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# An energy space finite element approach for elliptic Dirichlet boundary control problems

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**Abstract**

In this paper we present a finite element analysis for a Dirichlet boundary control problem where the Dirichlet control is considered in a convex closed subspace of the energy space $H^{1/2}(\Gamma)$. As an equivalent norm in $H^{1/2}(\Gamma)$ we use an energy norm which is induced by a hypersingular boundary integral operator. The presented stability and error analysis of the discretization of the resulting variational inequality is based on the mapping properties of the solution operators related to the primal and adjoint boundary value problems, and their finite element approximations. Some numerical results are given.

## 1 Introduction

In this paper, the focus is on the a priori error analysis of the finite element approximation to minimise the cost functional

$$\mathcal{J}(u,z) \,=\, \mathcal{F}(u) + \frac{1}{2}\varrho\|z\|^2_{\mathcal{V}}\,, \tag{1.1}$$

where the state $u$ is the unique solution of a second order elliptic partial differential equation in a bounded two– or three–dimensional domain $\Omega$, satisfying a Dirichlet boundary condition $u = z$ on $\Gamma = \partial\Omega$. The non–negative cost functional $\mathcal{F}(u)$ describes, e.g., some physical quantity to be minimised, while $\|z\|^2_{\mathcal{V}}$ reflects either the costs of the control $z \in \mathcal{U} \subset \mathcal{V}$, or represents some regularization when considering, for example, an inverse source problem. Note that $\mathcal{V}$ is an appropriate Hilbert space to be specified, and $\mathcal{U} \subset \mathcal{V}$ is a convex closed subset which describes additional constraints on the control.

For a general analysis on optimal control problems governed by partial differential equations we refer to, e.g., [27, 40], for numerical methods to solve related variational inequalities, see, e.g., [13], and for a more recent overview on the theory and numerics of optimal control problems, see, for example, [16].

Optimal control problems (1.1) with a Dirichlet boundary control play an important role, for example, in the context of computational fluid mechanics., see, e.g., [12, 15], and the references given therein. In [15], the cost functional $\mathcal{J}(u, z) = \mathcal{F}(u)$ is the domain integral over the strain tensor of the velocity field $u$ satisfying the steady state Navier–Stokes equations with the Dirichlet boundary condition $u = z \in \mathcal{U} \subset H^{1/2}(\Gamma)$. A similar minimisation problem is considered in [12], where the cost functional $\mathcal{J}(u, z)$ is either written as boundary integral to describe the work needed to overcome the drag exerted on a given body, or equivalently as a domain integral over the strain of the velocity, and some additional boundary integral. In both cases, the cost functional $\mathcal{J}(u, z)$ describes an energy in $H^1(\Omega)$, or equivalently, in the Sobolev trace space $H^{1/2}(\Gamma)$. Hence, when considering the minimisation problem (1.1) subject to a Dirichlet boundary value problem, $\mathcal{V} = H^{1/2}(\Gamma)$ appears as a natural choice [2]. To obtain smoother optimal solutions one may even consider more regular cost functionals: In [21], $\mathcal{V} = H^2(\Gamma)$ is considered where the norm $\| \cdot \|_{H^2(\Gamma)}$ is realized by using the Laplace–Beltrami operator on the boundary $\Gamma$. Note that such an approach requires a sufficient regularity of the domain $\Omega$ which is assumed to be of the class $C^{2,1}$. In [27], several Neumann and Dirichlet boundary control problems with observations in the domain $\Omega$ and on the boundary $\Gamma$ are considered from an analytic point of view. To describe the involved Sobolev norms on the boundary $\Gamma$, in particular $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, certain fractional powers of the Laplace–Beltrami operator are used which seems to be complicated from a numerical point of view. The situation simplifies when $\mathcal{U} = L_2(\Gamma)$ is used as control space. Although this choice allows also the use of a piecewise constant control function $z$, the associated partial differential equation has to be considered within an ultra–weak variational formulation, see, for example, [27], and [3] for an appropriate finite element approximation using standard piecewise linear basis functions. The use of the ultra–weak variational formulation of the primal Dirichlet boundary value problem in the context of an optimal control problem requires the adjoint variable $p$ to be sufficiently regular, i.e., $p \in H^2(\Omega) \cap H^1_0(\Omega)$. Since the adjoint variable $p$ itself is the unique solution of the adjoint partial differential equation with homogeneous Dirichlet boundary conditions, either a smooth boundary $\Gamma$, or a polygonal or polyhedral but convex domain $\Omega$ has to be assumed. For related finite element approximations, see, e.g., [7, 10, 14, 25, 29], or [42] in the case of a finite dimensional Dirichlet control. Several variational formulations of Dirichlet control problems are discussed in [24]. To include a Dirichlet boundary condition $u = z \in L_2(\Gamma)$ in a standard variational formulation, alternatively one may consider a penalty approximation of the Dirichlet boundary condition by using a Robin boundary condition, see, e.g., [2, 17, 19, 20]. Again, sufficient smoothness of the boundary $\Gamma$ has to be assumed.

The aim of this paper is to present a numerical analysis of an energy space finite element approach when the control $z$ is considered as an element of the boundary trace space $H^{1/2}(\Gamma)$, where an equivalent norm is realized by using a hypersingular boundary integral operator. Note that in this case the costs represent the energy of the harmonic extension of the Dirichlet control $z$. This approach allows the use of a standard finite element method in $H^1(\Omega)$ by using, e.g., piecewise linear basis functions, to solve both the primal and adjoint Dirichlet boundary value problems. The main difference to the more common approach when using $L_2(\Gamma)$ as control space appears in the optimality condition. In particular, the hypersingular boundary integral operator links the Dirichlet control with the normal derivative of the adjoint variable. The related optimality condition results then in a higher regularity of the control, and requires less assumptions on the smoothness of the adjoint variable, in fact, we may even consider general Lipschitz domains $\Omega$. Moreover, for polygonal or polyhedral bounded domains $\Omega$ we also obtain higher order convergence results for the approximate finite element solution. Indeed, the use of $H^{1/2}(\Gamma)$ as control space reflects the proper mapping properties of the Dirichlet to Neumann map which appears within the optimality condition. Instead, when using $L_2(\Gamma)$ as control space, Dirichlet and Neumann boundary conditions are identified with each other. In particular for polygonal and polyhedral domains $\Omega$ the control $z$ will then be zero in all corner points which seems to be not motivated from the application in mind.

The paper is organised as follows: In Section 2, we describe the considered Dirichlet boundary control problem, the primal boundary value problem, and the reduced cost functional as well as the related adjoint boundary value problem. The minimiser of the reduced cost functional is characterised as the unique solution of a variational inequality of the first kind. The finite element discretization of the variational inequality is described in Sect. 3, where also finite element approximations of both the primal and adjoint boundary value problems and related error estimates are given. These approximations are used to describe and to analyse the finite element discretization of the perturbed variational inequality. The main results of this paper are the error estimates as given in Corollary 3.9. Some numerical results are finally given in Section 4, where we also give a comparison with the more common approach when considering the control in $L_2(\Gamma)$.

For an overview on the used Sobolev spaces in the domain and on the boundary, see, for example, [1, 30, 36, 39].

