Chapter 3

Constitutive Relations

In this chapter we derive and discuss constitutive relations for the Cauchy stress tensor $\boldsymbol{T}$, for the first Piola transform $\boldsymbol{P}$, and for the second Piola transform $\boldsymbol{\Sigma}$, respectively. We will consider two different approaches, modelling elastic and hyperelastic materials.

3.1 Elastic Materials

In what follows we assume that the Cauchy stress tensor $\boldsymbol{T}(t, \boldsymbol{y})$ is time independent, i.e.

$$\boldsymbol{T}(t, \boldsymbol{y}) = \boldsymbol{T}(\boldsymbol{y}),$$

and that it is completely determined by the deformation gradient $\boldsymbol{F} = D_x \boldsymbol{\varphi}(t, \boldsymbol{x})$. In fact, a material is called *elastic*, if there exists a response function for the Cauchy stress tensor such that the constitutive equation

$$\boldsymbol{T}(\boldsymbol{y}) = \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{F})$$

is satisfied. A material is called *homogeneous* if its response function is independent of the particular material point $\boldsymbol{x}$, i.e.

$$\boldsymbol{T}(\boldsymbol{y}) = \boldsymbol{R}(\boldsymbol{F}).$$

The constitutive equations must be independent from the observation, i.e. independent of the particular choice of the coordinate system. Hence we formulate the principle of material frame indifference: *The constitutive laws governing the internal interactions between the parts of a physical system should not depend on whatever external frame of reference is used to describe them.* In particular, if $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$ is an orthogonal transformation satisfying

$$\boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}, \quad \det \boldsymbol{Q} = 1,$$

for the Cauchy stress vector we then have

$$\boldsymbol{t}(\boldsymbol{Q}\boldsymbol{y}, \boldsymbol{Q}\boldsymbol{n}) = \boldsymbol{Q}\,\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}).$$

For the Cauchy stress tensor we then conclude

$$\boldsymbol{T}(\boldsymbol{Q}\boldsymbol{y})\boldsymbol{Q}\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{Q}\boldsymbol{y}, \boldsymbol{Q}\boldsymbol{n}) = \boldsymbol{Q}\,\boldsymbol{t}(\boldsymbol{y}, \boldsymbol{n}) = \boldsymbol{Q}\boldsymbol{T}(\boldsymbol{y})\boldsymbol{n}$$

for all $\boldsymbol{n} \in \mathbb{R}^3$, and therefore

$$\boldsymbol{T}(\boldsymbol{Q}\boldsymbol{y}) = \boldsymbol{Q}\boldsymbol{T}(\boldsymbol{y})\boldsymbol{Q}^\top$$

follows. We finally restrict our considerations to *isotropic* materials where the material behavior does not depend on the directions, i.e.

$$\boldsymbol{R}(\boldsymbol{F}\boldsymbol{Q}) = \boldsymbol{R}(\boldsymbol{F}).$$

In the case of an elastic, homogeneous and isotropic material we are looking for a response function $\boldsymbol{R} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ satisfying

$$\boldsymbol{R}(\boldsymbol{F}) = [\boldsymbol{R}(\boldsymbol{F})]^\top, \quad \boldsymbol{R}(\boldsymbol{Q}\boldsymbol{F}) = \boldsymbol{Q}\boldsymbol{R}(\boldsymbol{F})\boldsymbol{Q}^\top, \quad \boldsymbol{R}(\boldsymbol{F}\boldsymbol{Q}) = \boldsymbol{R}(\boldsymbol{F}) \tag{3.1}$$

for all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$, and for all orthogonal transformations $\boldsymbol{Q} \in \mathbb{R}^{3\times 3}$. When considering an ansatz by means of a power series one easily concludes the symmetric representation

$$\boldsymbol{R}(\boldsymbol{F}) = \sum_{k=0}^{\infty} a_k \Big[\boldsymbol{F}\boldsymbol{F}^\top\Big]^k. \tag{3.2}$$

Although the ansatz

$$\boldsymbol{R}(\boldsymbol{F}) = \sum_{k=0}^{\infty} a_k \Big[\boldsymbol{F}^\top\boldsymbol{F}\Big]^k$$

is symmetric, due to

$$\boldsymbol{R}(\boldsymbol{Q}\boldsymbol{F}) \;=\; \sum_{k=0}^{\infty} a_k \Big[(\boldsymbol{Q}\boldsymbol{F})^\top\boldsymbol{Q}\boldsymbol{F}\Big]^k = \sum_{k=0}^{\infty} a_k \Big[\boldsymbol{F}^\top\boldsymbol{Q}^\top\boldsymbol{Q}\boldsymbol{F}\Big]^k = \sum_{k=0}^{\infty} a_k \Big[\boldsymbol{F}^\top\boldsymbol{F}\Big]^k = \boldsymbol{R}(\boldsymbol{F}),$$

we easily conclude, that the second requirement in (3.1) is violated. Hence we have to use the symmetric representation (3.2). Next we will consider a reformulation of the infinite power series (3.2) by means of a second order polynomial in $\boldsymbol{F}\boldsymbol{F}^\top$.

The principal invariants of a matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ are the coefficients $\iota_1(\boldsymbol{A})$, $\iota_2(\boldsymbol{A})$ and $\iota_3(\boldsymbol{A})$ of the characteristic polynomial

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \;=\; -\lambda^3 + \iota_1(\boldsymbol{A})\lambda^2 - \iota_2(\boldsymbol{A})\lambda + \iota_3(\boldsymbol{A})\,.$$

If the eigenvalues of the matrix $\boldsymbol{A}$ are given as λ_1, λ_2, and λ_3, we also have

$$\begin{aligned}\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \;&=\; (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \\ &=\; -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda + \lambda_1\lambda_2\lambda_3,\end{aligned}$$

and therefore we conclude

$$\begin{aligned}\iota_1(\boldsymbol{A}) &= \lambda_1+\lambda_2+\lambda_3,\\ \iota_2(\boldsymbol{A}) &= \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3,\\ \iota_3(\boldsymbol{A}) &= \lambda_1\lambda_2\lambda_3.\end{aligned}$$

On the other hand,

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$\begin{aligned}&= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}\\ &\qquad -(a_{11}-\lambda)a_{23}a_{32}-(a_{22}-\lambda)a_{13}a_{31}-(a_{33}-\lambda)a_{12}a_{21}\\ &= -\lambda^3+(a_{11}+a_{22}+a_{33})\lambda^2-(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{23}a_{32}-a_{13}a_{31}-a_{12}a_{21})\lambda\\ &\qquad +a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31}-a_{33}a_{12}a_{21}\end{aligned}$$

implies

$$\begin{aligned}\iota_1(\boldsymbol{A}) &= a_{11}+a_{22}+a_{33}\\ &= \mathrm{tr}(\boldsymbol{A}),\\ \iota_2(\boldsymbol{A}) &= a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{23}a_{32}-a_{13}a_{31}-a_{12}a_{21}\\ &= \frac{1}{2}\Big[(a_{11}+a_{22}+a_{33})^2-(a_{11}^2+a_{22}^2+a_{33}^2+2a_{12}a_{21}+2a_{13}a_{31}+2a_{23}a_{32})\Big]\\ &= \frac{1}{2}\Big[(\mathrm{tr}\boldsymbol{A})^2-\mathrm{tr}(\boldsymbol{A}^2)\Big],\\ \iota_3(\boldsymbol{A}) &= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{22}a_{13}a_{31}-a_{33}a_{12}a_{21}\\ &= \det(\boldsymbol{A}).\end{aligned}$$

We also recall the Cayley–Hamilton theorem, which states that a matrix satisfies its own characteristic polynomial, i.e.

$$-\boldsymbol{A}^3+\iota_1(\boldsymbol{A})\boldsymbol{A}^2-\iota_2(\boldsymbol{A})\boldsymbol{A}+\iota_3(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{0},$$

in particular we have

$$\boldsymbol{A}^3=\iota_1(\boldsymbol{A})\boldsymbol{A}^2-\iota_2(\boldsymbol{A})\boldsymbol{A}+\iota_3(\boldsymbol{A})\boldsymbol{I}.$$

