A Fourier Transformed

Boundary Element Method

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1. Motivation for the Fourier – BEM

2. Paradigmatic example: heat conduction

3. Some applications
   - isotropic heat conduction
   - anisotropic heat conduction

   - isotropic elasticity
   - anisotropic elasticity

4. Extended example : Kirchhoff plate

5. Conclusions

$$\int_{\Omega} u_i F^i \, d\Omega + \int_{\Gamma} u_i T^i \, d\Gamma = \int_{\Omega} U_i f^i \, d\Omega + \int_{\Gamma} U_i t^i \, d\Gamma$$

⟺

$$\int_{\Re^n} \hat{u}_i \hat{F}^i \, d\hat{x} + \int_{\Re^n} \hat{u}_i \hat{T}^i \, d\hat{x} = \int_{\Re^n} \hat{U}_i \hat{f}^i \, d\hat{x} + \int_{\Re^n} \hat{U}_i \hat{t}^i \, d\hat{x}$$

Fourier transform of the boundary integral equations

Motivation for the Fourier – BEM (theorem of Parseval)


J.-P. Fourier

problem : fundamental solution *U*

$$P(\partial)U(x) = F(x) = \delta(x)$$

$$U(x) = \quad ???$$

known only for simple cases

Fourier transform w.r.t. all coordinates

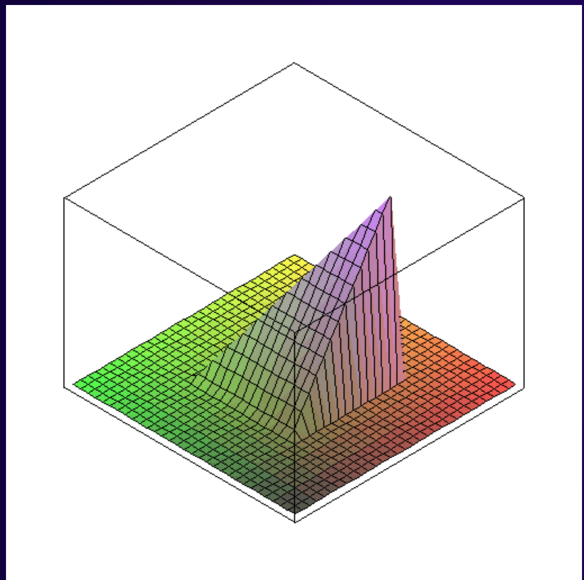
$$\hat{P}(\hat{x})\hat{U}(\hat{x}) = \hat{F}(\hat{x}) = 1$$

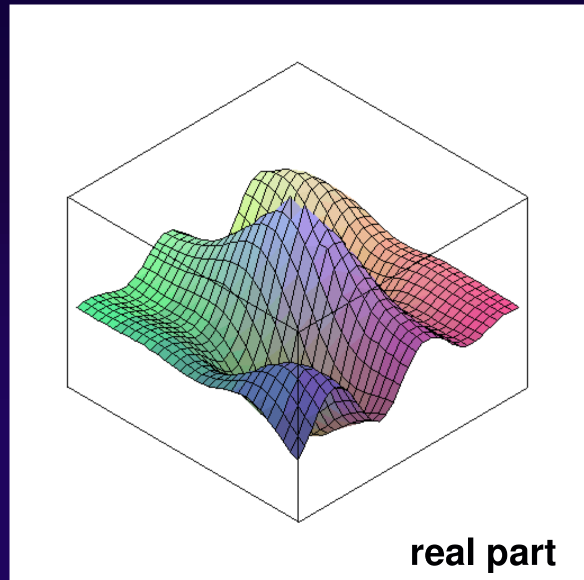
$$\hat{U}(\hat{x}) = \quad \hat{P}^{-1}(\hat{x})$$

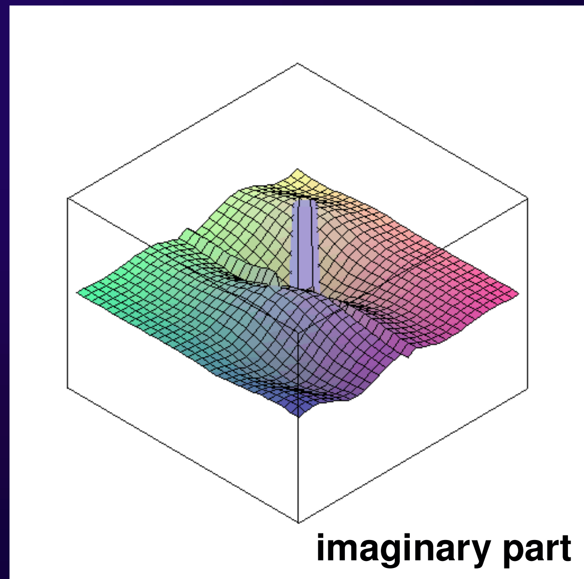
Fourier fundamental solution is known for all linear and homogeneous media

Motivation for the Fourier – BEM

Transform of the trial and test functions instead of a numerical inverse transformation of the Fourier fundamental solution.



