Coercive Combined Field Integral Equations

Ralf Hiptmair

Seminar für Angewandte Mathematik
ETH Zürich

(e–mail: ralf@hiptmair.de)

(Homepage: http://www.sam.math.ethz.ch/˜hiptmair)

joint work with A. Buffa, Pavia

Coercive Variational Problems

Coercivity

V = $\mathbb{C}$-Banach space with dual space V', duality pairing $\langle\cdot,\cdot\rangle$.

Definition:

Linear operator $A : V \mapsto V'$ coercive, if it satisfies a Gårding-type inequality

$$\exists c > 0: \quad |\langle Av, \overline{v}\rangle + \langle Kv, \overline{v}\rangle| \geq c\,\|v\|_V^2 \quad \forall v \in V\ .$$

for some compact operator $K : V \mapsto V'$.

$\longrightarrow$ Coercivity of bilinear forms $V \times V \mapsto \mathbb{C}$

Coercivity

V = $\mathbb{C}$-Banach space with dual space V', duality pairing $\langle \cdot, \cdot \rangle$.

Definition:

Linear operator $A : V \mapsto V'$ coercive, if it satisfies a Gårding-type inequality

$$\exists c > 0: \quad |\langle Av, \overline{v}\rangle + \langle Kv, \overline{v}\rangle| \geq c \, \|v\|_V^2 \quad \forall v \in V \; .$$

for some compact operator $K : V \mapsto V'$.

$\longrightarrow$ Coercivity of bilinear forms $V \times V \mapsto \mathbb{C}$

Theorem:

A continuous coercive operator is Fredholm with index zero.

A coercive $\quad \Rightarrow \quad$ (A injective $\quad \Rightarrow \quad A$ surjective)

Coercivity and Galerkin Discretization

V_n, $n \in \mathbb{N}$, sequence of closed subspaces of V (e.g., FEM/BEM spaces)

Assumption on V_n: Existence of linear projectors $P_n : V \mapsto V_n$ such that

$$\forall u \in V: \quad \lim_{n\to\infty} \|u - P_n u\|_V = 0 \ .$$

Given: Continuous, coercive and injective bilinear form $a : V \times V \mapsto \mathbb{C}$, that is $a(u,v) = 0$ for all $v \in V$ implies $u = 0$.

▶ $$\forall \varphi \in V' \quad \exists_1 u \in V: \quad a(u,v) = \langle \varphi, v \rangle \quad \forall v \in V \ .$$

For any fixed $\varphi \in V'$ there is an $N \in \mathbb{N}$ such that the variational problems

$$u_n \in V_n: \quad a(u_n, v_n) = \langle \varphi, v_n \rangle \quad \forall v_n \in V_n \ ,$$

have unique solutions u_n for all $n > N$. Those are asymptotically quasi-optimal in the sense that there is a constant $C > 0$ independent of φ such that

$$\|u - u_n\|_V \leq C \inf_{v_n \in V_n} \|u - v_n\|_V \quad \forall n > N \ .$$

Acoustic Scattering

Boundary Value Problem

Bounded Lipschitz domain/polyhedron $\Omega \subset \mathbb{R}^3$ (scatterer), complement $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$ (air region), connected boundary $\Gamma := \partial\Omega$, exterior unit normal vector field $\mathbf{n} \in L^\infty(\Gamma)$ points from Ω into Ω'.

Exterior Dirichlet problem for Helmholtz equation

$$\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega' \quad , \quad U = g \in H^{\frac{1}{2}}(\Gamma) \quad \text{on } \Gamma \,,$$
$$\frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) = o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty \,.$$

$\kappa > 0$ = wave number, g given Dirichlet boundary value (from incident wave)

A distribution U is called a *(radiating) Helmholtz solution*, if it satisfies $\Delta U + \kappa^2 U = 0$ in $\Omega \cup \Omega'$ and the Sommerfeld radiation conditions.

Boundary Value Problem

Bounded Lipschitz domain/polyhedron $\Omega \subset \mathbb{R}^3$ (scatterer), complement $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$ (air region), connected boundary $\Gamma := \partial\Omega$, exterior unit normal vector field $\mathbf{n} \in L^\infty(\Gamma)$ points from Ω into Ω'.

Exterior Dirichlet problem for Helmholtz equation

$$\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega' \quad , \quad U = g \in H^{\frac{1}{2}}(\Gamma) \quad \text{on } \Gamma \, ,$$
$$\frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) = o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty \, .$$

$\kappa > 0$ = wave number, g given Dirichlet boundary value (from incident wave)

A distribution U is called a *(radiating) Helmholtz solution*, if it satisfies $\Delta U + \kappa^2 U = 0$ in $\Omega \cup \Omega'$ and the Sommerfeld radiation conditions.

Existence and uniqueness of solutions

Potentials

Helmholtz kernel: $$\Phi_\kappa(\mathbf{x},\mathbf{y}) := \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Transmission representation formula for Helmholtz solution U:

$$U = -\Psi^\kappa_{\mathrm{SL}}([\gamma_N U]_\Gamma) + \Psi^\kappa_{\mathrm{DL}}([\gamma_D U]_\Gamma)$$

γ_D = Dirichlet trace, $\gamma_N := \frac{\partial}{\partial \mathbf{n}}$ Neumann trace, $[\cdot]_\Gamma$ = jump across Γ

single layer potential: $$\Psi^\kappa_{\mathrm{SL}}(\lambda)(\mathbf{x}) = \int_\Gamma \Phi_\kappa(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) ,$$

double layer potential: $$\Psi^\kappa_{\mathrm{DL}}(u)(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi_\kappa(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}u(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) .$$

Potentials

Helmholtz kernel: $\Phi_\kappa(\mathbf{x},\mathbf{y}) := \dfrac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

Transmission *representation formula* *for Helmholtz solution* U:

$$U = -\Psi^\kappa_{\mathrm{SL}}([\gamma_N U]_\Gamma) + \Psi^\kappa_{\mathrm{DL}}([\gamma_D U]_\Gamma)$$

γ_D = Dirichlet trace, $\gamma_N := \frac{\partial}{\partial\mathbf{n}}$ Neumann trace, $[\cdot]_\Gamma$ = jump across Γ

single layer potential: $\Psi^\kappa_{\mathrm{SL}}(\lambda)(\mathbf{x}) = \int_\Gamma \Phi_\kappa(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})\,\mathrm{d}S(\mathbf{y})$,

double layer potential: $\Psi^\kappa_{\mathrm{DL}}(u)(\mathbf{x}) = \int_\Gamma \dfrac{\partial\Phi_\kappa(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}u(\mathbf{y})\,\mathrm{d}S(\mathbf{y})$.