Then,

$$\begin{aligned}\boldsymbol{A}^4 = \boldsymbol{A}\,\boldsymbol{A}^3 &= \boldsymbol{A}\Big[\iota_1(\boldsymbol{A})\boldsymbol{A}^2-\iota_2(\boldsymbol{A})\boldsymbol{A}+\iota_3(\boldsymbol{A})\boldsymbol{I}\Big]\\ &= \iota_1(\boldsymbol{A})\boldsymbol{A}^3-\iota_2(\boldsymbol{A})\boldsymbol{A}^2+\iota_3(\boldsymbol{A})\boldsymbol{A}\\ &= \iota_1(\boldsymbol{A})\Big[\iota_1(\boldsymbol{A})\boldsymbol{A}^2-\iota_2(\boldsymbol{A})\boldsymbol{A}+\iota_3(\boldsymbol{A})\boldsymbol{I}\Big]-\iota_2(\boldsymbol{A})\boldsymbol{A}^2+\iota_3(\boldsymbol{A})\boldsymbol{A}\\ &= \Big([\iota_1(\boldsymbol{A})]^2-\iota_2(\boldsymbol{A})\Big)\boldsymbol{A}^2+\Big(\iota_3(\boldsymbol{A})-\iota_1(\boldsymbol{A})\iota_2(\boldsymbol{A})\Big)\boldsymbol{A}+\Big(\iota_1(\boldsymbol{A})\iota_3(\boldsymbol{A})\Big)\boldsymbol{I},\end{aligned}$$

and by induction we find

$$\boldsymbol{A}^k = q_{k,2}(\iota(\boldsymbol{A}))\boldsymbol{A}^2 + q_{k,1}(\iota(\boldsymbol{A}))\boldsymbol{A} + q_{k,0}(\iota(\boldsymbol{A}))\boldsymbol{I}, \qquad k \geq 0.$$

Hence we find the Rivlin–Ericksen representation theorem

$$\boldsymbol{R}(\boldsymbol{F}) = \beta_0(\iota(\boldsymbol{B}))\boldsymbol{I} + \beta_1(\iota(\boldsymbol{B}))\boldsymbol{B} + \beta_2(\iota(\boldsymbol{B}))\boldsymbol{B}^2 \tag{3.3}$$

where

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^\top = [D_x\boldsymbol{\varphi}(t,\boldsymbol{x})][D_x\boldsymbol{\varphi}(t,\boldsymbol{x})]^\top \tag{3.4}$$

is the left Cauchy–Green strain tensor, and the coefficients $\beta_k(\iota(\boldsymbol{B}))$ are functions in the invariants of $\boldsymbol{B}$.

For the second Piola transformation (2.19) we now obtain

$$\begin{aligned}
\boldsymbol{\Sigma} &= \det\boldsymbol{F}\,\boldsymbol{F}^{-1}\,\boldsymbol{T}\,\boldsymbol{F}^{-\top} \\
&= \det\boldsymbol{F}\,\boldsymbol{F}^{-1}\,\boldsymbol{R}(\boldsymbol{F})\,\boldsymbol{F}^{-\top} \\
&= \det\boldsymbol{F}\,\boldsymbol{F}^{-1}\left[\beta_0(\iota(\boldsymbol{B}))\boldsymbol{I} + \beta_1(\iota(\boldsymbol{B}))\boldsymbol{B} + \beta_2(\iota(\boldsymbol{B}))\boldsymbol{B}^2\right]\boldsymbol{F}^{-\top} \\
&= \det\boldsymbol{F}\,\boldsymbol{F}^{-1}\left[\beta_0(\iota(\boldsymbol{B}))\boldsymbol{I} + \beta_1(\iota(\boldsymbol{B}))\boldsymbol{F}\boldsymbol{F}^\top + \beta_2(\iota(\boldsymbol{B}))\boldsymbol{F}\boldsymbol{F}^\top\boldsymbol{F}\boldsymbol{F}^\top\right]\boldsymbol{F}^{-\top} \\
&= \det\boldsymbol{F}\;\left[\beta_0(\iota(\boldsymbol{B}))\boldsymbol{F}^{-1}\boldsymbol{F}^{-\top} + \beta_1(\iota(\boldsymbol{B}))\boldsymbol{I} + \beta_2(\iota(\boldsymbol{B}))\boldsymbol{F}^\top\boldsymbol{F}\right] \\
&= \det\boldsymbol{F}\;\left[\beta_0(\iota(\boldsymbol{B}))\boldsymbol{C}^{-1} + \beta_1(\iota(\boldsymbol{B}))\boldsymbol{I} + \beta_2(\iota(\boldsymbol{B}))\boldsymbol{C}\right]
\end{aligned}$$

where

$$\boldsymbol{C} = \boldsymbol{F}^\top\boldsymbol{F} = [D_x\boldsymbol{\varphi}(t,\boldsymbol{x})]^\top[D_x\boldsymbol{\varphi}(t,\boldsymbol{x})] \tag{3.5}$$

is the right Cauchy–Green strain tensor. With

$$\det\boldsymbol{F} = \det\boldsymbol{F}^\top, \quad [\det\boldsymbol{F}]^2 = \det\boldsymbol{F}^\top\det\boldsymbol{F} = \det\boldsymbol{F}^\top\boldsymbol{F} = \det\boldsymbol{C} = \iota_3(\boldsymbol{C})$$

and

$$\iota(\boldsymbol{B}) = \iota(\boldsymbol{F}\boldsymbol{F}^\top) = \iota(\boldsymbol{F}^\top\boldsymbol{F}) = \iota(\boldsymbol{C})$$

we find

$$\boldsymbol{\Sigma} = \sqrt{\iota_3(\boldsymbol{C})}\left[\beta_0(\iota(\boldsymbol{C}))\boldsymbol{C}^{-1} + \beta_1(\iota(\boldsymbol{C}))\boldsymbol{I} + \beta_2(\iota(\boldsymbol{C}))\boldsymbol{C}\right].$$

On the other hand, by using the Cayley–Hamilton theorem we have

$$-\boldsymbol{C}^3 + \iota_1(\boldsymbol{C})\boldsymbol{C}^2 - \iota_2(\boldsymbol{C})\boldsymbol{C} + \iota_3(\boldsymbol{C})\boldsymbol{I} \;=\; \boldsymbol{0},$$

and therefore

$$\boldsymbol{C}^{-1} \;=\; \frac{1}{\iota_3(\boldsymbol{C})}\Big[\boldsymbol{C}^2 - \iota_1(\boldsymbol{C})\boldsymbol{C} + \iota_2(\boldsymbol{C})\boldsymbol{I}\Big].$$

Hence we obtain

$$\begin{aligned}
\boldsymbol{\Sigma} &= \sqrt{\iota_3(\boldsymbol{C})}\left[\beta_0(\iota(\boldsymbol{C}))\boldsymbol{C}^{-1} + \beta_1(\iota(\boldsymbol{C}))\boldsymbol{I} + \beta_2(\iota(\boldsymbol{C}))\boldsymbol{C}\right] \\
&= \sqrt{\iota_3(\boldsymbol{C})}\left[\frac{\beta_0(\iota(\boldsymbol{C}))}{\iota_3(\boldsymbol{C})}\Big(\boldsymbol{C}^2 - \iota_1(\boldsymbol{C})\boldsymbol{C} + \iota_2(\boldsymbol{C})\boldsymbol{I}\Big) + \beta_1(\iota(\boldsymbol{C}))\boldsymbol{I} + \beta_2(\iota(\boldsymbol{C}))\boldsymbol{C}\right] \\
&= \gamma_0(\iota(\boldsymbol{C}))\boldsymbol{I} + \gamma_1(\iota(\boldsymbol{C}))\boldsymbol{C} + \gamma_2(\iota(\boldsymbol{C}))\boldsymbol{C}^2\,.
\end{aligned}$$

By using (1.4) we further conclude

$$\begin{aligned}\boldsymbol{C} &= [D_x\boldsymbol{\varphi}(t,\boldsymbol{x})]^\top[D_x\boldsymbol{\varphi}(t,\boldsymbol{x})]\\ &= [\boldsymbol{I}+D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top[\boldsymbol{I}+D_x\boldsymbol{u}(t,\boldsymbol{x})]\\ &= \boldsymbol{I}+[D_x\boldsymbol{u}(t,\boldsymbol{x})]+[D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top+[D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top[D_x\boldsymbol{u}(t,\boldsymbol{x})]\,.\end{aligned}$$

By using the Green–St. Venant strain tensor

$$\boldsymbol{E}=\frac{1}{2}[\boldsymbol{C}-\boldsymbol{I}]=\frac{1}{2}\Big[[D_x\boldsymbol{u}(t,\boldsymbol{x})]+[D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top+[D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top[D_x\boldsymbol{u}(t,\boldsymbol{x})]\Big] \tag{3.6}$$

we have

$$\boldsymbol{C}=\boldsymbol{I}+2\,\boldsymbol{E},$$

and therefore

$$\boldsymbol{\Sigma}=\gamma_0(\iota(\boldsymbol{I}+2\boldsymbol{E}))\boldsymbol{I}+\gamma_1(\iota(\boldsymbol{I}+2\boldsymbol{E}))(\boldsymbol{I}+2\boldsymbol{E})+\gamma_2(\iota(\boldsymbol{I}+2\boldsymbol{E}))(\boldsymbol{I}+2\boldsymbol{E})^2$$

follows. The aim is to find, for small deformations, a linear relation between $\boldsymbol{\Sigma}$ and $\boldsymbol{E}$. In particular we need to consider the principal invariants of $\boldsymbol{C}=\boldsymbol{I}+2\boldsymbol{E}$. We first have

$$\iota_1(\boldsymbol{C})=\iota_1(\boldsymbol{I}+2\boldsymbol{E})=\mathrm{tr}(\boldsymbol{I}+2\boldsymbol{E})=3+2\,\mathrm{tr}(\boldsymbol{E})\,.$$