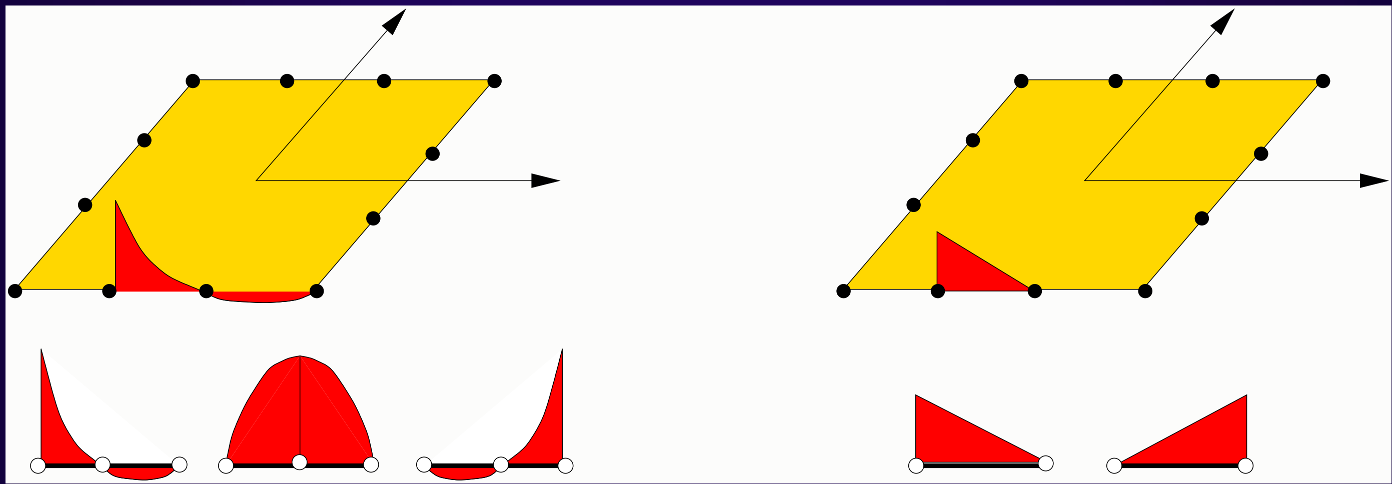




Motivation for the Fourier – BEM

Poisson's equation :

$$\begin{aligned} -\Delta u(x) &= f(x), && x \in \Omega \\ u(x) &= u_\Gamma(x), && x \in \Gamma_\mathrm{u} \\ t(x) &= t_\Gamma(x), && x \in \Gamma_\mathrm{t} \end{aligned}$$



$$u(x) \approx \sum_i^{N_\mathrm{u}} \mathrm{u}_i \phi_\mathrm{u}^i(x) \qquad t(x) \approx \sum_i^{N_\mathrm{t}} \mathrm{t}_i \phi_\mathrm{t}^i(x)$$

Paradigmatic example: heat conduction

Traditional Galerkin – BIE :

$$\int_{\Omega} \phi_{\mathrm{t}}^{j}(x)\kappa(x)u(x)\mathrm{d}\Gamma_{x} = \int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Omega} f(y)U(x-y)\mathrm{d}\Omega_{y}\mathrm{d}\Gamma_{x}$$

$$+\sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i} \int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Gamma} \phi_{\mathrm{t}}^{i}(y)U(x-y)\mathrm{d}\Gamma_{y}\mathrm{d}\Gamma_{x}$$

$$-\sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i} \int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Gamma} \phi_{\mathrm{u}}^{i}(y)\mathrm{A}_{\mathrm{t}}^{i}U(x-y)\mathrm{d}\Gamma_{y}\mathrm{d}\Gamma_{x}$$

mit $\mathrm{A}_{\mathrm{t}}^{i} = \nu^{i}\cdot\nabla$

Paradigmatic example: heat conduction

Derivative of the Galerkin – BIE :

$$-\int_{\Omega} \phi_{\mathrm{t}}^{j}(x)\,\mathrm{A}_{\mathrm{t}}^{j}\left\{\kappa(x)u(x)\right\}\mathrm{d}\Gamma_x = -\int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Omega} f(y)\,\mathrm{A}_{\mathrm{t}}^{j}U(x-y)\,\mathrm{d}\Omega_y\,\mathrm{d}\Gamma_x$$

$$-\sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i}\int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Gamma} \phi_{\mathrm{t}}^{i}(y)\,\mathrm{A}_{\mathrm{t}}^{j}U(x-y)\,\mathrm{d}\Gamma_y\,\mathrm{d}\Gamma_x$$

$$+\sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i}\int_{\Gamma} \phi_{\mathrm{t}}^{j}(x)\int_{\Gamma} \phi_{\mathrm{u}}^{i}(y)\,\mathrm{A}_{\mathrm{t}}^{j}\,\mathrm{A}_{\mathrm{t}}^{i}U(x-y)\,\mathrm{d}\Gamma_y\,\mathrm{d}\Gamma_x$$

mit $\mathrm{A}_{\mathrm{t}}^{i} = \nu^{i}\cdot\nabla$

Algebraic system of equations :

$$\sum_i K_{\mathrm{u}}^{ji}\mathrm{u}^i = F_{\mathrm{u}}^j + \sum_i H_{\mathrm{u}}^{ji}\mathrm{t}^i - \sum_i G_{\mathrm{u}}^{ji}\mathrm{u}^i$$

$$\sum_i K_{\mathrm{t}}^{ji}\mathrm{t}^i = F_{\mathrm{t}}^j + \sum_i H_{\mathrm{t}}^{ji}\mathrm{t}^i - \sum_i G_{\mathrm{t}}^{ji}\mathrm{u}^i$$

$$F_{\mathrm{u}}^j = \int_\Gamma \phi_{\mathrm{t}}^j(x)\int_\Omega f(y)U(x-y)\,\mathrm{d}\Omega_y\mathrm{d}\Gamma_x \qquad F_{\mathrm{t}}^j = \int_\Gamma \phi_{\mathrm{u}}^j(x)\int_\Omega f(y)\mathrm{A}_{\mathrm{t}}^j U(x-y)\,\mathrm{d}\Omega_y\mathrm{d}\Gamma_x$$

$$H_{\mathrm{u}}^{ji} = \int_\Gamma \phi_{\mathrm{t}}^j(x)\int_\Omega \phi_{\mathrm{t}}^i(y)U(x-y)\,\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \qquad H_{\mathrm{t}}^{ji} = \int_\Gamma \phi_{\mathrm{u}}^j(x)\int_\Omega \phi_{\mathrm{t}}^i(y)\mathrm{A}_{\mathrm{t}}^j U(x-y)\,\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x$$

$$G_{\mathrm{u}}^{ji} = \int_\Gamma \phi_{\mathrm{t}}^j(x)\int_\Omega \phi_{\mathrm{u}}^i(y)\mathrm{A}_{\mathrm{t}}^i U(x-y)\,\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \qquad G_{\mathrm{t}}^{ji} = \int_\Gamma \phi_{\mathrm{u}}^j(x)\int_\Omega \phi_{\mathrm{u}}^i(y)\mathrm{A}_{\mathrm{t}}^j\mathrm{A}_{\mathrm{t}}^i U(x-y)\,\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x$$

$$K_{\mathrm{u}}^{ji} = \int_\Gamma \phi_{\mathrm{t}}^j(x)\kappa(x)\phi_{\mathrm{u}}^i(x)\,\mathrm{d}\Gamma_x \qquad K_{\mathrm{t}}^{ji} = \int_\Gamma \phi_{\mathrm{u}}^j(x)\mathrm{A}_{\mathrm{t}}^j\left\{\kappa(x)\phi_{\mathrm{t}}^i(x)\right\}\mathrm{d}\Gamma_x$$