Continuity: $\Psi^\kappa_{\mathrm{SL}} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3)$, $\Psi^\kappa_{\mathrm{DL}} : H^{\frac{1}{2}}(\Gamma) \mapsto H_{\mathrm{loc}}(\Delta, \Omega\cup\Omega')$

$\Psi^\kappa_{\mathrm{SL}}$ and $\Psi^\kappa_{\mathrm{DL}}$ are radiating Helmholtz solutions

Boundary Integral Operators

Continuous boundary integral operators: ($\{\gamma\cdot\}_\Gamma := \frac{1}{2}(\gamma^+\cdot + \gamma^-\cdot)$ average)

$$
\begin{aligned}
\mathsf{V}_\kappa &: H^s(\Gamma) \mapsto H^{s+1}(\Gamma), \quad -1 \le s \le 0 , & \mathsf{V}_\kappa &:= \left\{\gamma_D \Psi^\kappa_{\mathrm{SL}}\right\}_\Gamma ,\\
\mathsf{K}_\kappa &: H^s(\Gamma) \mapsto H^{s}(\Gamma), \quad 0 \le s \le 1 , & \mathsf{K}_\kappa &:= \left\{\gamma_D \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma ,\\
\mathsf{D}_\kappa &: H^s(\Gamma) \mapsto H^{s-1}(\Gamma), \quad 0 \le s \le 1 , & \mathsf{D}_\kappa &:= \left\{\gamma_N \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma .
\end{aligned}
$$

Jump relations $\Rightarrow$ $\gamma_D^+ \Psi^\kappa_{\mathrm{SL}} = \mathsf{V}_\kappa$, $\gamma_D^+ \Psi^\kappa_{\mathrm{DL}} = \mathsf{K}_\kappa + \frac{1}{2} Id$

Boundary Integral Operators

Continuous boundary integral operators: $(\{\gamma\cdot\}_\Gamma := \frac{1}{2}(\gamma^+\cdot + \gamma^-\cdot)$ average)

$$\mathsf{V}_\kappa : H^s(\Gamma) \mapsto H^{s+1}(\Gamma), \quad -1 \le s \le 0 \quad , \qquad \mathsf{V}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{SL}}\right\}_\Gamma ,$$

$$\mathsf{K}_\kappa : H^s(\Gamma) \mapsto H^s(\Gamma), \qquad 0 \le s \le 1 \quad , \qquad \mathsf{K}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma ,$$

$$\mathsf{D}_\kappa : H^s(\Gamma) \mapsto H^{s-1}(\Gamma), \quad 0 \le s \le 1 \quad , \qquad \mathsf{D}_\kappa := \left\{\gamma_N \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma .0 < s <$$

Jump relations $\Rightarrow$ $\gamma_D^+ \Psi^\kappa_{\mathrm{SL}} = \mathsf{V}_\kappa \quad , \quad \gamma_D^+ \Psi^\kappa_{\mathrm{DL}} = \mathsf{K}_\kappa + \frac{1}{2} Id$

Compactness: $\mathsf{V}_\kappa - \mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is compact.

Boundary Integral Operators

Continuous boundary integral operators: $(\{\gamma\cdot\}_\Gamma := \frac{1}{2}(\gamma^+\cdot + \gamma^-\cdot)$ average)

$$\mathsf{V}_\kappa : H^s(\Gamma) \mapsto H^{s+1}(\Gamma), \quad -1 \le s \le 0 \quad , \qquad \mathsf{V}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{SL}}\right\}_\Gamma \ ,$$

$$\mathsf{K}_\kappa : H^s(\Gamma) \mapsto H^s(\Gamma), \quad 0 \le s \le 1 \quad , \qquad \mathsf{K}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma \ ,$$

$$\mathsf{D}_\kappa : H^s(\Gamma) \mapsto H^{s-1}(\Gamma), \quad 0 \le s \le 1 \quad , \qquad \mathsf{D}_\kappa := \left\{\gamma_N \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma \ .0 < s <$$

Jump relations $\Rightarrow$ $\gamma_D^+ \Psi^\kappa_{\mathrm{SL}} = \mathsf{V}_\kappa \quad , \quad \gamma_D^+ \Psi^\kappa_{\mathrm{DL}} = \mathsf{K}_\kappa + \frac{1}{2} Id$

Compactness: $\mathsf{V}_\kappa - \mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is compact.

Symmetry: $\langle \psi, \mathsf{V}_\kappa \varphi \rangle_\Gamma = \langle \varphi, \mathsf{V}_\kappa \psi \rangle_\Gamma \quad \forall \varphi, \psi \in H^{-\frac{1}{2}}(\Gamma)$.

Boundary Integral Operators

Continuous boundary integral operators: ($\{\gamma\cdot\}_\Gamma := \frac{1}{2}(\gamma^+\cdot + \gamma^-\cdot)$ average)

$$\mathsf{V}_\kappa : H^s(\Gamma) \mapsto H^{s+1}(\Gamma), \quad -1 \leq s \leq 0 \quad , \qquad \mathsf{V}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{SL}}\right\}_\Gamma \; ,$$

$$\mathsf{K}_\kappa : H^s(\Gamma) \mapsto H^s(\Gamma), \quad 0 \leq s \leq 1 \quad , \qquad \mathsf{K}_\kappa := \left\{\gamma_D \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma \; ,$$

$$\mathsf{D}_\kappa : H^s(\Gamma) \mapsto H^{s-1}(\Gamma), \quad 0 \leq s \leq 1 \quad , \qquad \mathsf{D}_\kappa := \left\{\gamma_N \Psi^\kappa_{\mathrm{DL}}\right\}_\Gamma \; .0 < s <$$

Jump relations $\Rightarrow$ $\gamma_D^+ \Psi^\kappa_{\mathrm{SL}} = \mathsf{V}_\kappa \quad , \quad \gamma_D^+ \Psi^\kappa_{\mathrm{DL}} = \mathsf{K}_\kappa + \frac{1}{2} Id$

Compactness: $\mathsf{V}_\kappa - \mathsf{V}_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ is compact.