# 2 Dirichlet control problems

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Dirichlet boundary control problem to minimise the cost functional

$$J(u, z) \,=\, \frac{1}{2}\int_\Omega [u(x) - \overline{u}(x)]^2 dx + \frac{1}{2}\varrho |z|^2_{H^{1/2}(\Gamma)} \tag{2.1}$$

subject to the constraint

$$-\Delta u(x) \,=\, f(x) \quad \text{for } x \in \Omega, \quad u(x) \,=\, z(x) \quad \text{for } x \in \Gamma, \tag{2.2}$$

and where the control satisfies the box constraints

$$z \in \mathcal{U} := \Big\{ w \in H^{1/2}(\Gamma) \,:\, g_a(x) \le w(x) \le g_b(x) \quad \text{for almost all } x \in \Gamma \Big\}. \tag{2.3}$$

We assume $f \in L_2(\Omega)$, $\overline{u} \in L_2(\Omega)$, $\varrho \in \mathbb{R}_+$, and $g_a, g_b \in H^{1/2}(\Gamma)$.

The choice $\mathcal{V} = H^{1/2}(\Gamma)$ ensures the well–posedness of the standard variational formulation of the Dirichlet boundary value problem (2.2). In particular, for any $z \in H^{1/2}(\Gamma)$ there exists the harmonic extension $u_z \in H^1(\Omega)$ as the unique solution of the homogeneous Dirichlet boundary value problem

$$-\Delta u_z(x) \;=\; 0 \quad \text{for } x \in \Omega, \quad u_z(x) = z(x) \quad \text{for } x \in \Gamma. \tag{2.4}$$

Using Green's first formula, this motivates the use of the semi–norm

$$|z|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \frac{\partial}{\partial n_x} u_z(x) u_z(x) ds_x \;=\; \int_\Omega |\nabla u_z(x)|^2 dx,$$

which describes the energy of the harmonic extension $u_z \in H^1(\Omega)$ of the Dirichlet control $z \in H^{1/2}(\Gamma)$. By introducing the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$,

$$(Sz)(x) := \frac{\partial}{\partial n_x} u_z(x) \quad \text{for } x \in \Gamma,$$

which realizes the Dirichlet to Neumann map related to the Dirichlet boundary value problem (2.4), we can rewrite the semi–norm as

$$|z|^2_{H^{1/2}(\Gamma)} \;=\; \langle Sz, z\rangle_\Gamma. \tag{2.5}$$

Note that the duality pairing

$$\langle \tau, w\rangle_\Gamma \;:=\; \int_\Gamma \tau(x) w(x) ds_x \quad \text{for all } w \in H^{1/2}(\Gamma), \tau \in H^{-1/2}(\Gamma)$$

has to be understood as an extension of the inner product in $L_2(\Gamma)$, see, e.g., [22, Chapt. 4].

By using boundary integral equations which are associated to the Dirichlet boundary value problem (2.4), we can represent the Steklov–Poincaré operator $S$ in an explicit form, see, e.g., [22, 36]. Since the hypersingular boundary integral operator

$$(Dz)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \quad \text{for } x \in \Gamma$$

with the Laplace fundamental solution

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } n = 3 \end{cases}$$

is spectrally equivalent to the Steklov–Poincaré operator $S$, instead of (2.5) we will use the equivalent semi–norm

$$|z|^2_{H^{1/2}(\Gamma)} \;:=\; \langle Dz, z\rangle_\Gamma. \tag{2.6}$$

By using integration by parts [32, 36], we can rewrite the bilinear form of the hypersingular integral operator $D$ as

$$\langle Dz, w\rangle_\Gamma \;=\; -\frac{1}{2\pi}\int_\Gamma \frac{d}{dt}w(x(t))\int_\Gamma \log|x(t)-y(s)|\frac{d}{ds}z(y(s))ds_{y(s)}ds_{x(t)} \quad \text{for } n=2 \tag{2.7}$$

and

$$\langle Dz, w\rangle_\Gamma \;=\; \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\underline{\mathrm{curl}}_\Gamma z(y)\cdot \underline{\mathrm{curl}}_\Gamma w(x)}{|x-y|}ds_y ds_x \quad \text{for } n=3 \tag{2.8}$$

where

$$\underline{\mathrm{curl}}_\Gamma w(x) \;=\; n_x \times \nabla_x \widetilde{w}(x)$$

defines the surface curl, and $\widetilde{w}$ is a suitable extension of the given $w$ on $\Gamma$ into a three–dimensional neighbourhood of $\Gamma$. Hence we can express the bilinear form of the hypersingular integral operator $D$ and therefore the energy semi–norm $|\cdot|^2_{H^{1/2}(\Gamma)}$ by using the weakly singular single layer integral operator $V$ only.

## 2.1 Primal boundary value problem

To rewrite the Dirichlet boundary control problem (2.1)–(2.3) by using a reduced cost functional, we introduce a linear solution operator describing the application of the constraint (2.2). Therefore we consider the homogeneous partial differential equation (2.4) with the control as Dirichlet boundary condition, and an inhomogeneous partial differential equation with zero Dirichlet boundary conditions to describe a particular solution. Let $u_f$ be the weak solution of the Dirichlet boundary value problem

$$-\Delta u_f(x) \;=\; f(x) \quad \text{for } x\in\Omega, \quad u_f(x) \;=\; 0 \quad \text{for } x \in \Gamma, \tag{2.9}$$

i.e., $u_f \in H^1_0(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla u_f(x)\cdot\nabla v(x)dx \;=\; \int_\Omega f(x)v(x)dx \quad \text{for all } v\in H^1_0(\Omega). \tag{2.10}$$

The solution of the Dirichlet boundary value problem (2.2) is then given by $u = u_z + u_f$, where $u_z \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem (2.4). By applying the inverse trace theorem, see, e.g., [22, 30, 39] or [36, Theorem 2.22], there exists a bounded extension $\mathcal{E}z \in H^1(\Omega)$ for $z \in H^{1/2}(\Gamma)$. With $u_z = u_0 + \mathcal{E}z$, the variational formulation of the Dirichlet boundary value problem (2.4) is to find $u_0 \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u_0(x)\cdot\nabla v(x)dx \;=\; -\int_\Omega \nabla\mathcal{E}z(x)\cdot\nabla v(x)dx \quad \text{for all } v\in H^1_0(\Omega). \tag{2.11}$$

By using standard arguments we can ensure the unique solvability of the variational formulations (2.10) and (2.11), respectively. Due to the compact imbedding $H^1(\Omega)\subset L_2(\Omega)$ we may introduce the bounded solution operator $\mathcal{S}: H^{1/2}(\Gamma)\to L_2(\Omega)$, $u_z = \mathcal{S}z$. Hence we can write the solution of the primal boundary value problem (2.2) as $u = \mathcal{S}z + u_f$.

## 2.2 Reduced cost functional and adjoint boundary value problem

By using $u = \mathcal{S}z + u_f$, we can write the cost functional (2.1) as the reduced cost functional

$$\widetilde{J}(z) \;=\; \frac{1}{2}\langle \mathcal{S}^*\mathcal{S}z, z\rangle_\Gamma + \langle \mathcal{S}^*(u_f - \overline{u}), z\rangle_\Gamma + \frac{1}{2}\,\|u_f - \overline{u}\|^2_{L_2(\Omega)} + \frac{1}{2}\varrho\langle Dz, z\rangle_\Gamma, \tag{2.12}$$

where $\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is the adjoint operator of $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$. For the application $\tau = \mathcal{S}^*\psi \in H^{-1/2}(\Gamma)$, $\psi \in L_2(\Omega)$, we have

$$\langle \tau, \varphi\rangle_\Gamma = \langle \mathcal{S}^*\psi, \varphi\rangle_\Gamma = -\int_\Gamma \frac{\partial}{\partial n_x} p(x)\varphi(x) ds_x \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \tag{2.13}$$

where the adjoint variable $p$ is the weak solution of the Dirichlet boundary value problem

$$-\Delta p(x) \;=\; \psi(x) \quad \text{for } x \in \Omega, \quad p(x) \;=\; 0 \quad \text{for } x \in \Gamma, \tag{2.14}$$

i.e., $p \in H^1_0(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla p(x) \cdot \nabla q(x) dx \;=\; \int_\Omega \psi(x) q(x) dx \quad \text{for all } q \in H^1_0(\Omega). \tag{2.15}$$

Finally, by using Green's first formula we can rewrite (2.13) for $\varphi \in H^{1/2}(\Gamma)$ as

$$\langle \tau, \varphi\rangle_\Gamma = -\int_\Gamma \frac{\partial}{\partial n_x} p(x)\varphi(x) ds_x = -\int_\Omega \nabla p(x) \cdot \nabla \mathcal{E}\varphi(x) dx + \int_\Omega \psi(x)\mathcal{E}\varphi(x) dx, \tag{2.16}$$

where $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega)$ is the bounded extension operator.