Paradigmatic example: heat conduction

Extension from

$\Omega \quad \rightarrow \quad \Re^n$

$$u(x) \rightarrow \chi(x)u(x); \qquad \chi(x) := \begin{cases} 1 \ldots x \in \Omega \\ \kappa \ldots x \in \partial\Omega \\ 0 \ldots x \notin \Omega \end{cases}$$

Introduction of a cut-off distribution

$$\kappa(x) = \int_{\Re^n} \chi(x)\delta(x-y)\,\mathrm{d}y; \qquad x \in \partial\Omega$$

Multi-dimensional Heaviside – distribution : $\chi(x) = \mathrm{H}(\psi(x))$

$\psi(x)$ **hyper-surface of the boundary**

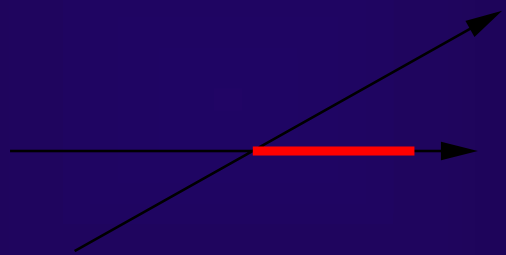
Paradigmatic example: heat conduction

Definition of a reference element :

$$\Re^2 : \quad \chi^0 := \mathrm{H}(x_1)\mathrm{H}(1-x_1)\delta(x_2)$$

$$\Re^3 : \quad \chi^0_\Diamond := \mathrm{H}(x_1)\mathrm{H}(1-x_1)\mathrm{H}(x_2)\mathrm{H}(1-x_2)\delta(x_3)$$

$$\Re^3 : \quad \chi^0_\Delta := \mathrm{H}(x_1)\mathrm{H}(1-x_1-x_2)\mathrm{H}(x_2)\delta(x_3)$$



Trial and test functions for the reference element :

$$\phi^0 = \chi^0(x) p^0(x)$$

$p^0(x)$ **arbitrary polynomial**

Trial and test functions for an arbitrary element :

$$\mathrm{T}^i : \quad \phi^0 \rightarrow \phi^i = \phi^0(x - b^i)$$

$$\mathrm{D}^i : \quad \phi^0 \rightarrow \phi^i = \phi^0(a^i x)$$

scalar product : $$\langle a,b\rangle = \int_{\Re^n} a(x)b(x)\,\mathrm{d}x$$

convolution : $$a*b = \int_{\Re^n} a(y)b(x-y)\,\mathrm{d}y$$

Galerkin – BIE for the $\Re^n$:

$$\langle \phi_{\mathrm{t}}^j, u_\chi\rangle = \langle \phi_{\mathrm{t}}^j, f_\chi * U\rangle + \sum_i^{N_{\mathrm{t}}} \mathrm{t}^i \langle \phi_{\mathrm{t}}^j, \phi_{\mathrm{t}}^i * U\rangle - \sum_i^{N_{\mathrm{u}}} \mathrm{u}^i \langle \phi_{\mathrm{t}}^j, \phi_{\mathrm{u}}^i * \mathrm{A}_{\mathrm{t}}^i U\rangle$$

$$-\langle \phi_{\mathrm{u}}^j, \mathrm{A}_{\mathrm{t}}^j u_\chi\rangle = -\langle \phi_{\mathrm{u}}^j, f_\chi * \mathrm{A}_{\mathrm{t}}^j U\rangle - \sum_i^{N_{\mathrm{t}}} \mathrm{t}^i \langle \phi_{\mathrm{u}}^j, \phi_{\mathrm{t}}^i * \mathrm{A}_{\mathrm{t}}^j U\rangle -$$

$$-\sum_i^{N_{\mathrm{u}}} \mathrm{u}^i \langle \phi_{\mathrm{u}}^j, \phi_{\mathrm{u}}^i * \mathrm{A}_{\mathrm{t}}^j \mathrm{A}_{\mathrm{t}}^i U\rangle$$

$$u_\chi = u\chi;\ f_\chi = f\chi$$

Fourier transform is now possible !!

Paradigmatic example: heat conduction

Parseval's equality :

$$\int_{\Re^n} \phi^j(x)u(x)\,\mathrm{d}x = \frac{1}{(2\pi)^n}\int_{\Re^n} \hat{\phi}^j(-\hat{x})\hat{u}(\hat{x})\,\mathrm{d}\hat{x}$$

Convolution theorem :

$$u(x) * \phi^i(x) \overset{\mathrm{F}}{\longleftrightarrow} \hat{u}(\hat{x})\hat{\phi}^i(\hat{x})$$

Sample entry of the BEM – matrices :

$$H_{\mathrm{u}}^{ij} = \left\langle \phi_{\mathrm{t}}^j, \phi_{\mathrm{t}}^i * U \right\rangle = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_{\mathrm{t}}^j(-.), \hat{\phi}_{\mathrm{t}}^i \hat{U} \right\rangle$$

The double integration over two boundary panels is replaced by a single integration over $\Re^n$

Galerkin BIE for the Fourier – BEM :

$$\left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{u}_{\chi} \right\rangle = \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{f}_{\chi}\hat{U} \right\rangle + \sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i} \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{\phi}_{\mathrm{t}}^{i}\hat{U} \right\rangle - \sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i} \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{\phi}_{\mathrm{u}}^{i}\hat{\mathrm{A}}_{\mathrm{t}}^{i}\hat{U} \right\rangle$$

Derivation of the Galerkin BIE for the Fourier – BEM :

$$-\left\langle \hat{\phi}_{\mathrm{u}}^{j}(-.), \hat{\mathrm{A}}_{\mathrm{t}}^{j}\hat{u}_{\chi} \right\rangle = -\left\langle \hat{\phi}_{\mathrm{u}}^{j}(-.), \hat{f}_{\chi}\hat{\mathrm{A}}_{\mathrm{t}}^{j}\hat{U} \right\rangle - \sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i} \left\langle \hat{\phi}_{\mathrm{u}}^{j}(-.), \hat{\phi}_{\mathrm{t}}^{i}\hat{\mathrm{A}}_{\mathrm{t}}^{j}\hat{U} \right\rangle - \sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i} \left\langle \hat{\phi}_{\mathrm{u}}^{j}(-.), \hat{\phi}_{\mathrm{u}}^{i}\hat{\mathrm{A}}_{\mathrm{t}}^{j}\hat{\mathrm{A}}_{\mathrm{t}}^{i}\hat{U} \right\rangle$$