Symmetry: $\langle \psi, \mathsf{V}_\kappa \varphi \rangle_\Gamma = \langle \varphi, \mathsf{V}_\kappa \psi \rangle_\Gamma \quad \forall \varphi, \psi \in H^{-\frac{1}{2}}(\Gamma)$.

Ellipticity: $\langle \bar{\varphi}, \mathsf{V}_0 \varphi \rangle_\Gamma \geq c_V \, \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)$.

Indirect CFIE

Spurious Resonances

Derivation of indirect boundary integral equations (BIE):

- Use potentials as trial expression for solution of exterior Helmholtz BVP.
- Apply jump relations + boundary values

Trial expression $U = \Psi_{\mathrm{SL}}^{\kappa}(\varphi)$, $\quad \varphi \in H^{-\frac{1}{2}}(\Gamma)$

▶ $$g = \mathsf{V}_{\kappa}\varphi \quad \text{in } H^{\frac{1}{2}}(\Gamma)$$

If κ^2 is Dirichlet eigenvalue of $-\Delta$ in Ω, then $\mathsf{Ker}(\mathsf{V}_{\kappa}) \neq \{0\}$

Trial expression $U = \Psi_{\mathrm{DL}}^{\kappa}(u)$, $\quad u \in H^{\frac{1}{2}}(\Gamma)$

▶ $$g = (\tfrac{1}{2}Id + \mathsf{K}_{\kappa})u \quad \text{in } H^{\frac{1}{2}}(\Gamma)$$

If κ^2 is Neumann eigenvalue of $-\Delta$ in Ω, then $\mathsf{Ker}(\frac{1}{2}Id + \mathsf{K}_{\kappa}) \neq \{0\}$

Classical Indirect CFIE

Indirect approach based on trial expression

$$U = \Psi^{\kappa}_{\mathrm{DL}}(u) + i\eta\Psi^{\kappa}_{\mathrm{SL}}(u) \;, \quad \eta \in \mathbb{R} \setminus \{0\} \;.$$

Classical Indirect CFIE

Indirect approach based on trial expression

$$U = \Psi^{\kappa}_{\mathrm{DL}}(u) + i\eta\Psi^{\kappa}_{\mathrm{SL}}(u)\ , \quad \eta \in \mathbb{R} \setminus \{0\}\ .$$

► Boundary integral equation for unknown density $u \in L^2(\Gamma)$:

$$g = (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u + i\eta\mathsf{V}_\kappa u$$

Classical Indirect CFIE

Indirect approach based on trial expression

$$U = \Psi_{\mathrm{DL}}^{\kappa}(u) + i\eta\Psi_{\mathrm{SL}}^{\kappa}(u)\ , \quad \eta \in \mathbb{R} \setminus \{0\}\ .$$

- Boundary integral equation for unknown density $u \in L^2(\Gamma)$:

$$g = (\tfrac{1}{2}Id + \mathsf{K}_{\kappa})u + i\eta\mathsf{V}_{\kappa}u$$

The classical CFIE has at most one solution

Classical Indirect CFIE

Indirect approach based on trial expression

$$U = \Psi^{\kappa}_{\mathrm{DL}}(u) + i\eta\Psi^{\kappa}_{\mathrm{SL}}(u)\ , \quad \eta \in \mathbb{R} \setminus \{0\}\ .$$

▶ Boundary integral equation for unknown density $u \in L^2(\Gamma)$:

$$g = (\tfrac{1}{2}Id + \mathsf{K}_\kappa)u + i\eta\mathsf{V}_\kappa u$$

The classical CFIE has at most one solution

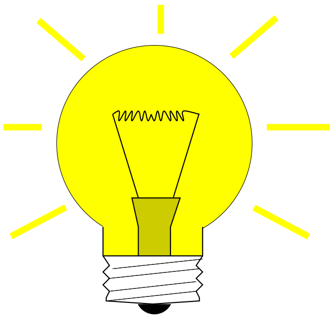
Lemma: If Γ C^2-smooth then $\mathsf{K}_\kappa : L^2(\Gamma) \mapsto H^1(\Gamma)$ continuous

▶ $L^2(\Gamma)$-coercivity of bilinear form associated with classical CFIE on smooth surfaces.

Problems: - Variational formulation lifted out of natural trace spaces

- No coercivity on non-smooth boundaries

Double Layer Regularization



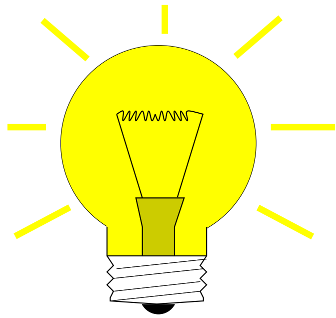
Devise CFIE set in natural trace spaces!

Tool: Compact regularizing operator $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$

Requirement: $\mathrm{Re}\{\langle \varphi, \mathsf{M}\overline{\varphi} \rangle_\Gamma\} > 0 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\}$

Trial expression: $$U = \Psi^{\kappa}_{\mathrm{DL}}(\mathsf{M}\varphi) + i\eta \Psi^{\kappa}_{\mathrm{SL}}(\varphi)\,, \quad \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

Double Layer Regularization



Devise CFIE set in natural trace spaces!

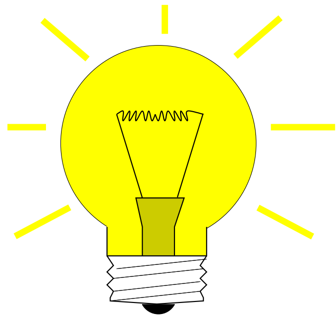
Tool: Compact regularizing operator $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$

Requirement: $\mathrm{Re}\{\langle \varphi, \mathsf{M}\overline{\varphi}\rangle_\Gamma\} > 0 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\}$

Trial expression: $$U = \Psi^\kappa_{\mathrm{DL}}(\mathsf{M}\varphi) + i\eta \Psi^\kappa_{\mathrm{SL}}(\varphi)\,, \quad \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

New CFIE: $$g = ((\tfrac{1}{2}Id + \mathsf{K}_\kappa) \circ \mathsf{M})(\varphi) + i\eta \mathsf{V}_\kappa \varphi$$

Double Layer Regularization



Devise CFIE set in natural trace spaces!