## 2.3 Optimality condition

To characterise the minimiser of the reduced cost functional (2.12) we introduce the self–adjoint and bounded operator

$$T_\varrho \;:=\; \mathcal{S}^*\mathcal{S} + \varrho D \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \tag{2.17}$$

satisfying

$$\|T_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_2^{T_\varrho}\, \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma),$$

and define

$$g \;:=\; \mathcal{S}^*(\overline{u} - u_f) \in H^{-1/2}(\Gamma).$$

Hence we can rewrite (2.12) as

$$\widetilde{J}(z) \;=\; \frac{1}{2}\,\langle T_\varrho z, z\rangle_\Gamma - \langle g, z\rangle_\Gamma + \frac{1}{2}\,\|u_f - \overline{u}\|^2_{L_2(\Omega)}. \tag{2.18}$$

**Lemma 2.1** *The operator $T_\varrho$ as defined in* (2.17) *is $H^{1/2}(\Gamma)$–elliptic, i.e., there exists a positive constant $c_1^{T_\varrho}$ such that*

$$\langle T_\varrho z, z\rangle_\Gamma \;\geq\; c_1^{T_\varrho}\,\|z\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

**Proof.** For $z \in H^{1/2}(\Gamma)$, let $u_z = \mathcal{S}z \in H^1(\Omega)$ be the harmonic extension satisfying the Dirichlet boundary value problem (2.4). Then,

$$u_z(x) = u_{z,0}(x) + \frac{1}{|\Omega|}\int_\Omega u_z(y)dy$$

defines an orthogonal splitting in $L_2(\Omega)$ implying

$$\|\mathcal{S}z\|^2_{L_2(\Omega)} = \|u_z\|^2_{L_2(\Omega)} = \|u_{z,0}\|^2_{L_2(\Omega)} + \left(\frac{1}{|\Omega|}\int_\Omega u_z(y)dy\right)^2.$$

Hence we obtain

$$\langle T_\varrho z, z\rangle_\Gamma = \|\mathcal{S}z\|^2_{L_2(\Omega)} + \varrho\langle Dz, z\rangle_\Gamma \geq \left(\frac{1}{|\Omega|}\int_\Omega u_z(y)dy\right)^2 + \varrho c_1^D |z|^2_{H^{1/2}(\Gamma)} \geq c\,\|z\|^2_{H^{1/2}(\Gamma),\Omega},$$

where

$$\|z\|^2_{H^{1/2}(\Gamma),\Omega} := \left(\int_\Omega (Sz)(y)dy\right)^2 + |z|^2_{H^{1/2}(\Gamma)}$$

defines an equivalent norm in $H^{1/2}(\Gamma)$. ∎

Since $\mathcal{U} \subset H^{1/2}(\Gamma)$ is convex and closed, and since $T_\varrho$ is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, the minimisation of (2.18) is equivalent to solving a variational inequality to find $z \in \mathcal{U}$ such that

$$\langle T_\varrho z, w - z\rangle_\Gamma \;\geq\; \langle g, w - z\rangle_\Gamma \quad \text{for all } w \in \mathcal{U}. \tag{2.19}$$

Since (2.19) is an elliptic variational inequality of the first kind, we can use standard arguments as given, for example in [5, 13, 27, 28], to establish unique solvability of the variational inequality (2.19).

# 3 Finite element approximations

Let

$$S_h^1(\Omega) := \text{span}\{\varphi_k\}_{k=1}^M \subset H^1(\Omega), \quad S_{h,0}^1(\Omega) := \text{span}\{\varphi_k\}_{k=1}^{M_\Omega} \subset H_0^1(\Omega) \tag{3.1}$$

be some conforming finite element spaces of, e.g., piecewise linear and continuous basis functions $\varphi_k$, which are defined with respect to some admissible domain triangulation $\Omega_h$ of mesh size $h$. Let

$$S_h^1(\Gamma) := S_h^1(\Omega)_{|\Gamma} = \text{span}\{\phi_i\}_{i=1}^{M_\Gamma} \subset H^{1/2}(\Gamma) \tag{3.2}$$

be the boundary finite element space of, e.g., piecewise linear and continuous basis functions $\phi_i$ which are the boundary traces of the domain basis functions $\varphi_{M_\Omega+i}$ as given in (3.1). Define the discrete convex set

$$\mathcal{U}_h \;:=\; \left\{ w_h \in S_h^1(\Gamma) \,:\, g_a(x_i) \leq w_h(x_i) \leq g_b(x_i) \quad \text{for all nodes } x_i \in \Gamma \right\}.$$

Then the Galerkin discretization of the variational inequality (2.19) is to find $z_h \in \mathcal{U}_h$ such that

$$\langle T_\varrho z_h, w_h - z_h \rangle_\Gamma \;\geq\; \langle g, w_h - z_h \rangle_\Gamma \quad \text{for all } w_h \in \mathcal{U}_h. \tag{3.3}$$

**Theorem 3.1** *Let $z \in \mathcal{U}$ and $z_h \in \mathcal{U}_h$ be the unique solutions of the variational inequalities* (2.19) *and* (3.3)*, respectively. If we assume $z, g_a, g_b \in H^s(\Gamma)$ for some $s \in [\frac{1}{2}, 2]$, then there hold the error estimates*

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \;\leq\; c\, h^{s-\frac{1}{2}}\, \|z\|_{H^s(\Gamma)} \tag{3.4}$$

*and*

$$\|z - z_h\|_{L_2(\Gamma)} \;\leq\; c\, h^s\, \|z\|_{H^s(\Gamma)}\,. \tag{3.5}$$

**Proof.** The error estimate (3.4) in the energy norm follows from the general abstract theory as presented, e.g., in [6, 11], see also [13]. The error estimate (3.5) follows from the Aubin–Nitsche trick for variational inequalities, see [31] for the case $\mathcal{U}_h \subset \mathcal{U}$, and [38] for the more general case $\mathcal{U}_h \not\subset \mathcal{U}$. ■

**Remark 3.1** *Note that the proof as given in* [6] *also covers the case of a polygonal or polyhedral approximation $\Omega_h$ of the computational domain $\Omega$, see also* [4, 8, 37].

Although the error estimates (3.4) and (3.5) seem to be optimal, the operator $T_\varrho$ as considered in the variational inequality (3.3) does not allow a practical implementation, since this would require the discretization of the composed solution operator $\mathcal{S}^*\mathcal{S}$, which is not possible in general. Hence, instead of (3.3) we need to consider a perturbed variational inequality to find $\widetilde{z}_h \in \mathcal{U}_h$ such that

$$\langle \widetilde{T}_\varrho \widetilde{z}_h, w_h - \widetilde{z}_h \rangle_\Gamma \;\geq\; \langle \widetilde{g}, w_h - \widetilde{z}_h \rangle_\Gamma \quad \text{for all } w_h \in \mathcal{U}_h, \tag{3.6}$$

where $\widetilde{T}_\varrho$ and $\widetilde{g}$ are appropriate approximations of $T_\varrho$ and $g$, respectively. The following theorem presents an abstract consistency result, which will later be used to analyse the finite element approximation of both the primal and adjoint boundary value problem.