Paradigmatic example: heat conduction

Galerkin – BIE :

$$\left\langle \phi_{\mathrm{t}}^{j}, u_{\chi} \right\rangle = \left\langle \phi_{\mathrm{t}}^{j}, f_{\chi} * U \right\rangle + \sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i} \left\langle \phi_{\mathrm{t}}^{j}, \phi_{\mathrm{t}}^{i} * U \right\rangle - \sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i} \left\langle \phi_{\mathrm{t}}^{j}, \phi_{\mathrm{u}}^{i} * \mathrm{A}_{\mathrm{t}}^{i} U \right\rangle$$

free term **strong singularity**

Fourier equivalent of the Galerkin – BIE :

$$\left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{u}_{\chi} \right\rangle = \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{f}_{\chi} \hat{U} \right\rangle + \sum_{i}^{N_{\mathrm{t}}} \mathrm{t}^{i} \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{\phi}_{\mathrm{t}}^{i} \hat{U} \right\rangle - \sum_{i}^{N_{\mathrm{u}}} \mathrm{u}^{i} \left\langle \hat{\phi}_{\mathrm{t}}^{j}(-.), \hat{\phi}_{\mathrm{u}}^{i} \hat{\mathrm{A}}_{\mathrm{t}}^{i} \hat{U} \right\rangle$$

free term ⟷ **strong singularity**

singularities cancel one another

$$u_{\chi} = u\chi; f_{\chi} = f\chi$$

Hypersingular Galerkin – BIE :

strong singularity

$$-\left\langle \phi_{u}^{j}, A_{t}^{j} u_{\chi} \right\rangle = -\left\langle \phi_{u}^{j}, f_{\chi} * A_{t}^{j} U \right\rangle - \sum_{i}^{N_t} t^{i} \left\langle \phi_{u}^{j}, \phi_{t}^{i} * A_{t}^{j} U \right\rangle -$$

two free terms due to

singularities cancel one another

$$-\sum_{i}^{N_u} u^{i} \left\langle \phi_{u}^{j}, \phi_{u}^{i} * A_{t}^{j} A_{t}^{i} U \right\rangle$$

$$A_{t}^{j} u_{\chi} = A_{t}^{j} \{u\chi\} = \chi A_{t}^{j} u + u A_{t}^{j} \chi$$

hyper singularity

Fourier equivalent of the hypersingular Galerkin – BIE :

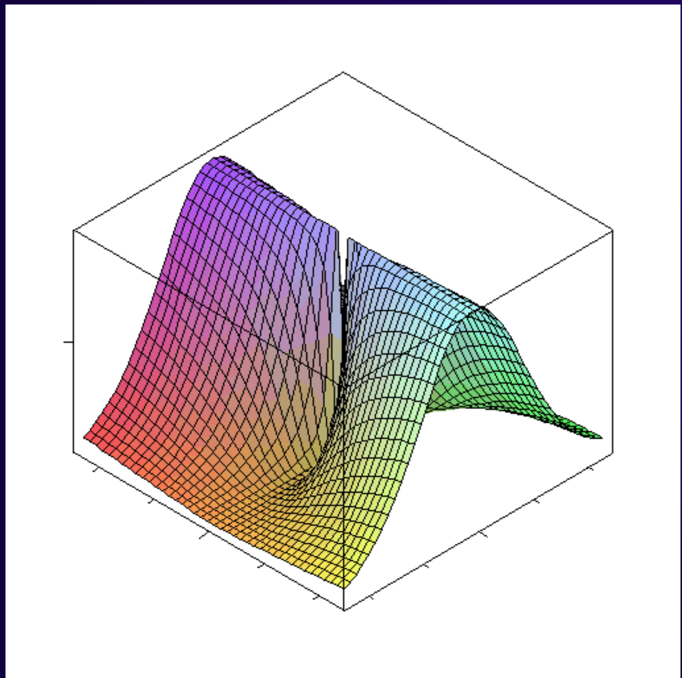
$$-\left\langle \hat{\phi}_{u}^{j}(-.), \hat{A}_{t}^{j} \hat{u}_{\chi} \right\rangle = -\left\langle \hat{\phi}_{u}^{j}(-.), \hat{f}_{\chi} \hat{A}_{t}^{j} \hat{U} \right\rangle - \sum_{i}^{N_t} t^{i} \left\langle \hat{\phi}_{u}^{j}(-.), \hat{\phi}_{t}^{i} \hat{A}_{t}^{j} \hat{U} \right\rangle -$$

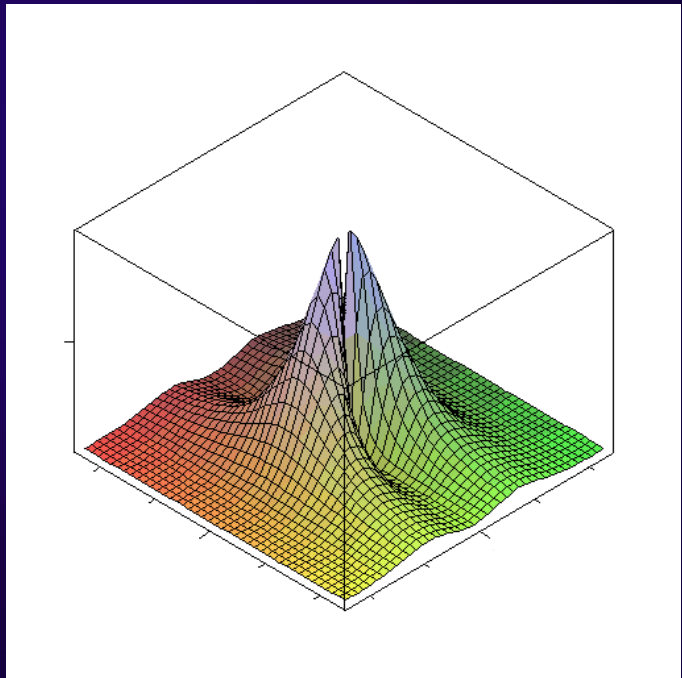
singularities cancel one another

$$-\sum_{i}^{N_u} u^{i} \left\langle \hat{\phi}_{u}^{j}(-.), \hat{\phi}_{u}^{i} \hat{A}_{t}^{j} \hat{A}_{t}^{i} \hat{U} \right\rangle$$