Tool: Compact regularizing operator $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$

Requirement: $\mathrm{Re}\{\langle \varphi, \mathsf{M}\overline{\varphi}\rangle_\Gamma\} > 0 \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\}$

Trial expression: $$U = \Psi^\kappa_{\mathrm{DL}}(\mathsf{M}\varphi) + i\eta\Psi^\kappa_{\mathrm{SL}}(\varphi)\,, \quad \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

New CFIE: $$g = ((\tfrac{1}{2}Id + \mathsf{K}_\kappa) \circ \mathsf{M})(\varphi) + i\eta \mathsf{V}_\kappa \varphi$$

Lemma: Uniqueness of solutions of new CFIE

Lemma: The operator associated with the new CFIE is $H^{-\frac{1}{2}}(\Gamma)$-coercive.

▶ Unique solvability of new CFIE for all κ, g

Regularizing Operator

Idea: $$\mathsf{M} = (-\Delta_\Gamma + Id)^{-1}$$

- Define $\mathsf{M} : H^{-1}(\Gamma) \mapsto H^1(\Gamma)$ by

$$\langle \mathbf{grad}_\Gamma \mathsf{M}\varphi, \mathbf{grad}_\Gamma v\rangle_\Gamma + \langle \mathsf{M}\varphi, v\rangle_\Gamma = \langle \varphi, v\rangle_\Gamma \quad \forall v \in H^1(\Gamma) \ .$$

- $\mathsf{M} : H^{-1}(\Gamma) \mapsto H^1(\Gamma)$ isomorphism and

$$\langle \varphi, \mathsf{M}\overline{\varphi}\rangle_\Gamma = \|\mathsf{M}\varphi\|^2_{H^1(\Gamma)} \geq c\, \|\varphi\|^2_{H^{-1}(\Gamma)} \ \forall \varphi \in H^{-1}(\Gamma) \ .$$

- $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ compact by Rellich's embedding theorem.

Remark. For piecewise smooth smooth Γ it is possible to choose product of $\Delta^{-1}_{\mathsf{Dir}}$ on faces as M (*cf.* Maxwell case).

Mixed Variational Problem

Avoid operator products by introducing new unknown $u := \mathsf{M}\varphi \in H^1(\Gamma)$

Saddle point problem: seek $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $u \in H^1(\Gamma)$,

$$
\begin{array}{rcccl}
i\eta \langle \mathsf{V}_\kappa \varphi, \xi \rangle_\Gamma & + & \left\langle (\tfrac{1}{2} Id + \mathsf{K}_\kappa) u, \xi \right\rangle_\Gamma & = & \langle g, \xi \rangle_\Gamma \quad \forall \xi \in H^{-\frac{1}{2}}(\Gamma) \\
-\langle \varphi, v \rangle_\Gamma & + & \langle \mathbf{grad}_\Gamma u, \mathbf{grad}_\Gamma v \rangle_\Gamma + \langle u, v \rangle_\Gamma & = & 0 \quad \forall v \in H^1(\Gamma) \,.
\end{array}
$$

Mixed Variational Problem

Avoid operator products by introducing new unknown $u := \mathsf{M}\varphi \in H^1(\Gamma)$

Saddle point problem: seek $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $u \in H^1(\Gamma)$,

$$\begin{array}{rcll} i\eta\,\langle \mathsf{V}_\kappa\varphi,\xi\rangle_\Gamma & + & \left\langle(\tfrac{1}{2}Id + \mathsf{K}_\kappa)u,\xi\right\rangle_\Gamma & = \langle g,\xi\rangle_\Gamma \quad \forall \xi \in H^{-\frac{1}{2}}(\Gamma) \\ -\langle\varphi,v\rangle_\Gamma & + & \langle \mathbf{grad}_\Gamma\, u, \mathbf{grad}_\Gamma\, v\rangle_\Gamma + \langle u,v\rangle_\Gamma & = 0 \quad \forall v \in H^1(\Gamma)\;. \end{array}$$

Off-diagonal terms in the variational problem are compact!

- $H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$-coercivity follows from coercivity of diagonal terms

Asymptotically optimal convergence of conforming Galerkin-BEM

Regularity

By jump relations: if $U = \Psi^{\kappa}_{\mathrm{DL}}(\mathsf{M}\varphi) + i\eta\Psi^{\kappa}_{\mathrm{SL}}(\varphi)$, then

$$[\gamma_D U]_\Gamma = \mathsf{M}\varphi \quad , \quad [\gamma_N U]_\Gamma = -i\eta\varphi \; .$$

- Elimination of unknown φ

$$\gamma_D^- U = i\eta^{-1}\mathsf{M}(\gamma_N^- U) + (g - i\eta^{-1}\mathsf{M}(\gamma_N^+ U)) \; .$$

Assume: $g - i\eta^{-1}\mathsf{M}(\gamma_N^+ U) \in H^r(\Gamma)\,, \quad r > \frac{1}{2}\,,$

$\mathsf{M} : H^{s-1}(\Gamma) \mapsto H^{s+1}(\Gamma)\,, \quad \forall 0 \leq s \leq s^*$, for some $s^* > 0$.

- "Bootstrap argument": first we see

$$\gamma_D^- U \in H^t(\Gamma)\,, \quad \frac{1}{2} \leq t \leq \min\{\frac{3}{2}, s^* + 1, r\} \; .$$

Next, use regularity of $-\Delta$ in Ω to gain more smoothness of $\gamma_N^- U$.

- Extra smoothness of φ from $[\gamma_N U]_\Gamma = -i\eta\varphi$

Direct CFIE

Classical CFIE

Exterior Helmholtz Calderón projector:

$$\gamma_D^+ U = (\mathsf{K}_\kappa + \tfrac{1}{2}Id)(\gamma_D^+ U) - \mathsf{V}_\kappa(\gamma_N^+ U) \ , \qquad (1)$$
$$\gamma_N^+ U = -\mathsf{D}_\kappa(\gamma_D^+ U) - (\mathsf{K}^*_\kappa - \tfrac{1}{2}Id)(\gamma_N^+ U) \ . \qquad (2)$$

Burton & Miller 1971: $i\eta\cdot$(1)$+$(2) ► CFIE:

$$(i\eta(\mathsf{K}_\kappa - \tfrac{1}{2}Id) - \mathsf{D}_\kappa)(\gamma_D^+ U) - (i\eta \mathsf{V}_\kappa + \tfrac{1}{2}Id + \mathsf{K}^*_\kappa)(\gamma_N^+ U) = 0 \ .$$