By using

$$\mathrm{tr}(\boldsymbol{C}^2)=\mathrm{tr}((\boldsymbol{I}+2\boldsymbol{E})^2)=\mathrm{tr}(\boldsymbol{I}+4\boldsymbol{E}+4\boldsymbol{E}^2)=3+4\,\mathrm{tr}\boldsymbol{E}+4\,\mathrm{tr}\boldsymbol{E}^2$$

we further conclude

$$\begin{aligned}\iota_2(\boldsymbol{C}) &= \frac{1}{2}\Big[(\mathrm{tr}\,\boldsymbol{C})^2-\mathrm{tr}\boldsymbol{C}^2\Big]\\ &= \frac{1}{2}\Big[(3+2\,\mathrm{tr}\boldsymbol{E})^2-(3+4\,\mathrm{tr}\boldsymbol{E}+4\,\mathrm{tr}\boldsymbol{E}^2)\Big]\\ &= \frac{1}{2}\Big[9+12\,\mathrm{tr}\boldsymbol{E}+4(\mathrm{tr}\boldsymbol{E})^2-(3+4\,\mathrm{tr}\boldsymbol{E}+4\,\mathrm{tr}\boldsymbol{E}^2)\Big]\\ &= 3+4\,\mathrm{tr}\boldsymbol{E}+2[(\mathrm{tr}\boldsymbol{E})^2-\mathrm{tr}\boldsymbol{E}^2]\\ &= 3+4\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)\,.\end{aligned}$$

Moreover, with

$$\begin{aligned}\mathrm{tr}(\boldsymbol{C}^3) &= \mathrm{tr}(\boldsymbol{I}+2\boldsymbol{E})^3\\ &= \mathrm{tr}(\boldsymbol{I}+6\boldsymbol{E}+12\boldsymbol{E}^2+8\boldsymbol{E}^3)\\ &= 3+6\,\mathrm{tr}\boldsymbol{E}+12\,\mathrm{tr}\boldsymbol{E}^2+8\,\mathrm{tr}\boldsymbol{E}^3\end{aligned}$$

we have

$$
\begin{aligned}
\iota_3(\boldsymbol{C}) \;&=\; \det \boldsymbol{C} = \frac{1}{6}\Big[(\mathrm{tr}\boldsymbol{C})^3 - 3\,\mathrm{tr}\boldsymbol{C}\mathrm{tr}\boldsymbol{C}^2 + 2\,\mathrm{tr}\boldsymbol{C}^3\Big] \\
&= \frac{1}{6}\Big[(3+2\,\mathrm{tr}\boldsymbol{E})^3 - 3(3+2\,\mathrm{tr}\boldsymbol{E})(3+4\,\mathrm{tr}\boldsymbol{E}+4\mathrm{tr}\boldsymbol{E}^2) + 2(3+6\,\mathrm{tr}\boldsymbol{E}+12\mathrm{tr}\boldsymbol{E}^2+8\mathrm{tr}\boldsymbol{E}^3)\Big] \\
&= \frac{1}{6}\Big[27+54\mathrm{tr}\boldsymbol{E}+36(\mathrm{tr}\boldsymbol{E})^2+8(\mathrm{tr}\boldsymbol{E})^3+6+12\mathrm{tr}\boldsymbol{E}+24\mathrm{tr}\boldsymbol{E}^2+16\mathrm{tr}\boldsymbol{E}^3 \\
&\qquad -\Big(27+54\,\mathrm{tr}\boldsymbol{E}+36\mathrm{tr}\boldsymbol{E}^2+24(\mathrm{tr}\boldsymbol{E})^2+24\mathrm{tr}\boldsymbol{E}\mathrm{tr}\boldsymbol{E}^2\Big)\Big] \\
&= 1+2\mathrm{tr}\boldsymbol{E}+22(\mathrm{tr}\boldsymbol{E})^2-22\mathrm{tr}\boldsymbol{E}^2-4\mathrm{tr}\boldsymbol{E}\mathrm{tr}\boldsymbol{E}^2+\frac{4}{3}(\mathrm{tr}\boldsymbol{E})^3+\frac{8}{3}\mathrm{tr}\boldsymbol{E}^3 \\
&= 1+2\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)\,.
\end{aligned}
$$

Hence we have, by a Taylor expansion, for $i=0,1,2$,

$$
\begin{aligned}
\gamma_i(\iota(\boldsymbol{C})) \;&=\; \gamma_i(\iota_1(\boldsymbol{C}),\iota_2(\boldsymbol{C}),\iota_3(\boldsymbol{C})) \\
&= \gamma_i(3+2\,\mathrm{tr}\boldsymbol{E}, 3+4\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|), 1+2\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)) \\
&= \gamma_i(3,3,1)+\frac{\partial}{\partial\iota_1}\gamma_i(3,3,1)\,2\,\mathrm{tr}\boldsymbol{E}+\frac{\partial}{\partial\iota_2}\gamma_i(3,3,1)\,(4\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)) \\
&\qquad +\frac{\partial}{\partial\iota_3}\gamma_i(3,3,1)\,(2\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)+o(\|\boldsymbol{E}\|) \\
&= \gamma_i(3,3,1)+\widetilde{\gamma}_i(3,3,1)\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{\Sigma} \;&=\; \gamma_0(\iota(\boldsymbol{C}))\boldsymbol{I}+\gamma_1(\iota(\boldsymbol{C}))(\boldsymbol{I}+2\boldsymbol{E})+\gamma_2(\iota(\boldsymbol{C}))(\boldsymbol{I}+4\boldsymbol{E}+4\boldsymbol{E}^2) \\
&= \Big[\gamma_0(3,3,1)+\widetilde{\gamma}_0(3,3,1)\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)\Big]\boldsymbol{I} \\
&\quad +\Big[\gamma_1(3,3,1)+\widetilde{\gamma}_1(3,3,1)\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)\Big](\boldsymbol{I}+2\boldsymbol{E}) \\
&\quad +\Big[\gamma_2(3,3,1)+\widetilde{\gamma}_2(3,3,1)\,\mathrm{tr}\boldsymbol{E}+o(\|\boldsymbol{E}\|)\Big](\boldsymbol{I}+4\boldsymbol{E}+4\boldsymbol{E}^2) \\
&= \Big[\gamma_0(3,3,1)+\gamma_1(3,3,1)+\gamma_2(3,3,1)\Big]\,\boldsymbol{I} \\
&\quad +\Big[\widetilde{\gamma}_0(3,3,1)+\widetilde{\gamma}_1(3,3,1)+\widetilde{\gamma}_2(3,3,1)\Big]\,\mathrm{tr}(\boldsymbol{E}) \\
&\quad +\Big[2\gamma_1(3,3,1)+4\gamma_2(3,3,1)\Big]\,\boldsymbol{E}+o(\|\boldsymbol{E}\|).
\end{aligned}
$$

For a homogeneous, isotropic, and elastic material we therefore conclude a representation of the form

$$\boldsymbol{\Sigma} = -p\boldsymbol{I}+\lambda\,\mathrm{tr}\boldsymbol{E}\,\boldsymbol{I}+2\mu\boldsymbol{E}\,. \tag{3.7}$$

In the natural state we have no stress when no strain is given, i.e. $\boldsymbol{E}=\boldsymbol{0}$ implies $\boldsymbol{\Sigma}=\boldsymbol{0}$. In fact, this implies $p=0$ and therefore

$$\boldsymbol{\Sigma} = \lambda\,\mathrm{tr}\boldsymbol{E}\,\boldsymbol{I}+2\mu\boldsymbol{E} \tag{3.8}$$

follows. Since the strain tensor $\boldsymbol{E}$ is nonlinear, in the case of small deformations we consider its linear part

$$\boldsymbol{e}(\boldsymbol{u}) = \frac{1}{2}\Big[[D_x\boldsymbol{u}(t,\boldsymbol{x})] + [D_x\boldsymbol{u}(t,\boldsymbol{x})]^\top\Big], \tag{3.9}$$

i.e.

