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# A new approach to space-time boundary integral equations for the wave equation

Olaf Steinbach* Carolina Urzúa-Torres†

**Abstract**

We present a new approach for boundary integral equations for the wave equation with zero initial conditions. Unlike previous attempts, our mathematical formulation allows us to prove that the associated boundary integral operators are continuous and satisfy inf-sup conditions in trace spaces of the same regularity, which are closely related to standard energy spaces with the expected regularity in space and time. This feature is crucial from a numerical perspective, as it provides the foundations to derive sharper error estimates and paves the way to devise efficient adaptive space-time boundary element methods, which will be tackled in future work. On the other hand, the proposed approach is compatible with current time dependent boundary element method's implementations and we predict that it explains many of the behaviours observed in practice but that were not understood with the existing theory.

## 1 Introduction

Different strategies have been used to derive variational methods for time domain boundary integral equations for the wave equation. The more established and successful ones include weak formulations derived via the Laplace transform, and also space-time energetic variational formulations, often referred as *energetic BEM* in the literature. These approaches started with the groundbreaking works of Bamberger and Ha Duong [4], and Aimi et al. [3], respectively. In spite of their extensive use [1, 2, 5, 12, 13, 14, 15, 16, 17, 18, 23, 24, 25] at the time of writing this article, the numerical analysis corresponding to these formulations was still incomplete and presents difficulties that are hard to overcome, if possible at all.

One of these difficulties is the fact that current approaches provide continuity and coercivity estimates which are not in the same space-time (Sobolev) norms. Indeed, there is a so-called *norm gap* arising from a loss of regularity in time of the related boundary integral operators. Yet, recent work by Joly and Rodríguez shows that these norm gaps are not present in 1D [18]. Moreover, to the best of the authors' knowledge, there is no proof nor numerical evidence that such loss of time regularity should hold for higher dimensions either. These two observations encouraged us to believe that one may be able to prove sharper results using different mathematical tools. Another disadvantage of current strategies is that they do not provide the foundations for space-time boundary element methods, which are basically boundary element discretizations where the time variable is treated simply as another space variable, in contrast to techniques such as time-stepping methods and convolution quadrature methods [25].

Space-time discretization methods offer an increasingly popular alternative, since they allow the treatment of moving boundaries, adaptivity in space and time simultaneously, and space-time parallelization

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[11, 26, 27, 29]. However, in order to exploit these advantages, one needs to have a complete numerical analysis of the corresponding Galerkin methods.

We construct a new approach to boundary integral equations for the wave equation by working directly in the time domain. Furthermore, we develop a mathematical framework that not only overcomes the aforementioned difficulties, but also paves the way to stable space-time FEM/BEM coupling. We present these new results following the standard pieces and arguments from classical boundary integral equations. We hope this highlights some mathematical intuitions behind the obtained results and makes the article easier to read for those familiarized with the boundary integral equation literature. In addition to a new boundary integral equation formulation, we provide novel existence and uniqueness results for the Dirichlet and Neumann wave equation initial boundary value problems, when initial conditions are zero.

The structure of this article is as follows. Section 2 introduces notation and summarizes results from the literature that will be needed later in the paper. We begin by using some key ideas of recent work on the wave equation in $H^1(Q)$ [30, 34]. Then, in Section 3, we introduce trace spaces, trace operators and their corresponding properties for three different families of spaces. With this we aim, on the one hand, to emphasize the link between the existing space-time (volumetric) variational formulations and our new results. On the other hand, we prove that the related trace spaces are indeed connected, which provides a new and deeper understanding of the different existing boundary integral formulations and their relation. Section 4 presents some required results on initial boundary value problems for the wave equation, while all the remaining building blocks of the new boundary integral equation formulation are presented in Section 5. This final section concludes with the existence and uniqueness results for solutions of related boundary integral equations.