Fourier equivalent of the hypersingular Galerkin – BIE :

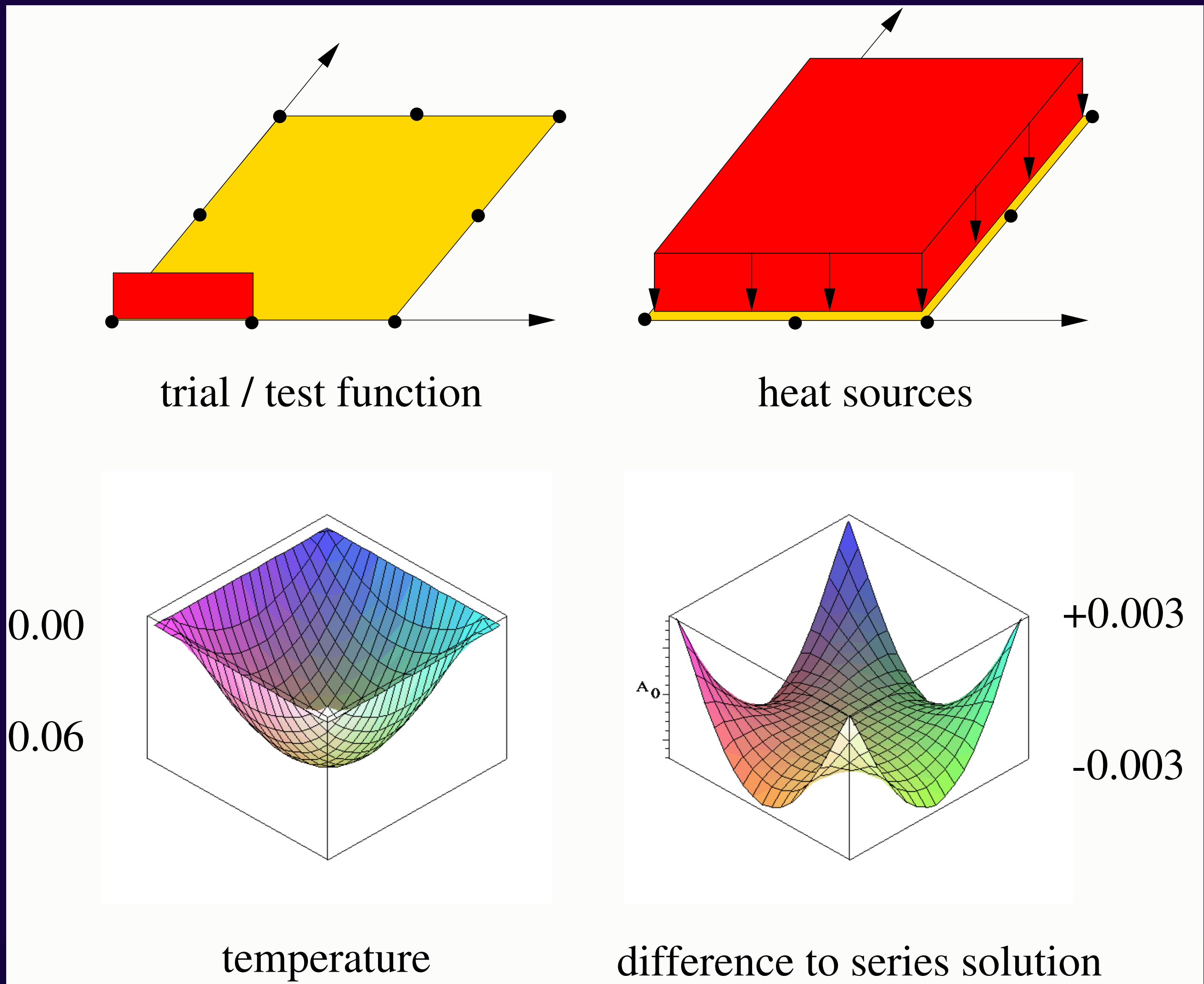
$$G^{ji} = \left\langle \phi_u^j, \phi_u^i * A_t^j A_t^i U \right\rangle = \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_u^j(-.), \hat{\phi}_u^i \hat{A}_t^j \hat{A}_t^i \hat{U} \right\rangle$$