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left[\frac{\partial}{\partial x_i}u_j(\boldsymbol{x}) + \frac{\partial}{\partial x_j}u_i(\boldsymbol{x})\right] \quad \text{for } i,j = 1,2,3.$$

When replacing in (3.8) the strain tensor $\boldsymbol{E}$ by the linearized strain tensor $\boldsymbol{e}$, this gives the linearized stress tensor

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda \operatorname{div}\boldsymbol{u}\,\boldsymbol{I} + 2\mu\boldsymbol{e}(\boldsymbol{u})\,. \tag{3.10}$$

Note that the linear stress–strain relation (3.10) is known as Hooke's law, and λ and μ are the Lamé parameters.

3.2 Conservation of Energy

The conservation of energy for a mechanical system states that the rate of change of the total energy of the system is equal to the power input of the external forces, i.e.

$$\frac{d}{dt}\Big[\mathcal{K}(t) + \mathcal{U}(t)\Big] = \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} + \int_{\partial\omega(t)} \boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,ds_y. \tag{3.11}$$

Here,

$$\mathcal{K}(t) = \frac{1}{2}\int_{\omega(t)} \varrho(t,\boldsymbol{y})[\boldsymbol{v}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})]\,d\boldsymbol{y} \tag{3.12}$$

is the kinetic energy in the material region $\omega(t)$, and the internal energy for the control volumen $\omega(t)$ is given by

$$\mathcal{U}(t) = \int_{\omega(t)} \varrho(t,\boldsymbol{y})w(t,\boldsymbol{y})\,d\boldsymbol{y},$$

where $w(t,\boldsymbol{y})$ is the specific internal energy, i.e. the internal energy per unit mass.

The application of Reynold's transport theorem (Theorem 1.1) for $f(t,\boldsymbol{y}) = [v_i(t,\boldsymbol{y})]^2$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y})[v_i(t,\boldsymbol{y})]^2\,d\boldsymbol{y} &= \frac{1}{2}\int_{\omega(t)} \varrho(t,\boldsymbol{y})\frac{d}{dt}[v_i(t,\boldsymbol{y})]^2\,d\boldsymbol{y} \\ &= \int_{\omega(t)} \varrho(t,\boldsymbol{y})v_i(t,\boldsymbol{y})\frac{d}{dt}v_i(t,\boldsymbol{y})\,d\boldsymbol{y}\,.\end{aligned}$$

By inserting the Cauchy equations of motion, see (2.14), this gives

$$\frac{1}{2}\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y})[v_i(t,\boldsymbol{y})]^2\,d\boldsymbol{y} = \int_{\omega(t)} v_i(t,\boldsymbol{y})\Big[\varrho(t,\boldsymbol{y})f_i(t,\boldsymbol{y}) + \sum_{j=1}^{3}\frac{\partial}{\partial y_j}T_{ij}(t,\boldsymbol{y})\Big]\,d\boldsymbol{y}.$$

Hence we conclude, by summing up, by applying integration by parts, and by using the symmetry of the Cauchy stress tensor,

$$\begin{aligned}
\frac{d}{dt}\mathcal{K}(t) &= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} + \int_{\omega(t)} \sum_{i=1}^{3}\sum_{j=1}^{3} v_i(t,\boldsymbol{y})\frac{\partial}{\partial y_j}T_{ij}(t,\boldsymbol{y})d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} \\
&\quad + \int_{\omega(t)} \sum_{i=1}^{3}\sum_{j=1}^{3} \left\{ \frac{\partial}{\partial y_j}\Big[v_i(t,\boldsymbol{y})T_{ij}(t,\boldsymbol{y})\Big] - T_{ij}(t,\boldsymbol{y})\frac{\partial}{\partial y_j}v_i(t,\boldsymbol{y}) \right\} d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} + \int_{\partial\omega(t)} \sum_{i=1}^{3}\sum_{j=1}^{3} v_i(t,\boldsymbol{y})T_{ij}(t,\boldsymbol{y})n_j\,ds_y \\
&\quad - \int_{\omega(t)} \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\Big[T_{ij}(t,\boldsymbol{y}) + T_{ji}(t,\boldsymbol{y})\Big]\frac{\partial}{\partial y_j}v_i(t,\boldsymbol{y})d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} + \int_{\partial\omega(t)} \boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,ds_y \\
&\quad - \int_{\omega(t)} \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(t,\boldsymbol{y})\frac{1}{2}\Big[\frac{\partial}{\partial y_j}v_i(t,\boldsymbol{y}) + \frac{\partial}{\partial y_i}v_j(t,\boldsymbol{y})\Big]d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\,\boldsymbol{f}(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,d\boldsymbol{y} + \int_{\partial\omega(t)} \boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,ds_y \\
&\quad - \int_{\omega(t)} \boldsymbol{T}(t,\boldsymbol{y}) : \boldsymbol{e}(\boldsymbol{v})d\boldsymbol{y}
\end{aligned}$$

where

$$\boldsymbol{T}(t,\boldsymbol{y}) : \boldsymbol{e}(\boldsymbol{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(t,\boldsymbol{y})e_{ij}(\boldsymbol{v})$$

is the associated tensor product, and

$$e_{ij}(\boldsymbol{v}) = \frac{1}{2}\left[\frac{\partial}{\partial y_i}v_j(\boldsymbol{y}) + \frac{\partial}{\partial y_j}v_i(\boldsymbol{y})\right] \tag{3.13}$$

is the linearized Green strain tensor. From the conservation of energy we therefore find

$$\frac{d}{dt}\mathcal{U}(t) = \int_{\omega(t)} \boldsymbol{T}(t,\boldsymbol{y}) : \boldsymbol{e}(\boldsymbol{v})d\boldsymbol{y}.$$

On the other hand, the application of (2.8) gives

$$\frac{d}{dt}\mathcal{U}(t) = \frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y})w(t,\boldsymbol{y})\,d\boldsymbol{y} = \int_{\omega(t)} \varrho(t,\boldsymbol{y})\frac{d}{dt}w(t,\boldsymbol{y})\,d\boldsymbol{y},$$

and hence we conclude

$$\int_{\omega(t)} \varrho(t,\boldsymbol{y})\frac{d}{dt}w(t,\boldsymbol{y})\,d\boldsymbol{y} = \int_{\omega(t)} \boldsymbol{T}(t,\boldsymbol{y}) : \boldsymbol{e}(\boldsymbol{v})d\boldsymbol{y}$$

for all test volumina $\omega(t)$. In the case of continuous functions we finally obtain the energy equation

$$\varrho(t,\boldsymbol{y})\frac{d}{dt}w(t,\boldsymbol{y}) = \boldsymbol{T}(t,\boldsymbol{y}) : \boldsymbol{e}(\boldsymbol{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(t,\boldsymbol{y})\frac{\partial}{\partial y_j}v_i(t,\boldsymbol{y}). \tag{3.14}$$

3.3 Hyperelastic Materials

By using the ansatz

$$w(t,\boldsymbol{y}) = W(\boldsymbol{F}) \tag{3.15}$$

we obtain, by applying the chain rule,

$$\frac{d}{dt}w(t,\boldsymbol{y}) = \frac{d}{dt}W(\boldsymbol{F}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial F_{ij}}W(\boldsymbol{F})\frac{d}{dt}F_{ij}\,.$$

From $\boldsymbol{F} = D_x\boldsymbol{\varphi}(t,\boldsymbol{x})$ we further find

$$\begin{aligned}\frac{d}{dt}F_{ij} &= \frac{d}{dt}\frac{\partial}{\partial x_j}\varphi_i(t,\boldsymbol{x}) = \frac{\partial}{\partial x_j}\frac{d}{dt}y_i(t) = \frac{\partial}{\partial x_j}v_i(t,\boldsymbol{y})\\ &= \frac{\partial}{\partial x_j}v_i(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) = \sum_{k=1}^{3}\frac{\partial}{\partial y_k}v_i(t,\boldsymbol{y})\frac{\partial}{\partial x_j}\varphi_k(t,\boldsymbol{x}) = \sum_{k=1}^{3}\frac{\partial}{\partial y_k}v_i(t,\boldsymbol{y})F_{kj}\,.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\varrho(t,\boldsymbol{y})\frac{d}{dt}w(t,\boldsymbol{y}) &= \varrho(t,\boldsymbol{y})\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial F_{ij}}W(\boldsymbol{F})\sum_{k=1}^{3}\frac{\partial}{\partial y_k}v_i(t,\boldsymbol{y})F_{kj}\\ &= \varrho(t,\boldsymbol{y})\sum_{i=1}^{3}\sum_{k=1}^{3}\left(\sum_{j=1}^{3}\frac{\partial}{\partial F_{ij}}W(\boldsymbol{F})F_{kj}\right)\frac{\partial}{\partial y_k}v_i(t,\boldsymbol{y})\\ &= \varrho(t,\boldsymbol{y})\sum_{i=1}^{3}\sum_{k=1}^{3}T_{ik}(t,\boldsymbol{y})\frac{\partial}{\partial y_k}v_i(t,\boldsymbol{y}),\end{aligned}$$

and therefore

$$T_{ik}(t,\boldsymbol{y}) = \varrho(t,\boldsymbol{y})\sum_{j=1}^{3}\frac{\partial}{\partial F_{ij}}W(\boldsymbol{F})F_{kj},$$

i.e.