**Theorem 3.2** *Let $\widetilde{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be a bounded and $S_h^1(\Gamma)$–elliptic approximation of $T_\varrho$ satisfying*

$$\langle \widetilde{T}_\varrho z_h, z_h \rangle_\Gamma \;\geq\; c_1^{\widetilde{T}_\varrho}\, \|z_h\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z_h \in S_h^1(\Gamma)$$

*and*

$$\|\widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \leq c_2^{\widetilde{T}_\varrho}\,\|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

*Let* $\widetilde{g} \in H^{-1/2}(\Gamma)$ *be some approximation of* $g$. *For the unique solution* $\widetilde{z}_h \in \mathcal{U}_h$ *of the perturbed variational inequality* (3.6) *there holds the error estimate*

$$\begin{aligned}\|z-\widetilde{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\; & \left(1+\frac{1}{c_1^{\widetilde{T}_\varrho}}[c_2^{T_\varrho}+c_2^{\widetilde{T}_\varrho}]\right)\|z-z_h\|_{H^{1/2}(\Gamma)} \\ & +\frac{1}{c_1^{\widetilde{T}_\varrho}}\left[\|(T_\varrho-\widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)}+\|g-\widetilde{g}\|_{H^{-1/2}(\Gamma)}\right],\end{aligned} \tag{3.7}$$

*where* $z_h \in \mathcal{U}_h$ *is the unique solution of the discrete variational inequality* (3.3).

**Proof.** The unique solvability of the discrete variational inequality (3.6) follows from the $S_h^1(\Gamma)$–ellipticity of $\widetilde{T}_\varrho$. From this we further obtain

$$\begin{aligned} c_1^{\widetilde{T}_\varrho}\,\|z_h-\widetilde{z}_h\|^2_{H^{1/2}(\Gamma)} &\leq \langle \widetilde{T}_\varrho(z_h-\widetilde{z}_h), z_h-\widetilde{z}_h\rangle_\Gamma \\ &\leq \langle \widetilde{T}_\varrho z_h, z_h-\widetilde{z}_h\rangle_\Gamma + \langle \widetilde{g}-g, \widetilde{z}_h - z_h\rangle_\Gamma + \langle T_\varrho z_h, \widetilde{z}_h - z_h\rangle_\Gamma \\ &\leq \left[\|(\widetilde{T}_\varrho - T_\varrho)z_h\|_{H^{-1/2}(\Gamma)} + \|g-\widetilde{g}\|_{H^{-1/2}(\Gamma)}\right]\|z_h-\widetilde{z}_h\|_{H^{1/2}(\Gamma)}\,. \end{aligned}$$

The assertion now follows from the triangle inequality, and by using the boundedness of $T_\varrho$ and $\widetilde{T}_\varrho$, respectively. ■

It remains to define a suitable approximation $\widetilde{T}_\varrho := \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}} + \varrho D$ which is based on finite element approximations $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}^*$ of the primal and adjoint boundary value problem, respectively.

## 3.1 Primal boundary value problem

The application of the solution operator $u_z = \mathcal{S}z = u_0 + \mathcal{E}z$, $u_0 \in H_0^1(\Omega)$, is given as the unique solution of the variational problem (2.11). The finite element approximation of (2.11) is to find $u_{0,h} \in S_{h,0}^1(\Omega)$ such that

$$\int_\Omega \nabla u_{0,h}(x)\cdot\nabla v_h(x)dx \;=\; -\int_\Omega \nabla\mathcal{E}z(x)\cdot\nabla v_h(x)dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega). \tag{3.8}$$

By using Cea's lemma, see, e.g., [4] or [36, Theorem 8.1], and by using the approximation property of $S_{h,0}^1(\Omega)$, see, e.g., [36, Theorem 9.10], we conclude the unique solvability of the Galerkin formulation (3.8) as well as the error estimate

$$\|u_0-u_{0,h}\|_{H^1(\Omega)} \leq c\,h^s\,|u_0|_{H^{1+s}(\Omega)} \tag{3.9}$$

when assuming $u_0 \in H^{1+s}(\Omega)$ for some $s \in [\frac{1}{2},1]$. By applying the Aubin–Nitsche trick, see, e.g., [4] or [36, Theorem 11.1], we also obtain an error estimate in $L_2(\Omega)$,

$$\|u_0-u_{0,h}\|_{L_2(\Omega)} \leq c\,h^{2s}\,|u_0|_{H^{1+s}(\Omega)}. \tag{3.10}$$

However, instead of the variational problem (3.8), we will consider a perturbed variational problem to find an approximate solution $\widetilde{u}_{0,h} \in S^1_{h,0}(\Omega)$ such that

$$\int_\Omega \nabla \widetilde{u}_{0,h}(x) \cdot \nabla v_h(x) dx \;=\; - \int_\Omega \nabla Q_h \mathcal{E} z(x) \cdot \nabla v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega), \qquad (3.11)$$

where $Q_h : H^1(\Omega) \to S^1_h(\Omega)$ denotes a quasi–interpolation operator [35]. Since the perturbed variational problem (3.11) admits a unique solution $\widetilde{u}_{0,h} \in S^1_{h,0}(\Omega)$, a finite element approximation $\widetilde{S}z$ of the solution operator $\mathcal{S}z = u_0 + \mathcal{E}z$ can be defined by

$$\widetilde{S}z \;:=\; \widetilde{u}_{0,h} + Q_h \mathcal{E} z. \qquad (3.12)$$

**Lemma 3.3** *The approximate solution operator $\widetilde{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$ as defined in* (3.12) *is bounded. Moreover, if we assume $z \in H^{1/2+s}(\Gamma)$ and $u_0 \in H^{1+s}(\Omega)$ for some $s \in [\frac{1}{2}, 1]$, then there holds the approximation error estimate*

$$\|\mathcal{S}z - \widetilde{\mathcal{S}}z\|_{L_2(\Omega)} \;\leq\; c\, h^{2s} \left[ |u_0|_{H^{1+s}(\Omega)} + \|z\|_{H^{1/2+s}(\Gamma)} \right]. \qquad (3.13)$$

**Proof.** The boundedness of $\widetilde{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$ follows from the stability of the finite element approximation scheme, and from the boundedness of the quasi–interpolation operator $Q : H^1(\Omega) \to S^1_h(\Omega) \subset H^1(\Omega)$, see [35]. The error estimate (3.13) follows from the application of the Strang lemma, see, e.g., [4] or [36, Theorem 8.2], the Aubin–Nitsche trick, see, e.g., [36, Lemma 11.3], and by using the $L_2$ error estimate of the linear quasi–interpolation operator $Q_h$, see [35]. ■

In the same way as above we may define a finite element approximation of the particular solution $u_f$, i.e., $u_{f,h} \in S^1_{h,0}(\Omega)$ is the unique solution of the Galerkin variational problem

$$\int_\Omega \nabla u_{f,h}(x) \cdot \nabla v_h(x) dx \;=\; \int_\Omega f(x) v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega)$$

satisfying the error estimate

$$\|u_f - u_{f,h}\|_{L_2(\Omega)} \;\leq\; c\, h^{2s}\, \|u_f\|_{H^{1+s}(\Omega)} \qquad (3.14)$$

when assuming $u_f \in H^{1+s}(\Omega)$ for some $s \in [\frac{1}{2}, 1]$.