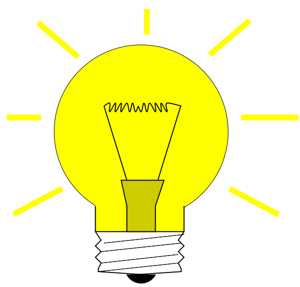
► Asscoiated boudary integral operator: $i\eta \mathsf{V}_\kappa + \tfrac{1}{2}Id + \mathsf{K}^*_\kappa$

▼

Uniqueness of solutions of CFIE
Coercivity in $L^2(\Gamma)$ on *smooth* Γ
Lack of coercivity in natural trace spaces

Regularization

Problem: Equations of the Calderón projector set in different trace spaces



Lift equation (2) set in $H^{-\frac{1}{2}}(\Gamma)$ into $H^{\frac{1}{2}}(\Gamma)$ by applying regularizing operator M before adding it to $i\eta\cdot$(1), $\eta \in \mathbb{R} \setminus \{0\}$.

► Regularized direct CFIE:

$$\mathsf{S}_\kappa(\varphi) := (\mathsf{M} \circ (\mathsf{K}_\kappa^* + \tfrac{1}{2}Id) + i\eta \mathsf{V}_\kappa)\varphi = (i\eta(\mathsf{K}_\kappa - \tfrac{1}{2}Id) - \mathsf{M} \circ \mathsf{D}_\kappa)g$$

Lemma: Uniqueness of solutions of new CFIE

Lemma: The operator associated with the new CFIE is $H^{-\frac{1}{2}}(\Gamma)$-coercive.

► Unique solvability of new CFIE for all κ, g

Mixed Variational Formulation

Concrete choice: $\mathsf{M} = (-\Delta_\Gamma + Id)^{-1}$

Introduce new "unknown" $u := \mathsf{M}((\frac{1}{2}Id + \mathsf{K}_\kappa^*)\varphi + \mathsf{D}_\kappa g) \in H^{\frac{1}{2}}(\Gamma)$.

Note: $u = 0$ (dummy variable), because from second equation of Calderón projector $\gamma_N^+ U = -\mathsf{D}_\kappa(\gamma_D^+ U) - (\mathsf{K}_\kappa^* - \frac{1}{2}Id)(\gamma_N^+ U)$.

Saddle point problem: seek $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, $u \in H^1(\Gamma)$,

$$\begin{aligned} i\eta\,\langle \xi, \mathsf{V}_\kappa \varphi\rangle_\Gamma \quad &+ \quad \langle \xi, u\rangle_\Gamma \quad = \quad i\eta \left\langle \xi, (\mathsf{K}_\kappa - \tfrac{1}{2}Id)g \right\rangle_\Gamma \;, \\ -\left\langle (\tfrac{1}{2}Id + \mathsf{K}_\kappa^*)\varphi, v \right\rangle_\Gamma \quad &+ \quad \langle \mathbf{grad}_\Gamma\, u, \mathbf{grad}_\Gamma\, v\rangle_\Gamma + \langle u, v\rangle_\Gamma = \langle \mathsf{D}_\kappa g, v\rangle_\Gamma \;. \end{aligned}$$

$H^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)$-coercivity & asymptotically optimal convergence of conforming Galerkin-BEM

Summary and References

New direct/indirect CFIE for acoustic scattering have been obtrained that possess coercive mixed variational formulations.

- Dummy multiplier & potential of FEM-BEM coupling makes direct CFIE particularly attractive.

References:

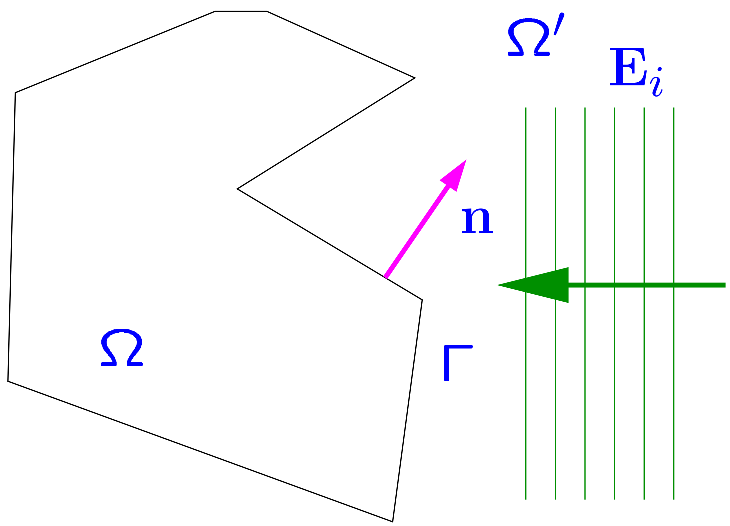
A. BUFFA AND R. HIPTMAIR, *A coercive combined field integral equation for electromagnetic scattering*, Preprint NI03003-CPD, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, 2003. Submitted.

R. HIPTMAIR, *Coercive combined field integral equations*, J. Numer. Math., 11 (2003), pp. 115–134.

R. HIPTMAIR AND A. BUFFA, *Coercive combined field integral equations*, Report 2003-06, SAM, ETH Zürich, Zürich, Switzerland, 2003. Submitted.

Electromagnetic Scattering

Scattering at PEC Obstacle



Exterior Dirichlet problem for electric wave equation (excited by incident wave)

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} &= 0 && \text{in } \Omega' , \\ \gamma_t\mathbf{E} &= \mathbf{g} := \gamma_t\mathbf{E}_i && \text{on } \Gamma , \end{aligned}$$

+ Silver-Müller radiation conditions

Wave number $\kappa = \omega\sqrt{\epsilon_0\mu_0} > 0$ fixed

Existence and uniqueness of solution for all $\mathbf{E}_i$

A distribution $\mathbf{U}$ is called a radiating Maxwell solution, if it satisfies $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \kappa^2\mathbf{U} = 0$ in $\Omega \cup \Omega'$ and the Silver-Müller radiation conditions at infinity.