$$\boldsymbol{T}(t,\boldsymbol{y}) = \varrho(t,\boldsymbol{y})\frac{\partial}{\partial\boldsymbol{F}}W(\boldsymbol{F})\,\boldsymbol{F}^\top = \frac{\varrho_0(\boldsymbol{x})}{J(t)}\,\frac{\partial}{\partial\boldsymbol{F}}W(\boldsymbol{F})\,\boldsymbol{F}^\top, \tag{3.16}$$

where we have used

$$\frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F}) = \begin{pmatrix} \frac{\partial}{\partial F_{11}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{12}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{13}} W(\boldsymbol{F}) \\ \frac{\partial}{\partial F_{21}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{22}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{23}} W(\boldsymbol{F}) \\ \frac{\partial}{\partial F_{31}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{32}} W(\boldsymbol{F}) & \frac{\partial}{\partial F_{33}} W(\boldsymbol{F}) \end{pmatrix}.$$

From this we also find a representation for the first Piola transformation

$$\varrho_0(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F}) = J(t)\, \boldsymbol{T}(t, \boldsymbol{\varphi}(t, \boldsymbol{x}))\, \boldsymbol{F}^{-\top} = \boldsymbol{P}(t, \boldsymbol{x}), \tag{3.17}$$

and for the second Piola transformation (2.19)

$$\boldsymbol{\Sigma}(t, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}) \boldsymbol{F}^{-1} \frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F}). \tag{3.18}$$

From the symmetry of the Cauchy stress tensor $\boldsymbol{T}$ we have to ensure

$$\frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F})\, \boldsymbol{F}^\top = \boldsymbol{F} \left(\frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F}) \right)^\top, \tag{3.19}$$

which implies restrictions on the choice of the energy function $W(\boldsymbol{F})$. In fact, we write

$$\varrho_0(\boldsymbol{x}) W(\boldsymbol{F}) = \Psi(\boldsymbol{E}), \tag{3.20}$$

where

$$\boldsymbol{E} = \frac{1}{2} \left[\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I} \right]$$

is the Green–St. Venant strain tensor.

Lemma 3.1 *Assume*

$$\frac{\partial}{\partial \boldsymbol{E}} \Psi(\boldsymbol{E}) = \left(\frac{\partial}{\partial \boldsymbol{E}} \Psi(\boldsymbol{E}) \right)^\top \tag{3.21}$$

Then,

$$\varrho_0(\boldsymbol{x}) \frac{\partial}{\partial \boldsymbol{F}} W(\boldsymbol{F}) = \boldsymbol{F} \frac{\partial}{\partial \boldsymbol{E}} \Psi(\boldsymbol{E})$$

Proof: Let us consider the two–dimensional case $n = 2$ first, where we have

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2} \left[\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I} \right] \\ &= \frac{1}{2} \left(\begin{pmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} F_{11}^2 + F_{21}^2 - 1 & F_{11} F_{12} + F_{21} F_{22} \\ F_{11} F_{12} + F_{21} F_{22} & F_{12}^2 + F_{22}^2 - 1 \end{pmatrix}. \end{aligned}$$

With the chain rule we then conclude

$$
\begin{aligned}
\varrho_0(\boldsymbol{x})\frac{\partial}{\partial F_{11}}W(\boldsymbol{F}) &= \frac{\partial}{\partial F_{11}}\Psi(\boldsymbol{E}(\boldsymbol{F})) \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})\frac{\partial E_{11}}{\partial F_{11}} + \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})\frac{\partial E_{12}}{\partial F_{11}} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})\frac{\partial E_{21}}{\partial F_{11}} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})\frac{\partial E_{22}}{\partial F_{11}} \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})F_{11} + \frac{1}{2}\frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{12} + \frac{1}{2}\frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{12} \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})F_{11} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{12}, \\
\varrho_0(\boldsymbol{x})\frac{\partial}{\partial F_{12}}W(\boldsymbol{F}) &= \frac{\partial}{\partial F_{12}}\Psi(\boldsymbol{E}(\boldsymbol{F})) \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})\frac{\partial E_{11}}{\partial F_{12}} + \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})\frac{\partial E_{12}}{\partial F_{12}} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})\frac{\partial E_{21}}{\partial F_{12}} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})\frac{\partial E_{22}}{\partial F_{12}} \\
&= \frac{1}{2}\frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{11} + \frac{1}{2}\frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{11} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})F_{12} \\
&= \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{11} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})F_{12}, \\
\varrho_0(\boldsymbol{x})\frac{\partial}{\partial F_{21}}W(\boldsymbol{F}) &= \frac{\partial}{\partial F_{21}}\Psi(\boldsymbol{E}(\boldsymbol{F})) \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})\frac{\partial E_{11}}{\partial F_{21}} + \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})\frac{\partial E_{12}}{\partial F_{21}} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})\frac{\partial E_{21}}{\partial F_{21}} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})\frac{\partial E_{22}}{\partial F_{21}} \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})F_{21} + \frac{1}{2}\frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{22} + \frac{1}{2}\frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{22} \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})F_{21} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{22}, \\
\varrho_0(\boldsymbol{x})\frac{\partial}{\partial F_{22}}W(\boldsymbol{F}) &= \frac{\partial}{\partial F_{22}}\Psi(\boldsymbol{E}(\boldsymbol{F})) \\
&= \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E})\frac{\partial E_{11}}{\partial F_{22}} + \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})\frac{\partial E_{12}}{\partial F_{22}} + \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})\frac{\partial E_{21}}{\partial F_{22}} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})\frac{\partial E_{22}}{\partial F_{22}} \\
&= \frac{1}{2}\frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{21} + \frac{1}{2}\frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E})F_{21} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})F_{22} \\
&= \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E})F_{21} + \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E})F_{22},
\end{aligned}
$$

i.e. we have

$$
\varrho_0(\boldsymbol{x})\begin{pmatrix} \frac{\partial}{\partial F_{11}}W(\boldsymbol{F}) & \frac{\partial}{\partial F_{12}}W(\boldsymbol{F}) \\ \frac{\partial}{\partial F_{21}}W(\boldsymbol{F}) & \frac{\partial}{\partial F_{22}}W(\boldsymbol{F}) \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial E_{11}}\Psi(\boldsymbol{E}) & \frac{\partial}{\partial E_{12}}\Psi(\boldsymbol{E}) \\ \frac{\partial}{\partial E_{21}}\Psi(\boldsymbol{E}) & \frac{\partial}{\partial E_{22}}\Psi(\boldsymbol{E}) \end{pmatrix}.
$$

■

For the Cauchy stress tensor we therefore find

$$\boldsymbol{T}(t,\boldsymbol{y}) = \frac{\varrho_0(\boldsymbol{x})}{J(t)}\frac{\partial}{\partial \boldsymbol{F}}W(\boldsymbol{F})\,\boldsymbol{F}^\top = \frac{1}{J(t)}\,\boldsymbol{F}\frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E})\,\boldsymbol{F}^\top$$

which is symmetric if (3.21) is satisfied. For the first Piola transformation we then conclude

$$\boldsymbol{P}(t,\boldsymbol{x}) = J(t)\boldsymbol{T}(t,\boldsymbol{y})\boldsymbol{F}^{-\top} = \boldsymbol{F}\frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E}),$$

while for the second Piola transformation we finally obtain

$$\boldsymbol{\Sigma}(t,\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E}). \tag{3.22}$$

The constitutive law (3.22) obviously depends on the particular definition of the potential function $\Psi(\boldsymbol{E})$. For a general linear material law we may consider a second order Taylor expansion of $\Psi(\boldsymbol{E})$.