## 3.2 Adjoint boundary value problem

Next we consider a finite element approximation of the adjoint solution operator $\tau = \mathcal{S}^* \psi$ as defined in (2.13), i.e., of

$$\tau(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma,$$

where $p \in H^1_0(\Omega)$ is the unique solution of the variational problem (2.15). The finite element approximation of (2.15) is to find $p_h \in S^1_{h,0}(\Omega)$ such that

$$\int_\Omega \nabla p_h(x) \cdot \nabla v_h(x) dx \; = \; \int_\Omega \psi(x) v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega). \tag{3.15}$$

Again, we conclude the unique solvability of the Galerkin formulation (3.15) by means of Cea's lemma, as well as the quasi–optimal error estimate

$$\|p - p_h\|_{H^1(\Omega)} \; \le \; c\, h^\sigma \, |p|_{H^{1+\sigma}(\Omega)} \; \le \; c\, h^{1+\sigma} \, \|\psi\|_{L_2(\Omega)}, \tag{3.16}$$

when assuming $p \in H^{1+\sigma}(\Omega)$ for some $\sigma \in [\frac{1}{2}, 1]$. Now we are able to define an approximation $\widetilde{\tau} = \widetilde{\mathcal{S}}^* \psi$ of $\tau = \mathcal{S}^* \psi$, $\psi \in L_2(\Omega)$, by

$$\langle \widetilde{\tau}, w \rangle_\Gamma = - \int_\Omega \nabla p_h(x) \cdot \nabla \mathcal{E} w(x) dx + \int_\Omega \psi(x) \mathcal{E} w(x) dx \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.17}$$

**Lemma 3.4** *For $\psi \in L_2(\Omega)$ let $\widetilde{\tau} = \widetilde{\mathcal{S}}^* \psi$ be the approximation as defined in* (3.17)*. Then, $\widetilde{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is bounded, and there holds the error estimate*

$$\|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\psi\|_{H^{-1/2}(\Gamma)} \; = \; \|\tau - \widetilde{\tau}\|_{H^{-1/2}(\Gamma)} \; \le \; c\, h^\sigma \, \|\psi\|_{L_2(\Omega)} \tag{3.18}$$

*if $p \in H^{1+\sigma}(\Omega)$ for some $\sigma \in [\frac{1}{2}, 1]$ is satisfied.*

**Proof.** Both the boundedness of $\widetilde{\mathcal{S}}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ and the error estimate (3.18) follow from the finite element error estimate (3.16) when using duality arguments and the inverse trace theorem. ∎

## 3.3 Approximation error estimates

By using the finite element approximations $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}^*$ as defined in (3.12) and (3.17), respectively, we can introduce the finite element approximations

$$\widetilde{T}_\varrho := \varrho D + \widetilde{\mathcal{S}}^* \widetilde{\mathcal{S}}, \quad \widetilde{g} := \widetilde{\mathcal{S}}^* (\overline{u} - u_{f,h}) \tag{3.19}$$

to be used within the perturbed variational inequality (3.6). For the application of Theorem 3.2 we need to estimate the related approximation error.

**Lemma 3.5** *For $z \in H^{1/2+s}(\Gamma)$, $s \in [\frac{1}{2}, 1]$, let $\mathcal{S} z = u_0 + \mathcal{E} z \in H^{1+s}(\Omega)$ be the solution of the Dirichlet boundary value problem* (2.4)*. Let $p \in H^{1+\sigma}(\Omega)$, $\sigma \in [\frac{1}{2}, 1]$, be the weak solution of the adjoint boundary value problem* (2.14) *with $\psi = \mathcal{S} z$. Then there holds the error estimate*

$$\|(T_\varrho - \widetilde{T}_\varrho) z\|_{H^{-1/2}(\Gamma)} \; \le \; c_1\, h^\sigma \, \|\mathcal{S} z\|_{L_2(\Omega)} + c_2\, h^{2s} \left[ \|u_0\|_{H^{1+s}(\Omega)} + \|z\|_{H^{1/2+s}(\Gamma)} \right]. \tag{3.20}$$

**Proof.** By the triangle inequality and by using the boundedness of $\widetilde{\mathcal{S}}^*: L_2(\Omega) \to H^{-1/2}(\Gamma)$ we have

$$\begin{aligned} \|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} &= \|(\mathcal{S}^*\mathcal{S} - \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}})z\|_{H^{-1/2}(\Gamma)} \\ &\leq \|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\mathcal{S}z\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathcal{S}}^*(\mathcal{S} - \widetilde{\mathcal{S}})z\|_{H^{-1/2}(\Gamma)} \\ &\leq \|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\mathcal{S}z\|_{H^{-1/2}(\Gamma)} + c_2^{\widetilde{\mathcal{S}}^*}\,\|(\mathcal{S} - \widetilde{\mathcal{S}})z\|_{L_2(\Omega)}. \end{aligned}$$

The assertion now follows from the error estimates (3.13) and (3.18). ∎

**Lemma 3.6** *Let $u_{f,h} \in S^1_{h,0}(\Omega)$ be the finite element approximation of the particular solution $u_f \in H^{1+s}(\Omega)$ for some $s \in [\frac{1}{2}, 1]$. Let $p_{\overline{u}}, p_{u_f} \in H^{1+\sigma}(\Omega)$, $\sigma \in [\frac{1}{2}, 1]$, be the weak solutions of the adjoint boundary value problem* (2.14) *with $\psi = \overline{u}$ and $\psi = u_f$, respectively. Then there holds the error estimate*

$$\|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \leq c_1\, h^{2s}\, \|u_f\|_{H^{1+s}(\Omega)} + c_2\, h^\sigma\, \left[\|\overline{u}\|_{L_2(\Omega)} + \|u_f\|_{L_2(\Omega)}\right]. \tag{3.21}$$

**Proof.** From the triangle inequality we first have

$$\begin{aligned} \|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} &= \|\mathcal{S}^*(\overline{u} - u_f) - \widetilde{\mathcal{S}}^*(\overline{u} - u_{f,h})\|_{H^{-1/2}(\Gamma)} \\ &\leq \|(\mathcal{S}^* - \widetilde{\mathcal{S}}^*)\overline{u}\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathcal{S}}^*(u_{f,h} - u_f)\|_{H^{-1/2}(\Gamma)} + \|(\widetilde{\mathcal{S}}^* - \mathcal{S}^*)u_f\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

The assertion now follows from the boundedness of $\widetilde{\mathcal{S}}^*: L_2(\Omega) \to H^{-1/2}(\Gamma)$, and the finite element error estimates (3.14) and (3.18). For the application of Lemma 3.4 we need to assume $p_{\overline{u}}, p_{u_f} \in H^{1+\sigma}(\Omega)$ for some $\sigma \in [\frac{1}{2}, 1]$, where $p_{\overline{u}}$ and $p_{u_f}$ are the weak solutions of the adjoint boundary value problem (2.14) with $\psi = \overline{u}$ and $\psi = u_f$, respectively. ∎

It remains to prove the $S^1_h(\Gamma)$–ellipticity of the finite element approximation $\widetilde{T}_\varrho$. This will be a consequence of the matrix representation of $\widetilde{T}_\varrho$ as discussed in the next subsection.