# 2 Preliminaries

## 2.1 Model problem

Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, with boundary $\Gamma := \partial\Omega$. We assume $\Omega$ to be an interval for $n = 1$, or a bounded Lipschitz domain for $n = 2, 3$. Let $0 < T < \infty$. For a finite time interval $(0, T)$, we define the *space-time cylinder* $Q := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, and its lateral boundary $\Sigma := \Gamma \times [0, T]$. We also introduce the initial boundary $\Sigma_0 := \Omega \times \{0\}$, and the final boundary $\Sigma_T := \Omega \times \{T\}$. We denote the D'Alembert operator by $\Box := \partial_{tt} - \Delta_x$, and write the *interior Dirichlet initial boundary value problem for the wave equation* as

$$\begin{aligned} \Box u(x,t) &= f(x,t) && \text{for } (x,t) \in Q, \\ u(x,t) &= g(x,t) && \text{for } (x,t) \in \Sigma, \\ u(x,0) = \partial_t u(x,t)_{|t=0} &= 0 && \text{for } x \in \Omega. \end{aligned} \tag{2.1}$$

## 2.2 Notation and mathematical framework

Let $O \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$. We stick to the usual notation for the space $C^\infty(O)$ of functions which are bounded and infinitely often continuously differentiable; the subspace $C_0^\infty(O)$ of compactly supported smooth functions; the spaces $L^p(O)$ of Lebesgue integrable functions; and the Sobolev spaces $H^s(O)$. Moreover, inner products of Hilbert spaces $X$ are denoted by standard brackets $(\cdot, \cdot)_X$, while angular brackets $\langle \cdot\,, \cdot \rangle_O$ are used for the duality pairing induced by the extension of the inner product $(\cdot, \cdot)_{L^2(O)}$. For a Hilbert space $X$ we denote by $X'$ its dual with the norm

$$\|f\|_{X'} = \sup_{0 \neq v \in X} \frac{|\langle f, v\rangle_O|}{\|v\|_X} \quad \text{for } f \in X'.$$

In particular, we will use

$$H^1(O) := \overline{C^\infty(O)}^{\|\cdot\|_{H^1(O)}}, \quad H^1_0(O) := \overline{C^\infty_0(O)}^{\|\cdot\|_{H^1(O)}},$$

where

$$\|\phi\|_{H^1(O)} := \left( \|\phi\|^2_{L^2(O)} + \sum_{i=1}^{m} \|\partial_{x_i}\phi\|^2_{L^2(O)} \right)^{1/2}.$$

In the specific case $O = Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ we identify $H^1(Q)$ with the Sobolev space

$$H^{1,1}(Q) := L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$$

using Bochner spaces, see, e.g., [20, Sect. 1.3, Chapt. 1] and [21, Sect. 2, Chapt. 4]. Furthermore, let

$$\begin{aligned} H^1_{0,}(0,T; L^2(\Omega)) &:= \Big\{ v \in L^2(Q) : \partial_t v \in L^2(Q),\ v(x,0) = 0 \quad \text{for } x \in \Omega \Big\}, \\ H^1_{,0}(0,T; L^2(\Omega)) &:= \Big\{ v \in L^2(Q) : \partial_t v \in L^2(Q),\ v(x,T) = 0 \quad \text{for } x \in \Omega \Big\}. \end{aligned}$$

With this we introduce

$$\begin{aligned} H^{1,1}_{;0,}(Q) &:= L^2(0,T; H^1(\Omega)) \cap H^1_{0,}(0,T; L^2(\Omega)), \\ H^{1,1}_{;,0}(Q) &:= L^2(0,T; H^1(\Omega)) \cap H^1_{,0}(0,T; L^2(\Omega)), \end{aligned}$$

with norms

$$\begin{aligned} \|u\|_{H^{1,1}_{;0,}(Q)} &:= \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}}, \\ \|v\|_{H^{1,1}_{;,0}(Q)} &:= \sqrt{\|\partial_t v\|^2_{L^2(Q)} + \|\nabla_x v\|^2_{L^2(Q)}}. \end{aligned}$$

Note that the space $H^1_{;0,}(Q)$ corresponds to ${}_0H^1(Q)$ as used in [16, 20, 21]. In the case of zero Dirichlet boundary data along the lateral boundary $\Sigma$ we define the subspaces

$$\begin{aligned} H^{1,1}_{0;0,}(Q) &:= L^2(0,T; H^1_0(\Omega)) \cap H^1_{0,}(0,T; L^2(\Omega)), \\ H^{1,1}_{0;,0}(Q) &:= L^2(0,T; H^1_0(\Omega)) \cap H^1_{,0}(0,T; L^2(\Omega)). \end{aligned}$$

We remark that $H^{1,1}_{;0,}(Q)$ and $H^{1,1}_{0;0,}(Q)$ prescribe zero initial values at $t = 0$, while $H^{1,1}_{;,0}(Q)$ and $H^{1,1}_{0;,0}(Q)$ have zero final values at $t = T$.