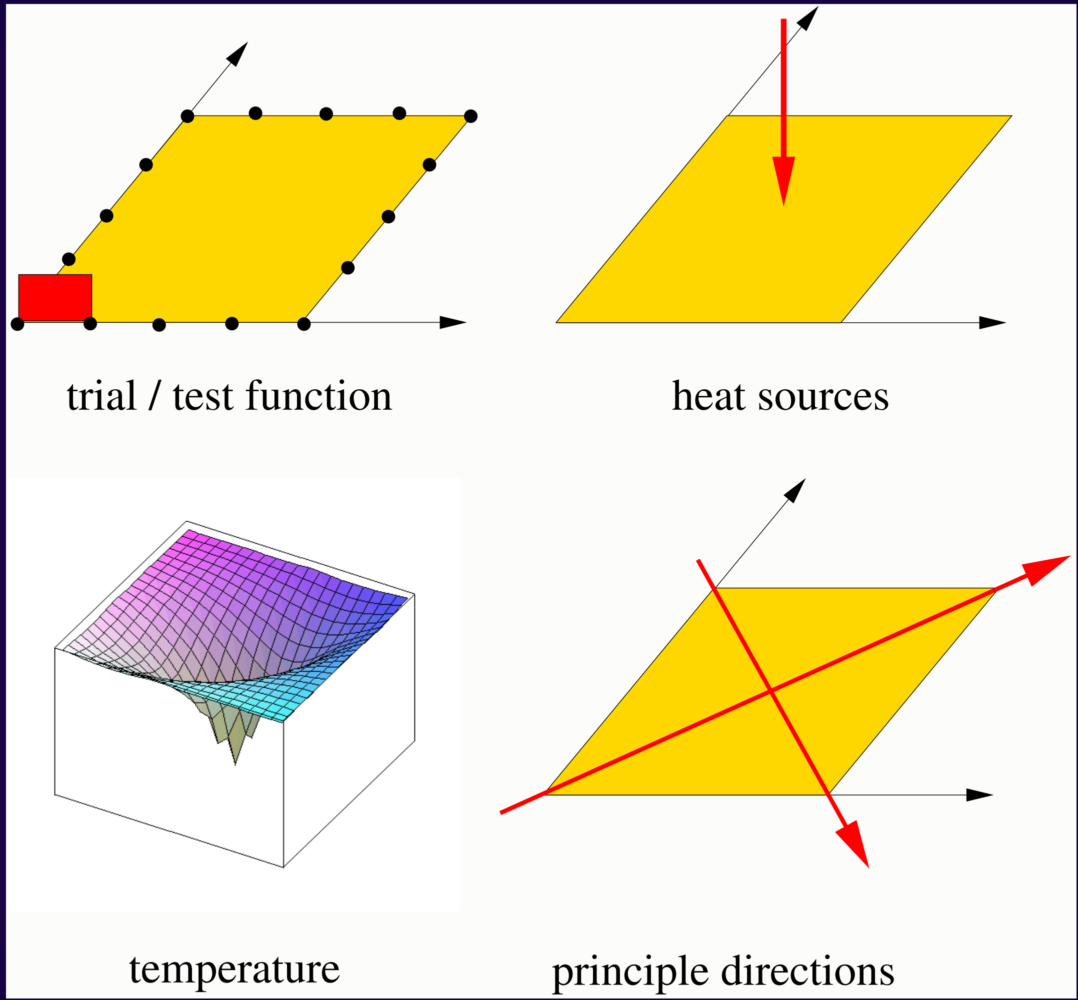


Some remarks on singular integrals

Isotropic heat conduction



Anisotropic heat conduction



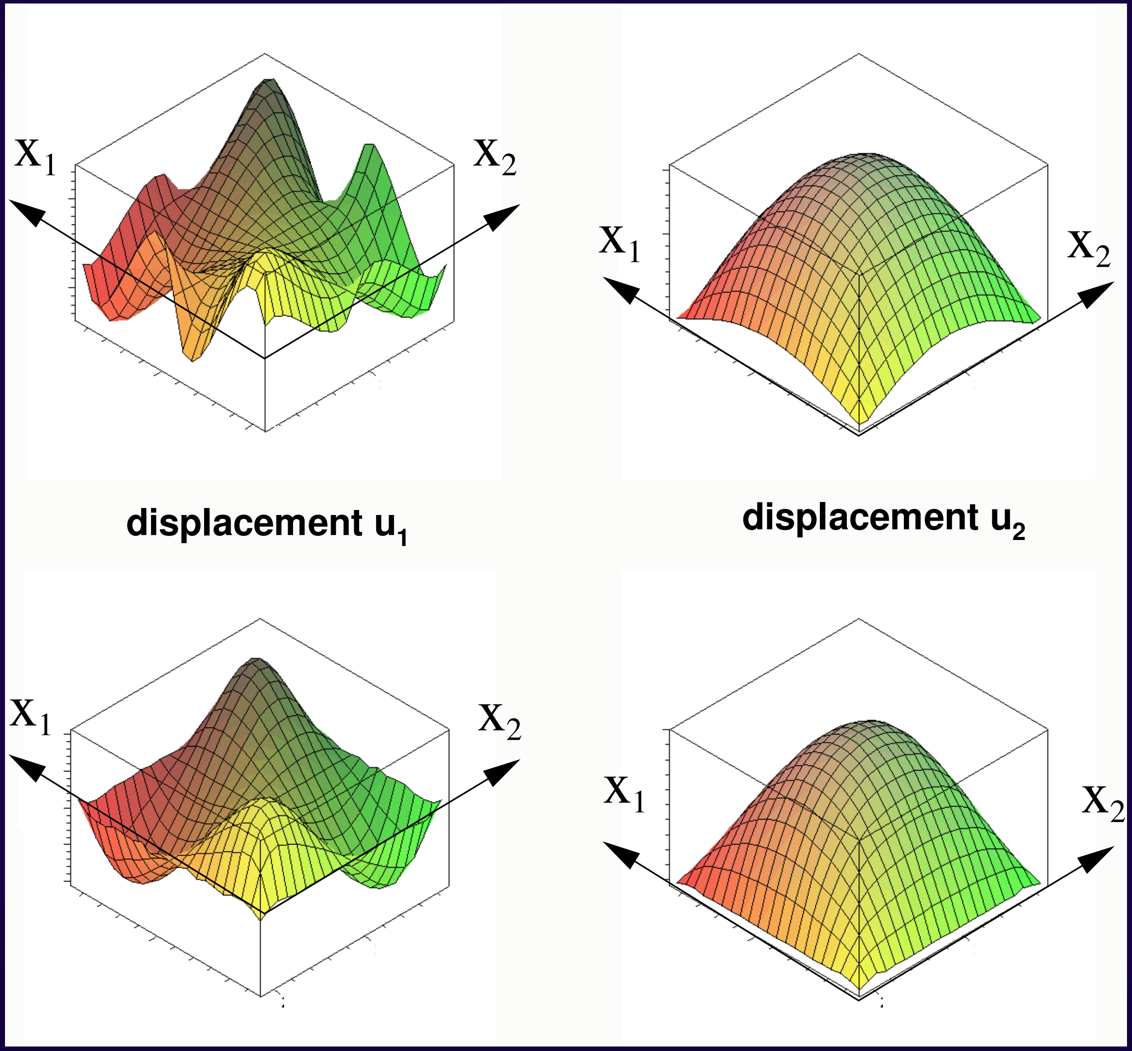
Some applications

Isotropic elasticity

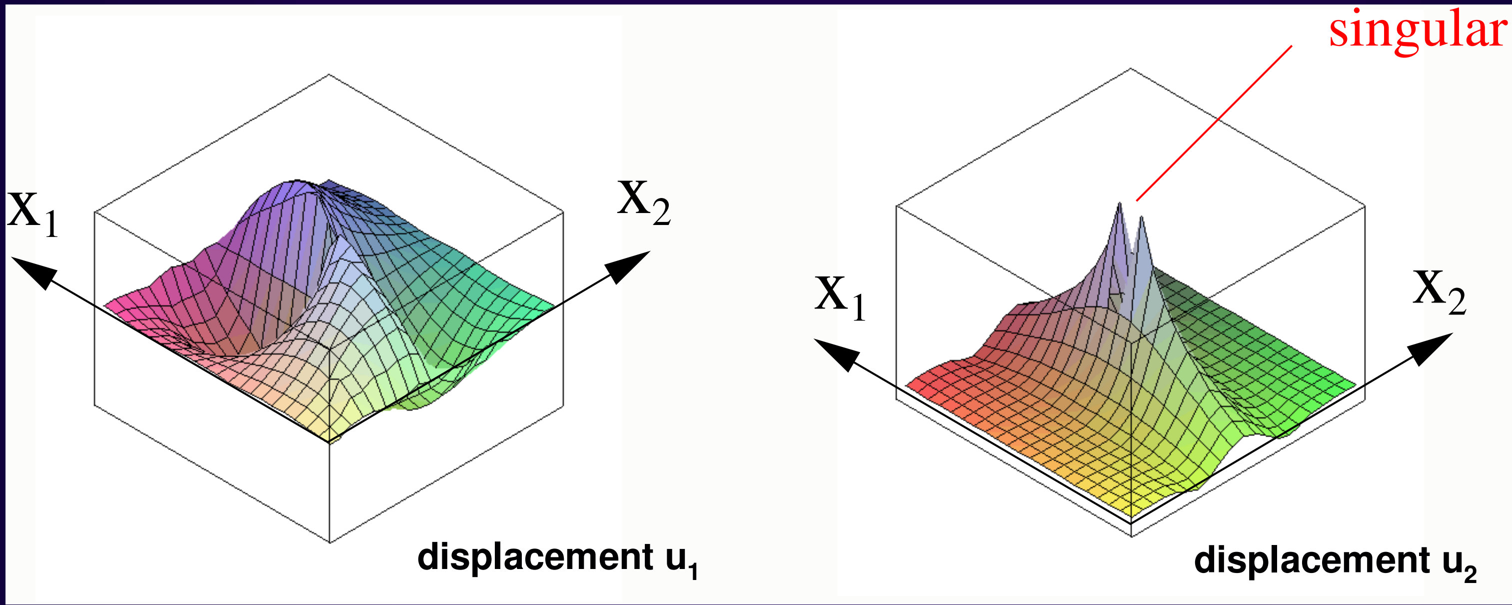
G. Lamé

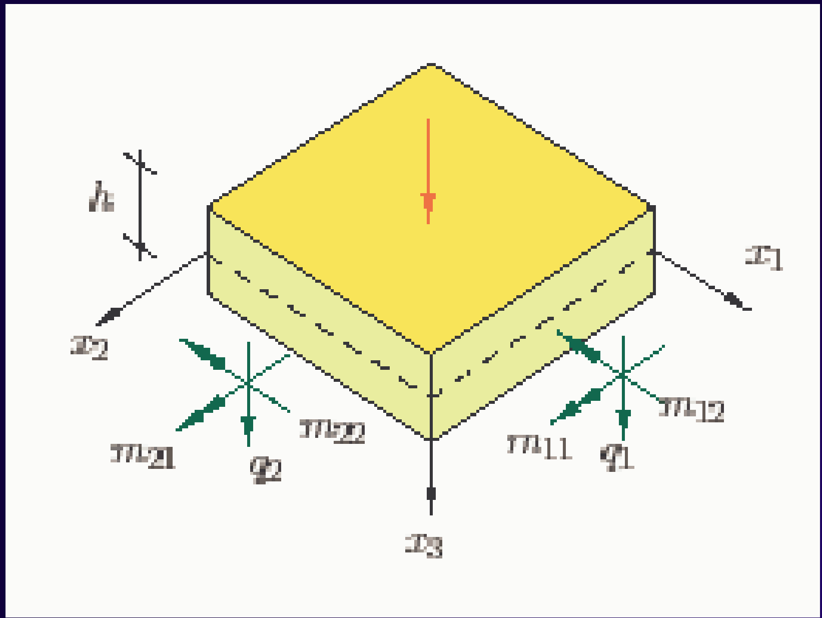
2 elements per side

8 elements per side



Anisotropic elasticity





rigidity :

$$K_{klmn} = D[(1-\bar{\nu})\delta_{km}\delta_{nl} + \bar{\nu}\delta_{kl}\delta_{mn}]$$

$$D = \frac{Eh^3}{12(1-\bar{\nu}^2)}$$

moment : $m_{kl} = -K_{klmn}\partial_{mn} w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{m}_{kl} = K_{klmn}\hat{x}_m\hat{x}_n\hat{w}$

shear force : $q_k = -D\partial_{kll} w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{q}_k = iD\hat{x}_k\hat{x}_l\hat{x}_l\hat{w}$

Extended example: Kirchhoff plate

differential equation $D\Delta\Delta w = f \quad \overset{F}{\longleftrightarrow} \quad D(\hat{x}_1^2 + \hat{x}_2^2)^2\,\hat{w} = \hat{f}$

fundamental solution:

$$\hat{W}(\hat{x}) = \frac{1}{D(\hat{x}_1^2 + \hat{x}_2^2)^2}$$