## 3.4 Approximate variational inequality

Now we are in a position to describe the finite element approximation of the perturbed variational inequality (3.6) to find $\widetilde{z}_h \in \mathcal{U}_h$ such that

$$\varrho\langle D\widetilde{z}_h, w_h - \widetilde{z}_h\rangle_\Gamma + \langle \widetilde{\mathcal{S}}^*(\widetilde{\mathcal{S}}\widetilde{z}_h + u_{f,h} - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma \geq 0 \quad \text{for all } w_h \in \mathcal{U}_h\,.$$

Note that $u_h = \widetilde{\mathcal{S}}\widetilde{z}_h + u_{f,h} \in S^1_h(\Omega)$ is the unique solution of the Galerkin variational formulation

$$\int_\Omega \nabla u_h(x) \cdot \nabla v_h(x) dx = \int_\Omega f(x) v_h(x) dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega). \tag{3.22}$$

By using

$$u_h(x) = \sum_{k=1}^{M_\Omega} u_{I,k}\varphi_k(x) + \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \varphi_{M_\Omega+\ell}(x), \quad \widetilde{z}_h(x) = \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \phi_\ell(x), \quad \phi_\ell = \varphi_{M_\Omega+\ell|\Gamma},$$

the variational formulation (3.22) is equivalent to the linear system

$$K_{II}\underline{u}_I + K_{CI}\underline{\widetilde{z}} = \underline{f}_I, \tag{3.23}$$

where

$$K_{II}[j,k] = \int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_j(x)dx, \quad K_{CI}[j,\ell] = \int_\Omega \nabla\varphi_{M_\Omega+\ell}\cdot\nabla\varphi_j(x)dx,$$

for $k,j=1,\ldots,M_\Omega$, $\ell=1,\ldots,M_\Gamma$, describe the standard finite element stiffness matrices, and

$$f_{I,j} = \int_\Omega f(x)\varphi_j(x)dx \quad \text{for } j=1,\ldots,M_\Omega.$$

Next we consider, by using (3.17),

$$\begin{aligned}\langle\widetilde{\mathcal{S}}^*(\widetilde{\mathcal{S}}\widetilde{z}_h + u_{f,h} - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma &= \langle\widetilde{\mathcal{S}}^* u_h, w_h - \widetilde{z}_h\rangle_\Gamma \\ &= -\int_\Omega \nabla p_h(x)\cdot\nabla\mathcal{E}(w_h-\widetilde{z}_h)(x)dx + \int_\Omega [u_h(x)-\overline{u}(x)]\mathcal{E}(w_h-\widetilde{z}_h)(x)dx,\end{aligned}$$

where $p_h \in S^1_{h,0}(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla p_h(x)\cdot\nabla q_h(x)dx = \int_\Omega [u_h(x)-\overline{u}(x)]q_h(x)dx \quad \text{for all } q_h\in S^1_{h,0}(\Omega).$$

Hence we conclude the matrix representation

$$\langle\widetilde{\mathcal{S}}^*(\widetilde{\mathcal{S}}\widetilde{z}_h + u_{f,h} - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma = (-K_{IC}\underline{p} + M_{IC}\underline{u}_I + M_{CC}\underline{\widetilde{z}} - \underline{g}_C, \underline{w} - \underline{\widetilde{z}}),$$

where

$$K_{II}\underline{p} = M_{II}\underline{u}_I + M_{CI}\underline{\widetilde{z}} - \underline{g}_I\,.$$

Note that the matrices defined by

$$M_{II}[j,k] = \int_\Omega \varphi_k(x)\varphi_j(x)dx, \quad M_{CI}[j,\ell] = M_{IC}[\ell,j] = \int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_j(x)dx$$

for $k,j=1,\ldots,M_\Omega$, $\ell=1,\ldots,M_\Gamma$ and

$$M_{CC}[j,\ell] = \int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_{M_\Omega+j}(x)dx \quad \text{for } j,\ell=1,\ldots,M_\Gamma$$

are the standard finite element mass matrices. Moreover,

$$g_{I,j} = \int_\Omega \overline{u}(x)\varphi_j(x)dx \text{ for } j=1,\ldots,M_\Omega, \quad g_{C,\ell} = \int_\Omega \overline{u}(x)\varphi_{M_\Omega+\ell}(x)dx \text{ for } \ell=1,\ldots,M_\Gamma.$$

Let $D_h$ be the Galerkin matrix of the hypersingular boundary integral operator $D$, i.e.,

$$D_h[j,\ell] \;=\; \langle D\phi_\ell, \phi_j\rangle_\Gamma \quad \text{for } j,\ell = 1,\ldots,M_\Gamma,$$

which can be computed by using integration by parts, see (2.7) and (2.8). The matrix representation of the variational inequality (3.6) is then given by the discrete variational inequality

$$\varrho(D_h\underline{\widetilde{z}}, \underline{w}-\underline{\widetilde{z}}) + (-K_{IC}\underline{p} + M_{IC}\underline{u}_I + M_{CC}\underline{\widetilde{z}} - \underline{g}_C, \underline{w}-\underline{\widetilde{z}}) \;\geq\; 0 \quad \text{for all } \underline{w} \in \mathbb{R}^{M_\Gamma} \leftrightarrow w_h \in \mathcal{U}_h,$$

or

$$(\widetilde{T}_{\varrho,h}\underline{\widetilde{z}} - \underline{g}, \underline{w}-\underline{\widetilde{z}}) \;\geq\; 0 \quad \text{for all } \underline{w} \in \mathbb{R}^{M_\Gamma} \leftrightarrow w_h \in \mathcal{U}_h, \tag{3.24}$$

where

$$\widetilde{T}_{\varrho,h} := \varrho D_h + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} \tag{3.25}$$

and

$$\underline{g} := \underline{g}_C - K_{IC}K_{II}^{-1}\underline{g}_I + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f}_I - M_{IC}K_{II}^{-1}\underline{f}_I.$$

**Lemma 3.7** *The approximate operator $\widetilde{T}_\varrho = \varrho D + \widetilde{\mathcal{S}}^*\widetilde{\mathcal{S}}$ is $S_h^1(\Gamma)$–elliptic. In particular, the matrix $\widetilde{T}_{\varrho,h}$ as defined in* (3.25) *is positive definite, i.e.,*

$$\langle \widetilde{T}_\varrho z_h, z_h\rangle_\Gamma = (\widetilde{T}_{\varrho,h}\underline{z},\underline{z}) \;\geq\; c_1^{\widetilde{T}_\varrho}\,\|z_h\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z_h \in S_h^1(\Gamma) \leftrightarrow \underline{z} \in \mathbb{R}^{M_\Gamma}.$$

**Proof.** For $\underline{z} \in \mathbb{R}^{M_\Gamma}$ and by defining $\underline{u} = -K_{II}^{-1}K_{CI}\underline{z}$ we have

$$\begin{aligned}
(\widetilde{T}_{\varrho,h}\underline{z},\underline{z}) &= ([K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho D_h]\underline{z},\underline{z}) \\
&= (M_{II}K_{II}^{-1}K_{CI}\underline{z}, K_{II}^{-1}K_{CI}\underline{z}) - (M_{CI}\underline{z}, K_{II}^{-1}K_{CI}\underline{z}) - (M_{IC}K_{II}^{-1}K_{CI}\underline{z},\underline{z}) \\
&\qquad\qquad + (M_{CC}\underline{z},\underline{z}) + \varrho(D_h\underline{z},\underline{z}) \\
&= (M_{II}\underline{u},\underline{u}) + (M_{CI}\underline{z},\underline{u}) + (M_{IC}\underline{u},\underline{z}) + (M_{CC}\underline{z},\underline{z}) + \varrho(D_h\underline{z},\underline{z}) \\
&= \left(\begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC}\end{pmatrix}\begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix}, \begin{pmatrix}\underline{u}\\ \underline{z}\end{pmatrix}\right) + \varrho(D_h\underline{z},\underline{z}).
\end{aligned}$$

Since the mass matrix

$$M_h \;=\; \begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC}\end{pmatrix}$$

is positive definite, the assertion follows. ■

In the particular case of a non–constrained minimization problem, instead of the discrete variational inequality (3.24) we have to solve the linear system

$$\widetilde{T}_{\varrho,h}\underline{z} = \underline{g},$$

which is equivalent to the system

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho D_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}. \tag{3.26}$$

Note that the coupled linear system (3.26) also results from a direct finite element approximation of the adjoint boundary value problem, of the primal boundary value problem, and of the optimality condition.