Example 3.1 *A second order Taylor expansion of the potential function $\Psi(\boldsymbol{E})$ gives*

$$\Psi(\boldsymbol{E}) \simeq \Psi(\mathbf{0}) + \sum_{i=1}^3\sum_{j=1}^3 E_{ij}\frac{\partial}{\partial E_{ij}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}} + \frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{\ell=1}^3 E_{ij}E_{k\ell}\frac{\partial^2}{\partial E_{ij}\partial E_{k\ell}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}}\,.$$

For simplicity we assume

$$\Psi(\mathbf{0}) = 0$$

and in the natural state we have

$$\Sigma_{ij} = \frac{\partial}{\partial E_{ij}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}} = 0\,.$$

Hence we have

$$\Psi(\boldsymbol{E}) \simeq \frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{\ell=1}^3 E_{ij}E_{k\ell}\frac{\partial^2}{\partial E_{ij}\partial E_{k\ell}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}}$$

and therefore

$$\Sigma_{ij} = \frac{\partial}{\partial E_{ij}}\Psi(\boldsymbol{E}) = \sum_{k=1}^3\sum_{\ell=1}^3 E_{k\ell}\frac{\partial^2}{\partial E_{ij}\partial E_{k\ell}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}} = \sum_{k=1}^3\sum_{\ell=1}^3 C_{ijk\ell}E_{k\ell}$$

with

$$C_{ijk\ell} = \frac{\partial^2}{\partial E_{ij}\partial E_{k\ell}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\mathbf{0}}$$

follows. The material law

$$\Sigma_{ij} = \sum_{k=1}^{3}\sum_{\ell=1}^{3} C_{ijk\ell}E_{k\ell}$$

includes $3^4 = 81$ *material parameters* $C_{ijk\ell}$*, but due to*

$$C_{ijk\ell} = \frac{\partial^2}{\partial E_{ij}\partial E_{k\ell}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\boldsymbol{0}} = \frac{\partial^2}{\partial E_{k\ell}\partial E_{ij}}\Psi(\boldsymbol{E})_{|\boldsymbol{E}=\boldsymbol{0}} = C_{k\ell ij}$$

we have some symmetry relations. Moreover, due to the symmetry relations $\Sigma_{ij} = \Sigma_{ji}$ *and* $E_{k\ell} = E_{\ell k}$ *we can use the Voigt notation*

$$\begin{pmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{33} \\ \Sigma_{12} \\ \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{1122} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{1133} & C_{2233} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1112} & C_{2212} & C_{3312} & C_{1212} & C_{1213} & C_{1223} \\ C_{1113} & C_{2213} & C_{3313} & C_{1213} & C_{1313} & C_{1323} \\ C_{1123} & C_{2223} & C_{3323} & C_{1223} & C_{1323} & C_{2323} \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{13} \\ E_{23} \end{pmatrix}$$

with 21 *parameters to be chosen. In the most simple case we have*

$$\begin{pmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{33} \\ \Sigma_{12} \\ \Sigma_{13} \\ \Sigma_{23} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{12} \\ E_{13} \\ E_{23} \end{pmatrix},$$

and a linearization again gives Hooke's law (3.10).

For the potential $\Psi(\boldsymbol{E})$ we may use a function in the invariants of $\boldsymbol{E}$, i.e.

$$\psi(\boldsymbol{E}) = \widetilde{\Psi}(\iota(\boldsymbol{E})) = \widetilde{\Psi}(\iota_1(\boldsymbol{E}), \iota_2(\boldsymbol{E}), \iota_3(\boldsymbol{E}))\,.$$

For the components of the second Piola stress tensor we then obtain from (3.22)

$$\Sigma_{ij} = \frac{\partial}{\partial E_{ij}}\widetilde{\Psi}(\iota(\boldsymbol{E})) = \sum_{k=1}^{3}\frac{\partial}{\partial \iota_k}\widetilde{\Psi}(\iota)\frac{\partial}{\partial E_{ij}}\iota_k(\boldsymbol{E})\,.$$

Hence we need to compute the partial derivatives of the invariants $\iota_k(\boldsymbol{E})$, $k = 1, 2, 3$.

Lemma 3.2 *The partial derivatives of the invariants $\iota_k(\boldsymbol{E})$, $k = 1, 2, 3$, are given as*

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{E}} \iota_1(\boldsymbol{E}) &= \boldsymbol{I}, && (3.23)\\ \frac{\partial}{\partial \boldsymbol{E}} \iota_2(\boldsymbol{E}) &= \operatorname{tr}(\boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E}, && (3.24)\\ \frac{\partial}{\partial \boldsymbol{E}} \iota_3(\boldsymbol{E}) &= \det \boldsymbol{E}\, \boldsymbol{E}^{-1}. && (3.25) \end{aligned}$$

Proof: For the first invariant

$$\iota_1(\boldsymbol{E}) = \operatorname{tr} \boldsymbol{E} = E_{11} + E_{22} + E_{33}$$

we obtain

$$\frac{\partial}{\partial E_{ij}} \iota_1(\boldsymbol{E}) = \frac{\partial}{\partial E_{ij}} \Big[E_{11} + E_{22} + E_{33} \Big] = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

i.e. (3.23). For the second invariant

$$\iota_2(\boldsymbol{E}) = E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21}$$

we compute

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{E}} \iota_2(\boldsymbol{E}) &= \frac{\partial}{\partial \boldsymbol{E}} \Big[E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21} \Big] \\ &= \begin{pmatrix} E_{22} + E_{33} & -E_{21} & -E_{31} \\ -E_{12} & E_{11} + E_{33} & -E_{32} \\ -E_{13} & -E_{23} & E_{11} + E_{22} \end{pmatrix} = \operatorname{tr}(\boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E}, \end{aligned}$$

i.e. (3.24). To prove (3.25), we first consider the case $n = 2$ where we have

$$\boldsymbol{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}, \quad \iota_3(\boldsymbol{E}) = \det \boldsymbol{E} = E_{11}E_{22} - E_{12}E_{21},$$

and therefore

$$\frac{\partial}{\partial \boldsymbol{E}} \det \boldsymbol{E} = \begin{pmatrix} E_{22} & -E_{21} \\ -E_{12} & E_{11} \end{pmatrix}$$

follows. On the other hand we have

$$\boldsymbol{E}^{-1} = \frac{1}{\det \boldsymbol{E}} \begin{pmatrix} E_{22} & -E_{12} \\ -E_{21} & E_{11} \end{pmatrix}.$$

Since $\boldsymbol{E}$ is symmetric, we finally conclude

$$\frac{\partial}{\partial \boldsymbol{E}} \det \boldsymbol{E} = \det \boldsymbol{E}\, \boldsymbol{E}^{-1}.$$

Similarly, for $n = 3$ we have

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{E}} \iota_3(\boldsymbol{E}) \\
&\quad = \frac{\partial}{\partial \boldsymbol{E}} \Big[E_{11}E_{22}E_{33} + E_{12}E_{23}E_{31} + E_{13}E_{32}E_{21} - E_{11}E_{23}E_{32} - E_{22}E_{13}E_{31} - E_{33}E_{12}E_{21} \Big] \\
&\quad = \begin{pmatrix} E_{22}E_{33} - E_{23}E_{32} & E_{23}E_{31} - E_{33}E_{21} & E_{32}E_{21} - E_{22}E_{31} \\ E_{13}E_{32} - E_{33}E_{12} & E_{11}E_{33} - E_{13}E_{31} & E_{12}E_{31} - E_{11}E_{32} \\ E_{12}E_{23} - E_{22}E_{13} & E_{13}E_{21} - E_{11}E_{23} & E_{11}E_{22} - E_{12}E_{21} \end{pmatrix} \\
&\quad = \det \boldsymbol{E}\, \boldsymbol{E}^{-1},
\end{aligned}$$

due to

$$\boldsymbol{E}^{-1} = \frac{1}{\det \boldsymbol{E}} \begin{pmatrix} E_{22}E_{33} - E_{23}E_{32} & E_{32}E_{13} - E_{33}E_{12} & E_{12}E_{23} - E_{22}E_{13} \\ E_{31}E_{23} - E_{33}E_{21} & E_{11}E_{33} - E_{13}E_{31} & E_{21}E_{13} - E_{11}E_{23} \\ E_{21}E_{32} - E_{22}E_{31} & E_{31}E_{12} - E_{11}E_{32} & E_{11}E_{22} - E_{12}E_{21} \end{pmatrix}.$$

■

Hence we obtain

$$\boldsymbol{\Sigma} = \frac{\partial}{\partial \iota_1} \widetilde{\Psi}(\iota(\boldsymbol{E}))\, \boldsymbol{I} + \frac{\partial}{\partial \iota_2} \widetilde{\Psi}(\iota(\boldsymbol{E})) \Big[(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E} \Big] + \frac{\partial}{\partial \iota_3} \widetilde{\Psi}(\iota(\boldsymbol{E})) \det \boldsymbol{E}\, \boldsymbol{E}^{-1}.$$

Example 3.2 *For the St. Venant–Kirchhoff material model we define*

$$\widetilde{\Psi}(\iota(\boldsymbol{E})) = \frac{1}{2}(\lambda + 2\mu)\, [\iota_1(\boldsymbol{E})]^2 - 2\mu\, \iota_2(\boldsymbol{E})$$

for which we compute

$$\frac{\partial}{\partial \iota_1} \widetilde{\Psi}(\iota(\boldsymbol{E})) = (\lambda + 2\mu)\, \iota_1(\boldsymbol{E}), \quad \frac{\partial}{\partial \iota_2} \widetilde{\Psi}(\iota(\boldsymbol{E})) = -2\mu, \quad \frac{\partial}{\partial \iota_3} \widetilde{\Psi}(\iota(\boldsymbol{E})) = 0.$$