Cauchy Data

Transmission conditions for electromagnetic fields:

$$[\gamma_{\mathbf{t}}\mathbf{E}]_\Gamma = 0 \quad , \quad [\mathbf{H} \times \mathbf{n}]_\Gamma = 0 \, .$$

▶ Ensure continuity of Poynting-flux $\mathbf{E} \cdot (\overline{\mathbf{H}} \times \mathbf{n})$

▼

Cauchy data for electric wave equation $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0$:

"Electric trace" (Dirichlet data): $\gamma_D\mathbf{E}(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \times (\mathbf{E}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))$

"Magnetic trace" (Neumann data): $\gamma_N\mathbf{E}(\mathbf{x}) := \mathbf{curl}\,\mathbf{E}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$

▲

Integration by parts formula for $\mathbf{curl}$-operator

Traces

“$\mathbf{E}$-space”: $\boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl};\Omega) = \{\mathbf{u} \in \boldsymbol{L}^2_{\mathsf{loc}}(\Omega),\ \mathbf{curl}\,\mathbf{u} \in \boldsymbol{L}^2_{\mathsf{loc}}(\Omega)\}$

Spaces:

$$\boldsymbol{T}_{\mathsf{el}} := \{v \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\Gamma),\ \mathsf{curl}_{\Gamma}\, v \in H^{-\frac{1}{2}}(\Gamma)\},$$

$$\boldsymbol{T}_{\mathsf{mag}} := \{\zeta \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma),\ \mathsf{div}_{\Gamma}\zeta \in H^{-\frac{1}{2}}(\Gamma)\}$$

duality $\langle\cdot,\cdot\rangle_{\boldsymbol{\tau}}$

[Surface differential operators: $\mathsf{div}_{\Gamma} := \mathbf{grad}^*_{\Gamma}$, $\mathsf{curl}_{\Gamma} := (\mathbf{n} \times \mathbf{grad}_{\Gamma})^*$]

Trace theorem (Buffa, Ciarlet, 1999; Buffa, Costabel, Sheen, 2000):

$$\gamma_D : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl};\Omega) \mapsto \boldsymbol{T}_{\mathsf{el}},$$
$$\gamma_{\mathbf{t}} := \gamma_D \times \mathbf{n} : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl};\Omega) \mapsto \boldsymbol{T}_{\mathsf{mag}}$$

are continuous, surjective.

Magnetic traces ($\mathbf{H} \times \mathbf{n} \doteq \mathbf{curl}\,\mathbf{E} \times \mathbf{n}$) : $\gamma_N\mathbf{u} = \mathbf{curl}\,\mathbf{u} \times \mathbf{n}$, weakly defined

$$\mp \int_{\Omega} \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\overline{\mathbf{v}} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{x} = \langle \gamma_N \mathbf{u}, \gamma_D \mathbf{v}\rangle_{\boldsymbol{\tau}}\ \forall \mathbf{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$$

$\gamma_N : \boldsymbol{H}_{\mathsf{loc}}(\mathbf{curl}\,\mathbf{curl},\Omega) \mapsto \boldsymbol{T}_{\mathsf{mag}}$ continuous, surjective

Potentials

Stratton-Chu representation formula for radiating solution $\mathbf{E}$ of electric wave equation in Ω':

$$\mathbf{E} = -\Psi_{\mathsf{SL}}^{\kappa}(\gamma_N^{+}\mathbf{E}) + \Psi_{\mathsf{DL}}^{\kappa}(\gamma_D^{+}\mathbf{E}) \quad \text{in } \Omega'$$

Helmholtz kernel: $\Phi_\kappa(\mathbf{x},\mathbf{y}) := \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

Single layer potential : $\Psi_V^\kappa(\phi)(\mathbf{x}) := \int_\Gamma \Phi_\kappa(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, dS(\mathbf{y})$

Vectorial single layer potential : $\Psi_{\mathbf{A}}^\kappa(\boldsymbol{\lambda})(\mathbf{x}) := \int_\Gamma \Phi_\kappa(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\, dS(\mathbf{y})$

Maxwell double layer potential : $\Psi_{\mathsf{DL}}^\kappa(u)(\mathbf{x}) := \mathbf{curl}_{\mathbf{x}}\, \Psi_{\mathbf{A}}^\kappa(\mathbf{n}\times u)(\mathbf{x})$

Maxwell single layer potential : $\Psi_{\mathsf{SL}}^\kappa(\boldsymbol{\lambda}) := \Psi_{\mathbf{A}}^\kappa(\boldsymbol{\lambda}) + \mathbf{grad}_\Gamma\, \Psi_V^\kappa(\mathsf{div}_\Gamma \boldsymbol{\lambda})$

Both $\Psi_{\mathsf{DL}}^\kappa$ and $\Psi_{\mathsf{SL}}^\kappa$ provide radiating Maxwell solutions

Boundary Integral Operators

Traces + potentials $\Rightarrow$ continuous boundary integral operators:

$$\begin{aligned} \mathsf{S}_\kappa &:= \gamma_D \boldsymbol{\Psi}^\kappa_{\mathsf{SL}} &&: \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}} \;, \\ \mathsf{C}_\kappa &:= \tfrac{1}{2}(\gamma_D^+ + \gamma_D^-) \boldsymbol{\Psi}^\kappa_{\mathsf{DL}} &&: \boldsymbol{T}_{\mathsf{el}} \mapsto \boldsymbol{T}_{\mathsf{el}} \;. \end{aligned}$$

Jump relations:

$$[\gamma_D \boldsymbol{\Psi}^\kappa_{\mathsf{SL}}(\boldsymbol{\lambda})]_\Gamma = 0 \quad , \quad [\gamma_D \boldsymbol{\Psi}^\kappa_{\mathsf{DL}}(\boldsymbol{u})]_\Gamma = \boldsymbol{u}$$

$$\gamma_D^+ \boldsymbol{\Psi}^\kappa_{\mathsf{SL}} = \mathsf{S}_\kappa \quad , \quad \gamma_D^+ \boldsymbol{\Psi}^\kappa_{\mathsf{DL}} = \mathsf{C}_\kappa + \tfrac{1}{2} Id$$

Compactness:

$\mathsf{S}_\kappa - \mathsf{S}_0 : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$ compact

BUT S_0 is **not** $\boldsymbol{T}_{\mathsf{mag}}$-elliptic

Generalized Coercivity

There is an isomorphism $\mathsf{X} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{mag}}$ and a compact operator $\mathsf{K} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$ such that

$$\exists c > 0: \quad |\langle \mathbf{S}_\kappa \boldsymbol{\mu}, \mathsf{X}\overline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\tau}} + \langle \mathsf{K}\boldsymbol{\mu}, \overline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\tau}}| \geq c \, \|\boldsymbol{\mu}\|^2_{\boldsymbol{T}_{\mathsf{el}}} \quad \forall \boldsymbol{\mu} \in \boldsymbol{T}_{\mathsf{mag}} \, .$$