## 3.5 Regularity and error estimates

Since the operator $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible and self–adjoint, we can rewrite the reduced cost functional $\widetilde{J}$ as defined in (2.18) as

$$\widetilde{J}(z) = \frac{1}{2}\langle T_\varrho z - g, z - T_\varrho^{-1} g\rangle_\Gamma + \frac{1}{2}\|u_f - \overline{u}\|^2_{L_2(\Omega)} - \langle T_\varrho^{-1} g, g\rangle_\Gamma,$$

in particular we have

$$z \,=\, \arg\min_{w\in\mathcal{U}} \|w - T_\varrho^{-1} g\|_{T_\varrho} \,=\, P_{\mathcal{U}} T_\varrho^{-1} g\,,$$

where $P_{\mathcal{U}} : H^{1/2}(\Gamma) \to \mathcal{U} \subset H^{1/2}(\Gamma)$ is the associated projection operator in $H^{1/2}(\Gamma)$. By introducing $\xi = T_\varrho^{-1} g \in H^{1/2}(\Gamma)$, $u_\xi = \mathcal{S}\xi + u_f$, and $\tau = \mathcal{S}^*(u_\xi - \overline{u}) \in H^{-1/2}(\Gamma)$ we obtain

$$z = P_{\mathcal{U}}\xi, \quad \varrho D\xi + \tau = 0, \quad \tau = -\frac{\partial}{\partial n} p, \tag{3.27}$$

where

$$-\Delta p(x) = u_\xi(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma,$$

and

$$-\Delta u_\xi = f(x) \quad \text{for } x \in \Omega, \quad u_\xi(x) = \xi(x) \quad \text{for } x \in \Gamma.$$

**Proposition 3.8** *Let $\Omega$ be either a bounded domain with smooth boundary $\Gamma = \partial\Omega$, or a polygonal or polyhedral domain. Assume $\overline{u} \in L_2(\Omega)$ and $f \in L_2(\Omega)$. Let $z \in H^{1/2}(\Gamma)$ be the unique solution of the variational inequality (2.19), let $u \in H^1(\Omega)$ be the associated state which is defined as the unique solution of the primal boundary value problem* (2.2), *and let $p \in H^1_0(\Omega)$ be the unique solution of the adjoint boundary value problem* (2.14) *with $\psi = u - \overline{u}$. Then,*

$$p \in H^{\sigma+1}(\Omega), \quad z \in H^{\overline{\sigma}+1/2}(\Gamma), \quad u \in H^{\overline{\sigma}+1}(\Omega),$$

*where $\sigma = 1$, when $\Omega$ has either a smooth boundary $\Gamma$, or when $\Omega$ is polygonal or polyhedral but convex. For a polygonal or polyhedral but non–convex domain we have $\sigma < \sigma_0$ with $\sigma_0 \in [\frac{1}{2}, 1]$ which is limited by the largest interior angle. In particular for $n = 2$ we have*

$$\sigma_0 \,=\, \min_{j=1,\ldots,J} \frac{\pi}{\alpha_j},$$

*where $\alpha_j$ are the interior angles at all corner nodes $x_j \in \Gamma$.*

*In the case of a smooth boundary $\Gamma$ we also have $\overline{\sigma} = 1$, while in the case of a polygonal or polyhedral domain we have $\overline{\sigma} < \overline{\sigma}_0$ with $\overline{\sigma}_0 \in [\frac{1}{2}, 1]$, which is limited by the largest interior or exterior angle when considering a polygonal or polyhedral domain. In particular for $n = 2$ we have*

$$\overline{\sigma}_0 := \min_{j=1,\ldots,J} \left\{ \min \left[ \frac{\pi}{\alpha_j}, \frac{\pi}{2\pi - \alpha_j} \right] \right\}.$$

**Proof.** The solutions of the optimal control problem are characterised by the relations as given in (3.27). For the solution of the adjoint boundary value problem we first find $p \in H^{\sigma+1}(\Omega)$, where $\sigma = 1$ if $\Omega$ has a smooth boundary $\Gamma = \partial\Omega$, or $\Omega$ is polygonal or polyhedral but convex. For a polygonal or polyhedral but non–convex domain $\Omega$ we have $\sigma < \sigma_0$ with $\sigma_0 \in (\frac{1}{2}, 1)$ depending on the largest interior angle $\alpha_j$ of $\Gamma$. From the trace theorem we then conclude $\tau \in H^{\sigma-1/2}(\Gamma)$ where in the case of a polygonal or polyhedral domain this has to be considered piecewise. From the mapping properties of the hypersingular boundary integral operator $D$ we then conclude $\xi \in H^{\sigma+1/2}(\Gamma)$ in the case of a smooth boundary, and $\xi \in H^{\overline{\sigma}+1/2}(\Gamma)$ in the case of a polygonal or polyhedral domain $\Omega$, see [9]. Finally, the boundedness of the projection $P_{\mathcal{U}} : H^r(\Gamma) \to H^r(\Gamma)$ for all $r \in [\frac{1}{2}, \frac{3}{2}]$ follows as in [5], see also [6, 26]. ■

Now we are in the position to present the final error estimate for the approximate solution $\widetilde{z}_h$ of the variational inequality (3.6).

**Corollary 3.9** *Let $z \in H^{\overline{\sigma}+1/2}(\Gamma)$ for some $\overline{\sigma} \in [\frac{1}{2}, 1]$ be the unique solution of the variational inequality* (2.19)*. Let $\widetilde{z}_h \in \mathcal{U}_h$ be the unique solution of the perturbed variational inequality* (3.6)*. Then there holds the error estimate*

$$\|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \leq c(z, \overline{u}, f)\, h^{\overline{\sigma}}. \tag{3.28}$$

*Moreover, we have*

$$\|z - \widetilde{z}_h\|_{L_2(\Gamma)} \leq c(z, \overline{u}, f)\, h^{1/2+\overline{\sigma}}. \tag{3.29}$$

**Proof.** The error estimate (3.28) follows from Theorem 3.2 by using the approximation error estimates (3.20) and (3.21). By using the Aubin–Nitsche trick for the perturbed variational inequality (3.6) we obtain the error estimate (3.29). ■

From the error estimate (3.29) we also conclude, by applying the Aubin–Nitsche trick for the primal boundary value problem (2.2), the error estimate

$$\|u - u_h\|_{L_2(\Omega)} \leq c(z, \overline{u}, f)\, h^{1+\overline{\sigma}}. \tag{3.30}$$

Moreover, by using using the Aubin–Nitsche trick for the adjoint boundary value problem (2.14) with $\psi = u - \overline{u}$ we also obtain the error estimate

$$\|p - p_h\|_{L_2(\Omega)} \leq c(z, \overline{u}, f)\, h^{2\sigma}. \tag{3.31}$$

# 4 Numerical results

As numerical example we consider as in [7], see also [29], the Dirichlet boundary control problem (2.1)–(2.3) for the domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$ where

$$\overline{u}(x) = (x_1^2 + x_2^2)^{-1/3}, \quad f(x) = 0, \quad \varrho = 1, \quad [g_a, g_b] = [-1, 2]\,.$$