boundary quantities

$$w \quad \overset{F}{\longleftrightarrow} \quad \hat{w}$$

$$\varphi_\nu = \nu_k \partial_k w \quad \overset{F}{\longleftrightarrow} \quad \hat{\varphi}_\nu = i\,\nu_k \hat{x}_k \hat{w}$$

$$m_\nu = \nu_k \nu_l m_{kl} \quad \overset{F}{\longleftrightarrow} \quad \hat{m}_\nu = \nu_k \nu_l \hat{m}_{kl}$$

$$q_\nu = \nu_k q_k + \tau_k \partial_k m_{lm} \nu_l \tau_m \quad \overset{F}{\longleftrightarrow} \quad \hat{q}_\nu = \nu_k \hat{q}_k + i\,\tau_k \hat{x}_k \hat{m}_{lm} \nu_l \tau_m$$

Extended example: Kirchhoff plate

slope : $\varphi = \mathrm{A}_\varphi w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{\varphi} = \hat{\mathrm{A}}_\varphi \hat{w}$

moment : $m = \mathrm{A}_m w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{m} = \hat{\mathrm{A}}_m \hat{w}$

shear force : $q = \mathrm{A}_q w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{q} = \hat{\mathrm{A}}_q \hat{w}$

corner terms : $f_c = \mathrm{A}_c w \quad \overset{\mathrm{F}}{\longleftrightarrow} \quad \hat{f}_c = \hat{\mathrm{A}}_c \hat{w}$