▶ $\mathbf{S}_\kappa$ is Fredholm with index zero

Construction of X based on stable Hodge-type decomposition

$$\boldsymbol{T}_{\mathsf{mag}} = \mathbf{X} \oplus \mathbf{N} \quad , \quad \mathbf{N} \subset \mathsf{Ker}(\mathsf{div}_\Gamma) \quad , \quad \mathbf{X} \subset \gamma_{\mathbf{t}} \boldsymbol{H}^1(\Omega) \, .$$

→ Associated continuous projectors $P_{\mathbf{X}}$, $P_{\mathbf{N}}$: $\quad P_{\mathbf{X}} + P_{\mathbf{N}} = Id$

▶ $$\mathsf{X} = P_{\mathbf{X}} - P_{\mathbf{N}}$$

Note: $\mathbf{X} \hookrightarrow \boldsymbol{L}^2(\Gamma)$ compact

Regularized CFIE

Combined field trial expression $\mathrm{U} = \Psi^{\kappa}_{\mathsf{DL}}(\mathsf{M}\zeta) + i\eta\Psi^{\kappa}_{\mathsf{SL}}(\zeta)$, $\zeta \in \boldsymbol{T}_{\mathsf{mag}}$.
(with regularizing operator $\mathsf{M} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$)

- Regularized CFIE: $$\mathbf{g} = ((\tfrac{1}{2}Id + \mathbf{C}_{\kappa}) \circ \mathsf{M})(\zeta) + i\eta\mathbf{S}_{\kappa}\zeta$$

If $\eta \neq 0$ and $\mathsf{M} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$ satisfies $\langle \mathsf{M}\boldsymbol{\mu}, \overline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\tau}} > 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{T}_{\mathsf{mag}} \setminus \{0\}$, then the above regularized combined field integral equation has at most one solution for any $\kappa > 0$.

If $\eta \neq 0$ and $\mathsf{M} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$ is compact, then the operator mapping $\boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{T}_{\mathsf{el}}$ associated with the above regularized combined field integral equation is Fredholm with index zero.

- Existence and uniqueness of solutions for any $\mathbf{g}$, κ

Regularizing Operator

Assume that Ω is polyhedron with flat (smooth) faces $\Gamma_1, \ldots, \Gamma_p$, $p \in \mathbb{N}$. Write Σ for the union of all edges of Ω.

$$\boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma) := \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}_\Gamma, \Gamma),\ \gamma_\mathbf{t}\mathbf{u} = 0 \text{ on } \Sigma\}$$

Lemma:

$\boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)$ is dense in $\boldsymbol{T}_{\mathsf{el}}$
with compact embedding $\boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma) \hookrightarrow \boldsymbol{T}_{\mathsf{mag}}$

Define $\mathsf{M} : \boldsymbol{T}_{\mathsf{mag}} \mapsto \boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)$ by

$$\langle \mathbf{curl}_\Gamma \mathsf{M}\boldsymbol{\mu}, \mathbf{curl}_\Gamma \mathbf{v}\rangle_\Gamma + \langle \mathsf{M}\boldsymbol{\mu}, \mathbf{v}\rangle_{\boldsymbol{\tau}} = \langle \boldsymbol{\mu}, \mathbf{v}\rangle_{\boldsymbol{\tau}} \quad \forall \mathbf{v} \in \boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)\ .$$

- $\mathsf{M}\boldsymbol{\mu} = 0 \quad \Rightarrow \quad \boldsymbol{\mu} = 0 \quad , \quad \langle \mathsf{M}\boldsymbol{\mu}, \overline{\boldsymbol{\mu}}\rangle_{\boldsymbol{\tau}} = \{\mathsf{M}\boldsymbol{\mu}\}_{\mathbf{curl}_\Gamma,\Gamma} > 0$ if $\boldsymbol{\mu} \neq 0$.

Remark. Split regularizing operator enjoys better lifting properties $\rightarrow$ ζ more regular

Mixed Variational Formulation

Get rid of operator products by introducing new unknown $\mathbf{u} := \mathsf{M}\zeta$, $\mathbf{u} \in \boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)$, and incorporate variational definition of M:

Seek $\zeta \in \boldsymbol{T}_{\mathrm{mag}}$, $\mathbf{u} \in \boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)$ such that

$$\begin{aligned} i\eta \left\langle \mathsf{S}_\kappa \zeta, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau}} &+ \left\langle (\tfrac{1}{2} Id + \mathsf{C}_\kappa)\mathbf{u}, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau}} &= \left\langle \mathbf{g}, \boldsymbol{\mu} \right\rangle_{\boldsymbol{\tau}} , \\ \left\langle \mathbf{curl}_\Gamma \mathbf{u}, \mathbf{curl}_\Gamma \mathbf{v} \right\rangle_\Gamma + \left\langle \mathbf{u}, \mathbf{v} \right\rangle_{\boldsymbol{\tau}} &- \left\langle \boldsymbol{\mu}, \mathbf{v} \right\rangle_{\boldsymbol{\tau}} &= 0 , \end{aligned} \qquad (1)$$

for all $\boldsymbol{\mu} \in \boldsymbol{T}_{\mathrm{mag}}$, $\mathbf{v} \in \boldsymbol{H}_\Sigma(\mathbf{curl}_\Gamma, \Gamma)$.