From Proposition 3.8 we obtain with $\sigma_0 = 1$ and $\overline{\sigma}_0 = 2/3$ the regularity results

$$p \in H^2(\Omega), \quad z \in H^{7/6}(\Gamma), \quad u \in H^{5/3}(\Omega),$$

and therefore, by using the error estimates (3.29), (3.30), and (3.31),

$$\|z - z_h\|_{L_2(\Gamma)} \le c\,h^{7/6}, \quad \|u - u_h\|_{L_2(\Omega)} \le c\,h^{5/3}, \quad \|p - p_h\|_{L_2(\Omega)} \le c\,h^2\,. \tag{4.1}$$

For the finite element discretization we introduce a uniform triangulation of $\Omega$ on several refinement levels $L$ where the mesh size is $h_L = 2^{-(L+2)}$, where for $L = 0$ the coarsest level is given by 4 uniform triangles. Since the minimiser of (2.1) is not known for this example, we use the finite element solutions $(z_{h_9}, u_{h_9})$ on the 9th level as reference solutions.

| L | $\|z_h - z_{h_9}\|_{L_2(\Gamma)}$ | eoc | $\|u_h - u_{h_9}\|_{L_2(\Omega)}$ | eoc | $\|p_h - p_{h_9}\|_{L_2(\Omega)}$ | eoc |
|---|---|---|---|---|---|---|
| 2 | 1.34e-03 | | 2.73e-04 | | 1.45e-04 | |
| 3 | 5.33e-04 | 1.33 | 9.25e-05 | 1.56 | 4.71e-05 | 1.63 |
| 4 | 2.08e-04 | 1.36 | 2.76e-05 | 1.74 | 1.39e-05 | 1.76 |
| 5 | 8.00e-05 | 1.38 | 7.65e-06 | 1.85 | 3.85e-06 | 1.85 |
| 6 | 3.14e-05 | 1.35 | 2.08e-06 | 1.88 | 1.02e-06 | 1.92 |
| 7 | 1.25e-05 | 1.33 | 5.74e-07 | 1.86 | 2.55e-07 | 2.00 |
| 8 | 4.55e-06 | 1.45 | 1.49e-07 | 1.95 | 5.48e-08 | 2.22 |

Table 1: Finite element errors for the Dirichlet control problem in $H^{1/2}(\Gamma)$.

In Table 1 we give the finite element errors for the control variable $z$ in the $L_2(\Gamma)$–norm, for the primal variable $u$ and for the adjoint state $p$ in the $L_2(\Omega)$–norm. For the latter we observe an estimated order of convergence (eoc) of 2 as predicted in the error estimates (4.1), while the estimated order of convergence for the control $z$ and for the state $u$ behave slightly better than predicted.

For comparison we present in Table 2 the numerical results when considering the Dirichlet control problem in $L_2(\Gamma)$, see also [7, 29]. While we obtain a quadratic convergence behaviour for the adjoint state $p$ with almost the same values, the convergence rate for the control $z$ and therefore also for the primal variable $u$ changes significantly. The main reason for this different convergence behaviour lies in the optimality condition: Instead of (3.27) we now obtain the optimality condition

$$z = P_{\mathcal{U}}\xi, \quad \varrho\xi + \tau = 0, \quad \tau = -\frac{\partial}{\partial n}p. \tag{4.2}$$

| L | $\lVert z_h - z_{h_9} \rVert_{L_2(\Gamma)}$ | eoc | $\lVert u_h - u_{h_9} \rVert_{L_2(\Omega)}$ | eoc | $\lVert p_h - p_{h_9} \rVert_{L_2(\Omega)}$ | eoc |
|---|---|---|---|---|---|---|
| 2 | 3.20e-02 | 0.91 | 6.76e-03 | 1.45 | 4.92e-04 | 1.61 |
| 3 | 1.84e-02 | 0.80 | 2.66e-03 | 1.35 | 1.38e-04 | 1.84 |
| 4 | 1.05e-02 | 0.80 | 1.05e-03 | 1.34 | 3.63e-05 | 1.92 |
| 5 | 6.07e-03 | 0.80 | 4.20e-04 | 1.32 | 9.34e-06 | 1.96 |
| 6 | 3.53e-03 | 0.78 | 1.69e-04 | 1.31 | 2.35e-06 | 1.99 |
| 7 | 2.07e-03 | 0.77 | 6.85e-05 | 1.30 | 5.71e-07 | 2.04 |
| 8 | 1.14e-03 | 0.85 | 2.57e-05 | 1.41 | 1.20e-07 | 2.25 |

Table 2: Finite element errors for the Dirichlet control problem in $L_2(\Gamma)$.

Indeed, for the given $\overline{u}$ we obtain $z \in H^{2/3}(\Gamma)$ only, instead of $z \in H^{7/6}(\Gamma)$ when considering the optimality conditions (3.27). Moreover, from the optimality conditions (4.2) we conclude, that $z$ is zero in all corner points due to the zero Dirichlet boundary condition of the adjoint state $p$. Hence we obtain a zero control $z$ in all corner points independent of the target function $\overline{u}$. Instead, when considering the control in $H^{1/2}(\Gamma)$, the application of the hypersingular boundary integral operator $D$ reflects the proper mapping properties of the Dirichlet to Neumann map in (3.27). This results in a more feasible Dirichlet control in corner points. For illustration we give the plots of the states $u$ in Fig. 1 for the $H^{1/2}(\Gamma)$ setting, and in Fig. 2 for the $L_2(\Gamma)$ setting. Note that for this example we used a stabilized hypersingular boundary integral operator [36] and different values of $\varrho$ to ensure comparable values for the tracking functional $\lVert u - \overline{u} \rVert_{L_2(\Omega)}$.

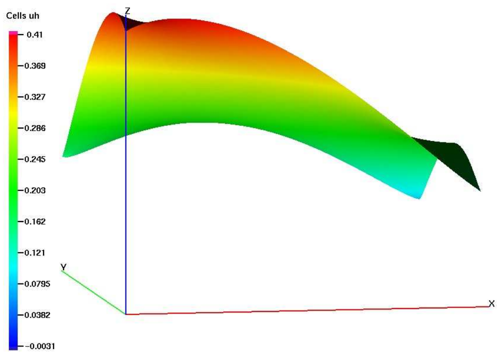


Fig. 1: State $u$ for the control in $H^{1/2}(\Gamma)$

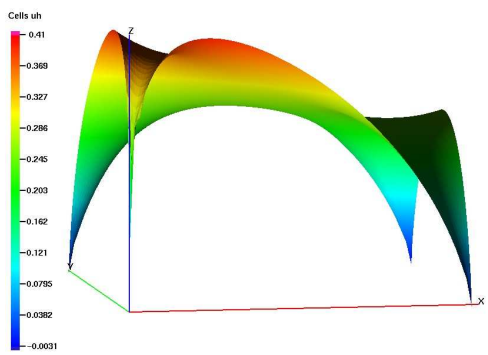


Fig. 2: State $u$ for the control in $L_2(\Gamma)$

Note that the box constraints $[g_a, g_b] = [-1, 2]$ as considered in this example are not active, i.e., we have to solve the coupled linear system (3.26), or the related Schur complement system. Since the Schur complement $\widetilde{T}_{\varrho,z}$ defines an equivalent discrete norm in $H^{1/2}(\Gamma)$, appropriate preconditioners can be constructed, e.g. by using multilevel methods. For the solution of the discrete variational inequality (3.24) one may consider either multigrid methods, see, e.g., [18, 34], or semi–smooth Newton methods, see, e.g., [23, 41]. Note that for the approximation of the operator $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as defined in (2.17) we may also consider boundary element methods which require an appropriate handling of the inhomogeneous adjoint partial differential equation in (2.14). The stability and error analysis of the resulting boundary element approach for the solution of Dirichlet control problems is discussed in [33].


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