Extended example: Kirchhoff plate

for the slope :

$$A_\varphi^i = \nu_k^i \partial_k \quad \overset{F}{\longleftrightarrow} \quad \hat{A}_\varphi^i = i\,\nu_k^i \hat{x}_k$$

for the moment :

$$A_m^i = -K_{klmn} \nu_k^i \nu_l^i \partial_{mn} \quad \overset{F}{\longleftrightarrow} \quad \hat{A}_m^i = K_{klmn} \nu_k^i \nu_l^i \hat{x}_m \hat{x}_n$$

for the shear force :

$$A_q^i = -D \nu_k^i \partial_{kll} - K_{klmn} \tau_p^i \nu_k^i \tau_l^i \partial_{mnp} \quad \overset{F}{\longleftrightarrow} \quad \hat{A}_q^i = iD \nu_k^i \hat{x}_k \hat{x}_l \hat{x}_l + iK_{klmn} \tau_p^i \nu_k^i \tau_l^i \hat{x}_m \hat{x}_n \hat{x}_p$$

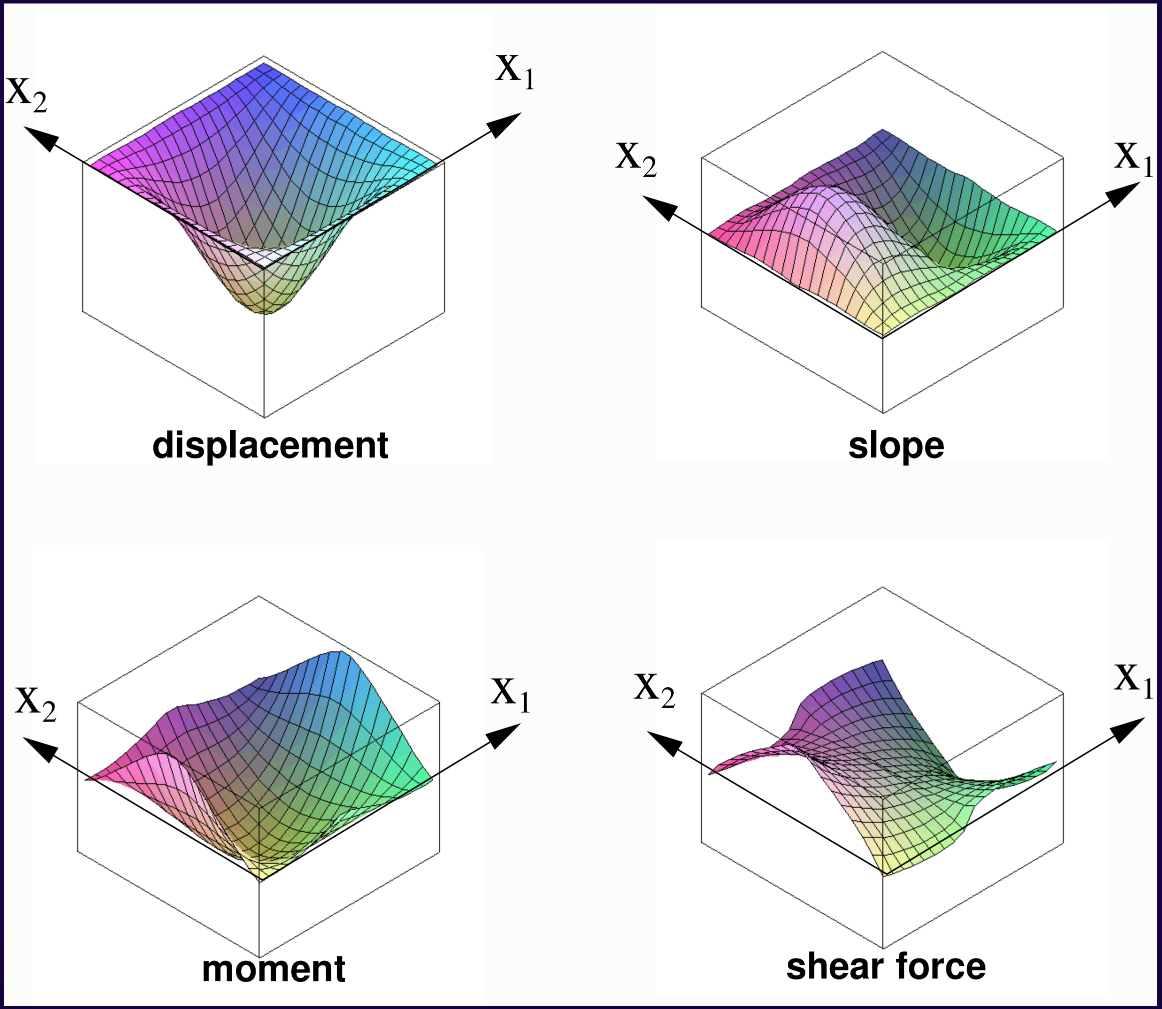
for the corner terms :

$$A_c^i = (1-\overline{\nu}) D (\nu_1^{+i} \nu_2^{+i} - \nu_1^{-i} \nu_2^{-i})(\partial_{11} - \partial_{22}) + 2D(\nu_1^{+i} \nu_1^{+i} - \nu_2^{+i} \nu_2^{+i} - \nu_1^{-i} \nu_1^{-i} + \nu_2^{-i} \nu_2^{-i}) \partial_{12}$$

$$\overset{F}{\longleftrightarrow} \quad \hat{A}_c^i = -(1-\overline{\nu}) D (\nu_1^{+i} \nu_2^{+i} - \nu_1^{-i} \nu_2^{-i})(\hat{x}_1 \hat{x}_1 - \hat{x}_2 \hat{x}_2) - 2D(\nu_1^{+i} \nu_1^{+i} - \nu_2^{+i} \nu_2^{+i} - \nu_1^{-i} \nu_1^{-i} + \nu_2^{-i} \nu_2^{-i}) \hat{x}_1 \hat{x}_2$$

Extended example: Kirchhoff plate

Kirchhoff plate



Kirchhoff

Extended example: Kirchhoff plate