\mathbb{N}_0^2\times N_0^2} (D^{\alpha+\beta}k)(x_0,y_0)$$

$$\frac{\int_\Gamma \psi_k(x)(x-x_0)^\alpha\partial\Gamma_x}{\alpha!}\frac{\int_\Gamma \psi_{k'}(y)(y-y_0)^\beta\partial\Gamma_y}{\beta!}$$

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$$\left|D^{\alpha+\beta}k(x_0,y_0)\right| \leq C\left(\frac{1}{\|x_0-y_0\|}\right)^{\alpha+\beta+1-2q}$$

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- $\begin{bmatrix} \Phi^{\nu,j-1} \\ \Psi^{\nu,j-1} \end{bmatrix} = \begin{bmatrix} V_0^{\nu,j-1} \\ V_0^{\nu,j-1} \end{bmatrix} \Phi^{\nu,j}$

- Let $M^{\nu,j-1}$ be the moment matrix of the cluster ν from level $j-1$

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- constant/linear ansatzfunctions $\Rightarrow$ Ψ is orthonormal/ Riesz-basis.

- complexity of computing cluster tree and wavelets:

$$O(N)$$

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- calculation of f^{Ψ} in $O(N)$ possible

$$\Omega^{\Psi,\Phi^\top} = \begin{pmatrix} \cdots & \psi^{\nu_1^2} & 0 & 0 & 0 & \psi^{\nu_1^1} & 0 & \psi^{\nu_1^0} \\ \cdots & 0 & \psi^{\nu_2^2} & 0 & 0 & \vdots & 0 & \vdots \\ \cdots & 0 & 0 & \psi^{\nu_3^2} & 0 & 0 & \psi^{\nu_2^1} & \vdots \\ \cdots & 0 & 0 & 0 & \psi^{\nu_4^2} & 0 & \vdots & \vdots \end{pmatrix}$$

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$\Rightarrow O(\log(N))$ columns

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- H.Harbecht: Error estimates for entries of A^{Ψ}

Numerical results

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- improved combination of multipole and wavelets