Hence we obtain

$$\begin{aligned}
\boldsymbol{\Sigma} &= \frac{\partial}{\partial \iota_1} \widetilde{\Psi}(\iota(\boldsymbol{E}))\, \boldsymbol{I} + \frac{\partial}{\partial \iota_2} \widetilde{\Psi}(\iota(\boldsymbol{E})) \Big[(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E} \Big] \\
&= (\lambda + 2\mu)\, \iota_1(\boldsymbol{E})\, \boldsymbol{I} - 2\mu \Big[(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E} \Big] \\
&= (\lambda + 2\mu) \operatorname{tr} \boldsymbol{E}\, \boldsymbol{I} - 2\mu \Big[(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{I} - \boldsymbol{E} \Big] \\
&= \lambda \operatorname{tr} \boldsymbol{E}\, \boldsymbol{I} + 2\mu\, \boldsymbol{E}.
\end{aligned}$$

On the other hand, by using

$$\iota_1(\boldsymbol{E}) = \operatorname{tr} \boldsymbol{E}, \quad \iota_2(\boldsymbol{E}) = \frac{1}{2} \Big[(\operatorname{tr} \boldsymbol{E})^2 - \operatorname{tr} \boldsymbol{E}^2 \Big]$$

we also find the alternative representation

$$\begin{aligned}\widetilde{\Psi}(\iota(\boldsymbol{E})) &= \frac{1}{2}(\lambda+2\mu)\,[\iota_1(\boldsymbol{E})]^2 - 2\mu\,\iota_2(\boldsymbol{E}) \\ &= \frac{1}{2}(\lambda+2\mu)\,[\operatorname{tr}\boldsymbol{E}]^2 - \mu\Big[(\operatorname{tr}\boldsymbol{E})^2 - \operatorname{tr}\boldsymbol{E}^2\Big] \\ &= \frac{\lambda}{2}\,[\operatorname{tr}\boldsymbol{E}]^2 + \mu\operatorname{tr}\boldsymbol{E}^2 = \Psi(\boldsymbol{E})\,.\end{aligned}$$

By using

$$\boldsymbol{C} = \boldsymbol{I} + 2\boldsymbol{E}, \quad \boldsymbol{E} = \frac{1}{2}\Big[\boldsymbol{C} - \boldsymbol{I}\Big]$$

and

$$\iota_1(\boldsymbol{E}) = \operatorname{tr}\boldsymbol{E} = \operatorname{tr}\Big(\frac{1}{2}[\boldsymbol{C}-\boldsymbol{I}]\Big) = \frac{1}{2}(\operatorname{tr}\boldsymbol{C} - 3) = \frac{1}{2}(\iota_1(\boldsymbol{C}) - 3)$$

as well as

$$\begin{aligned}\iota_2(\boldsymbol{E}) &= E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33} - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21} \\ &= \frac{1}{4}(C_{11}-1)(C_{22}-1) + \frac{1}{4}(C_{11}-1)(C_{33}-1) + \frac{1}{4}(C_{22}-1)(C_{33}-1) \\ &\quad -\frac{1}{4}C_{23}C_{32} - \frac{1}{4}C_{13}C_{31} - \frac{1}{4}C_{12}C_{21} \\ &= \frac{1}{4}\Big[(C_{11}C_{22} - C_{11} - C_{22} + 1) + (C_{11}C_{33} - C_{11} - C_{33} + 1) \\ &\quad +(C_{22}C_{33} - C_{22} - C_{33} + 1)\Big] - \frac{1}{4}\Big[C_{23}C_{32} + C_{13}C_{31} + C_{12}C_{21}\Big] \\ &= \frac{1}{4}\Big[C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33} - C_{23}C_{32} - C_{13}C_{31} - C_{12}C_{21}\Big] \\ &\quad +\frac{1}{4}\Big[3 - 2(C_{11} + C_{22} + C_{33})\Big] \\ &= \frac{1}{4}\Big[\iota_2(\boldsymbol{C}) - 2\iota_1(\boldsymbol{C}) + 3\Big]\end{aligned}$$

we also have

$$\begin{aligned}\widetilde{\Psi}(\iota(\boldsymbol{E})) &= \frac{1}{2}\,(\lambda+2\mu)\,[\iota_1(\boldsymbol{E})]^2 - 2\mu\,\iota_2(\boldsymbol{E}) \\ &= \frac{1}{8}\,(\lambda+2\mu)\,[\iota_1(\boldsymbol{C})-3]^2 - \frac{1}{2}\,\mu\,\Big[\iota_2(\boldsymbol{C}) - 2\iota_1(\boldsymbol{C}) + 3\Big] \\ &= \frac{1}{8}\,(\lambda+2\mu)\,[\iota_1(\boldsymbol{C})-3]^2 + \mu\,\Big[\iota_1(\boldsymbol{C}) - 3\Big] - \frac{1}{2}\,\mu\,\Big[\iota_2(\boldsymbol{C}) - 3\Big]\,.\end{aligned}$$

The previous considerations motivate to write the potential function $\widetilde{\Psi}(\iota(\boldsymbol{E}))$ in its general form

$$\widetilde{\Psi}(\iota(\boldsymbol{E})) = \widehat{\Psi}(\iota(\boldsymbol{C})) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} c_{pq}(\iota_1(\boldsymbol{C})-3)^p(\iota_2(\boldsymbol{C})-3)^q,$$

with the Mooney–Rivlin material model

$$\widehat{\Psi}(\iota(\boldsymbol{C})) = c_{10}[\iota_1(\boldsymbol{C}) - 3] + c_{01}[\iota_2(\boldsymbol{C}) - 3]$$

as simple example.

In general we may include the third invariant $\iota_3(\boldsymbol{C})$ as well, i.e. we can write

$$\widehat{\Psi}(\iota(\boldsymbol{C})) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} c_{pqr}(\iota_1(\boldsymbol{C}) - 3)^p(\iota_2(\boldsymbol{C}) - 3)^q(\iota_3(\boldsymbol{C}) - 1)^r \,.$$

Then, by using

$$\Psi(\boldsymbol{E}) = \Psi(\frac{1}{2}(\boldsymbol{F}^\top \boldsymbol{F} - \boldsymbol{I})) = \Psi(\frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})) =: \widehat{\Psi}(\boldsymbol{C}),$$

and by applying the chain rule,

$$\frac{\partial}{\partial \boldsymbol{C}}\widehat{\Psi}(\boldsymbol{C}) = \frac{\partial}{\partial \boldsymbol{C}}\Psi(\frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})) = \frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E})_{|E=\frac{1}{2}(C-I)}\frac{\partial}{\partial \boldsymbol{C}}\boldsymbol{E} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E})$$

we finally conclude

$$\boldsymbol{\Sigma} = \frac{\partial}{\partial \boldsymbol{E}}\Psi(\boldsymbol{E}) = 2\,\frac{\partial}{\partial \boldsymbol{C}}\widehat{\Psi}(\boldsymbol{C})\,.$$

3.4 Incompressible Materials

In what follows our main interest is in the modelling of (almost) incompressible materials with

$$J = \det \boldsymbol{F} \approx 1\,.$$

In fact, we consider a decoupling of the deformation gradient $\boldsymbol{F}$ into an isochoric, volume preserving part $\overline{\boldsymbol{F}}$, and a volumetric, volume changing part. From the requirement $\det \overline{\boldsymbol{F}} = 1$ we conclude

$$\boldsymbol{F} = (J^{1/3}\,\boldsymbol{I})\,\overline{\boldsymbol{F}}\,.$$

For the right Cauchy–Green strain tensor we then obtain

$$\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F} = J^{2/3}\,\overline{\boldsymbol{F}}^\top\overline{\boldsymbol{F}} = J^{2/3}\,\overline{\boldsymbol{C}}, \quad \overline{\boldsymbol{C}} = \overline{\boldsymbol{F}}^\top\overline{\boldsymbol{F}}\,.$$

With this we define the potential

$$\widehat{\Psi}(\boldsymbol{C}) = U(J) + \overline{\Psi}(\overline{\boldsymbol{C}})$$

with the volumetric elastic response $U(J)$, and the isochoric elastic response $\overline{\Psi}(\overline{\boldsymbol{C}})$. Then we need to compute

$$\boldsymbol{\Sigma} = 2\,\frac{\partial}{\partial \boldsymbol{C}}\Big[U(J) + \overline{\Psi}(\overline{\boldsymbol{C}})\Big] = 2U'(J)\frac{\partial}{\partial \boldsymbol{C}}J + 2\frac{\partial}{\partial \boldsymbol{C}}\overline{\Psi}(J^{2/3}\boldsymbol{C})\,.$$