Lemma: The off-diagonal forms in (1) are compact

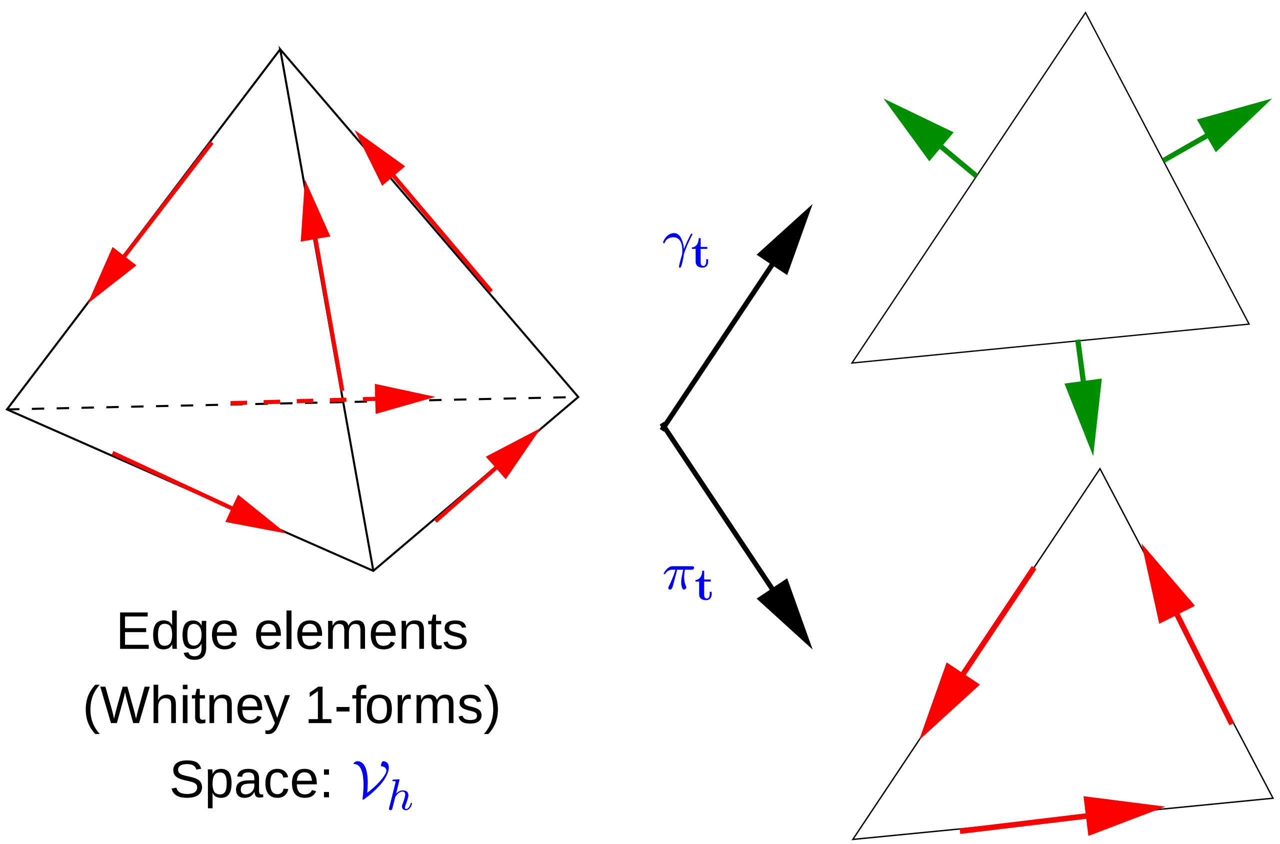
The bilinear form associated with (1) is coercive in the generalized sense.

Natural Boundary Elements

$\mathbf{E}$, $\mathbf{H}$ require $\mathbf{curl}$-conforming elements (e.g. edge element space $\mathcal{V}_h$)

- Discretize $\gamma_D \mathbf{E}$, $\gamma_N \mathbf{E} = \gamma_{\mathbf{t}} \mathbf{H}$ in $\gamma_D \mathcal{V}_h$, $\gamma_{\mathbf{t}} \mathcal{V}_h$ (on Γ-restricted mesh)

Example: Lowest order elements on simplicial triangulations of Ω (Γ):



Edge elements
(Whitney 1-forms)
Space: $\mathcal{V}_h$

Discrete surface currents $\in \boldsymbol{T}_{h,m}$

- ζ_h

D.o.f = edge fluxes

Discrete Dirichlet traces $\in \boldsymbol{T}_{h,\Sigma}$

- $\mathbf{u}_h$

D.o.f = edge voltages

(Set to zero on Σ)

- [Conforming spaces] $\Rightarrow$ Galerkin discretization

A Priori Error Estimates

Challenge: Mismatch of continuous and discrete Hodge-type decompositions

$$\boldsymbol{T}_{\mathrm{mag}} = \mathbf{X} \oplus \mathbf{N} \quad \leftrightarrow \quad \boldsymbol{T}_{h,m} = \mathbf{X}_h \oplus \mathbf{N}_h : \quad \mathbf{X}_h \not\subset \mathbf{X} \; .$$

Special properties of BEM-space $\boldsymbol{T}_{\mathrm{mag}}$ ensure "$\mathbf{X}_h \to \mathbf{X}$" as $h \to 0$:

There is $s > 0$ such that

$$\inf_{\boldsymbol{\mu}_h \in \mathbf{X}_h} \|\boldsymbol{\xi} - \boldsymbol{\mu}_h\|_{\boldsymbol{T}_{\mathrm{mag}}} \leq C h^s \, \|\boldsymbol{\xi}\|_{\boldsymbol{T}_{\mathrm{mag}}} \quad \forall \boldsymbol{\xi} \in \mathbf{X} \; ,$$

where $C > 0$ only depends on s and the shape regularity of the surface mesh.

Generalized coercivity ► asymptotic inf-sup condition for discrete problem

Asymptotic quasi-optimality of discrete Galerkin solutions.

Summary and References

Now a rigorous theoretical foundation for Galerkin-BEM for the CFIEs of direct acoustic and electromagnetic scattering has become available.

References:

A. BUFFA, *Remarks on the discretization of some non-positive operators with application to heterogeneous Maxwell problems*, preprint, IMATI-CNR, Pavia, Pavia, Italy, 2003.

A. BUFFA AND R. HIPTMAIR, *A coercive combined field integral equation for electromagnetic scattering*, Preprint NI03003-CPD, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, 2003.

——, *Galerkin boundary element methods for electromagnetic scattering*, in Computational Methods in Wave Propagation, M. Ainsworth, ed., Springer, New York, 2003, pp. 85–126. In print.

A. BUFFA, R. HIPTMAIR, T. VON PETERSDORFF, AND C. SCHWAB, *Boundary element methods for Maxwell equations on Lipschitz domains*, Numer. Math., (2002). To appear.

R. HIPTMAIR AND C. SCHWAB, *Natural boundary element methods for the electric field integral equation on polyhedra*, SIAM J. Numer. Anal., 40 (2002), pp. 66–86.