Lemma 3.3 *For the deformation gradient* $\boldsymbol{F}$ *we define* $J = \det \boldsymbol{F}$ *and* $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$*. Then,*

$$\frac{\partial}{\partial C} J \;=\; \frac{1}{2}\, J\, \boldsymbol{C}^{-1}\,.$$

Proof: From

$$\det \boldsymbol{C} = \det \boldsymbol{F}^\top \boldsymbol{F} = \det \boldsymbol{F}^\top \det \boldsymbol{F} = (\det \boldsymbol{F})^2 = J^2\,.$$

we first conclude

$$J = \det \boldsymbol{F} = \sqrt{\det \boldsymbol{C}}\,,$$

and by the chain rule we have

$$\frac{\partial}{\partial \boldsymbol{C}} J \;=\; \frac{1}{2} \frac{1}{\sqrt{\det \boldsymbol{C}}} \frac{\partial}{\partial \boldsymbol{C}} \det \boldsymbol{C} \;=\; \frac{1}{2} \frac{1}{J} \frac{\partial}{\partial \boldsymbol{C}} \det \boldsymbol{C}\,.$$

In the particular case $n = 2$ we have

$$\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad \det \boldsymbol{C} = C_{11}C_{22} - C_{12}C_{21}\,,$$

and therefore

$$\frac{\partial}{\partial \boldsymbol{C}} \det \boldsymbol{C} = \begin{pmatrix} C_{22} & -C_{21} \\ -C_{12} & C_{11} \end{pmatrix}$$

follows. On the other hand,

$$\boldsymbol{C}^{-1} = \frac{1}{\det \boldsymbol{C}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix}.$$

Since $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$ is symmetric, we finally conclude

$$\frac{\partial}{\partial \boldsymbol{C}} \det \boldsymbol{C} = \det \boldsymbol{C}\, \boldsymbol{C}^{-1} \;=\; J^2\, \boldsymbol{C}^{-1},$$

i.e.

$$\frac{\partial}{\partial \boldsymbol{C}} J = \frac{1}{2}\, J\, \boldsymbol{C}^{-1}\,.$$

Similarly, for $n = 3$ we have

$$\boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

and

$$\det \boldsymbol{C} \;=\; C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32} - C_{11}C_{23}C_{32} - C_{22}C_{13}C_{31} - C_{33}C_{12}C_{21}\,.$$

Hence,

$$\frac{\partial}{\partial \boldsymbol{C}}\det \boldsymbol{C} = \begin{pmatrix} C_{22}C_{33} - C_{23}C_{32} & C_{23}C_{31} - C_{33}C_{21} & C_{21}C_{32} - C_{22}C_{31} \\ C_{13}C_{32} - C_{33}C_{12} & C_{11}C_{33} - C_{13}C_{31} & C_{12}C_{31} - C_{11}C_{32} \\ C_{12}C_{23} - C_{22}C_{13} & C_{13}C_{21} - C_{11}C_{23} & C_{11}C_{22} - C_{12}C_{21} \end{pmatrix}.$$

Again, by using

$$\boldsymbol{C}^{-1} = \frac{1}{\det \boldsymbol{C}} \begin{pmatrix} C_{22}C_{33} - C_{23}C_{32} & C_{32}C_{13} - C_{33}C_{12} & C_{12}C_{23} - C_{22}C_{13} \\ C_{31}C_{23} - C_{33}C_{21} & C_{11}C_{33} - C_{13}C_{31} & C_{21}C_{13} - C_{11}C_{23} \\ C_{21}C_{32} - C_{22}C_{31} & C_{31}C_{12} - C_{11}C_{32} & C_{11}C_{22} - C_{12}C_{21} \end{pmatrix}$$

and the symmetry of $\boldsymbol{C}$ we conclude the assertion. ∎

Recall that

$$\frac{\partial}{\partial \boldsymbol{C}}\overline{\Psi}(\overline{\boldsymbol{C}}) = \begin{pmatrix} \frac{\partial}{\partial C_{11}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{12}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{13}}\overline{\Psi}(\overline{\boldsymbol{C}}) \\ \frac{\partial}{\partial C_{21}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{22}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{23}}\overline{\Psi}(\overline{\boldsymbol{C}}) \\ \frac{\partial}{\partial C_{31}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{32}}\overline{\Psi}(\overline{\boldsymbol{C}}) & \frac{\partial}{\partial C_{33}}\overline{\Psi}(\overline{\boldsymbol{C}}) \end{pmatrix}.$$

By the chain rule we then have

$$\begin{aligned}
\frac{\partial}{\partial C_{ij}}\overline{\Psi}(\overline{\boldsymbol{C}}) &= \sum_{k=1}^{3}\sum_{\ell=1}^{3} \frac{\partial}{\partial \overline{C}_{k\ell}}\overline{\Psi}(\overline{\boldsymbol{C}})\frac{\partial}{\partial C_{ij}}\overline{C}_{k\ell} \\
&= \sum_{k=1}^{3}\sum_{\ell=1}^{3} \frac{\partial}{\partial \overline{C}_{k\ell}}\overline{\Psi}(\overline{\boldsymbol{C}})\frac{\partial}{\partial C_{ij}}\left[J^{-2/3}C_{k\ell}\right] \\
&= \sum_{k=1}^{3}\sum_{\ell=1}^{3} \frac{\partial}{\partial \overline{C}_{k\ell}}\overline{\Psi}(\overline{\boldsymbol{C}})\left[J^{-2/3}\frac{\partial}{\partial C_{ij}}C_{k\ell} - \frac{2}{3}J^{-5/3}\frac{\partial}{\partial C_{ij}}J\, C_{k\ell}\right] \\
&= J^{-2/3}\frac{\partial}{\partial \overline{C}_{ij}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{2}{3}J^{-5/3}\frac{\partial}{\partial C_{ij}}J\,\sum_{k=1}^{3}\sum_{\ell=1}^{3}\frac{\partial}{\partial \overline{C}_{k\ell}}\overline{\Psi}(\overline{\boldsymbol{C}})C_{k\ell}.
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{C}}\overline{\Psi}(\overline{\boldsymbol{C}}) &= J^{-2/3}\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{2}{3}J^{-5/3}\frac{\partial}{\partial \boldsymbol{C}}J\,\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) : \boldsymbol{C} \\
&= J^{-2/3}\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{2}{3}J^{-5/3}\frac{1}{2}J\boldsymbol{C}^{-1}\,\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) : \boldsymbol{C} \\
&= J^{-2/3}\left[\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{1}{3}\left(\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) : \boldsymbol{C}\right)\boldsymbol{C}^{-1}\right].
\end{aligned}$$

We then conclude the constitutive relation

$$\begin{aligned}\boldsymbol{\Sigma} &= 2\,U'(J)\,\frac{\partial}{\partial \boldsymbol{C}}J + 2\,\frac{\partial}{\partial \boldsymbol{C}}\overline{\Psi}(J^{2/3}\boldsymbol{C}) \\ &= U'(J)\,J\,\boldsymbol{C}^{-1} + 2\,J^{-2/3}\left[\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{1}{3}\left(\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) : \boldsymbol{C}\right)\boldsymbol{C}^{-1}\right].\end{aligned}$$

When introducing the hydrostatic pressure

$$p = U'(J),$$

we finally obtain

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{vol} + \boldsymbol{\Sigma}_{isc}, \quad \boldsymbol{\Sigma}_{vol} = J\,p\,\boldsymbol{C}^{-1}, \quad \boldsymbol{\Sigma}_{isc} = 2J^{-2/3}\left[\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) - \frac{1}{3}\left(\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) : \boldsymbol{C}\right)\boldsymbol{C}^{-1}\right].$$

Example 3.3 *For the volumetric elastic response we may consider one of the following two choices:*

i.

$$U(J) = \kappa\frac{1}{2}(J-1)^2, \quad U'(J) = \kappa(J-1), \quad p = \kappa(J-1)$$

ii.

$$U(J) = \kappa\frac{1}{2}(\ln J)^2, \quad U'(J) = \kappa\frac{\ln J}{J}, \quad p = \kappa\frac{\ln J}{J}$$

As example for the isochoric elastic response we consider the Neo–Hooke model

$$\overline{\Psi}(\overline{\boldsymbol{C}}) = \frac{c}{2}(\iota_1(\overline{\boldsymbol{C}}) - 3) = \frac{c}{2}(\overline{C}_{11} + \overline{C}_{22} + \overline{C}_{33} - 3))$$

for which we compute

$$\frac{\partial}{\partial \overline{\boldsymbol{C}}}\overline{\Psi}(\overline{\boldsymbol{C}}) = \frac{c}{2}\,\boldsymbol{I}\,.$$