In this paper we will consider, as in [31], a generalized variational formulation to describe solutions of the wave equation (2.1) also for $f \in [H^{1,1}_{0;,0}(Q)]'$, instead of $f \in L^2(Q)$, as usually considered, e.g., [19]. Therefore we introduce the *extended* space-time cylinder $Q_- := \Omega \times (-T,T)$. For $u \in L^2(Q)$, we define $\widetilde{u} \in L^2(Q_-)$ as zero extension,

$$\widetilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in Q, \\ 0 & \text{for } (x,t) \in Q_- \backslash Q. \end{cases}$$

The application of the wave operator $\Box$ to $\widetilde{u} \in L^2(Q_-)$ is defined as a distribution on $Q_-$, i.e., for all test functions $\varphi \in C^\infty_0(Q_-)$, we define

$$\langle \Box \widetilde{u}, \varphi \rangle_{Q_-} := \int_{-T}^{T} \int_\Omega \widetilde{u}(x,t)\, \Box\varphi(x,t)\, dx\, dt = \int_0^T \int_\Omega u(x,t)\, \Box\varphi(x,t)\, dx\, dt.$$

This motivates to consider the Sobolev space $H^1_0(Q_-)$ with the norm

$$\|\phi\|_{H^1_0(Q_-)} = \sqrt{\|\partial_t \phi\|^2_{L^2(Q_-)} + \|\nabla_x \phi\|^2_{L^2(Q_-)}} \quad \text{for } \phi \in H^1_0(Q_-),$$

the dual space $[H^1_0(Q_-)]'$, and the duality pairing $\langle \cdot, \cdot \rangle_{Q_-}$ as extension of the inner product in $L^2(Q_-)$. We also introduce the restriction operator $\mathcal{R} : H^1_0(Q_-) \to H^{1,1}_{0;,0}(Q)$, i.e., $\mathcal{R}\phi := \phi_{|Q}$, and its adjoint $\mathcal{R}' : [H^{1,1}_{0;,0}(Q)]' \to [H^1_0(Q_-)]'$. Moreover, let $\mathcal{E} : H^{1,1}_{0;,0}(Q) \to H^1_0(Q_-)$ be any continuous and injective extension operator with norm

$$\|\mathcal{E}\|_{H^{1,1}_{0;,0}(Q), H^1_0(Q_-)} := \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{\|\mathcal{E}v\|_{H^1_0(Q_-)}}{\|v\|_{H^{1,1}_{0;,0}(Q)}},$$

and its adjoint $\mathcal{E}' : [H^1_0(Q_-)]' \to [H^{1,1}_{0;,0}(Q)]'$, satisfying $\mathcal{R}\mathcal{E}\phi = \phi$ for all $\phi \in H^{1,1}_{0;,0}(Q)$.

As in [31] we introduce the Banach space

$$\mathcal{H}(Q) := \left\{ u = \widetilde{u}_{|Q} : \widetilde{u} \in L^2(Q_-),\ \widetilde{u}_{|\Omega \times (-T,0)} = 0,\ \Box \widetilde{u} \in [H^1_0(Q_-)]' \right\},$$

with the norm $\|u\|_{\mathcal{H}(Q)} := \sqrt{\|u\|^2_{L^2(Q)} + \|\Box \widetilde{u}\|^2_{[H^1_0(Q_-)]'}}$, where

$$\|\Box \widetilde{u}\|_{[H^1_0(Q_-)]'} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box \widetilde{u}, \mathcal{E}v \rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}}.$$

By completion, we finally define the Hilbert spaces

$$\mathcal{H}_{0;0,}(Q) := \overline{H^{1,1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}_{;0,}(Q) := \overline{H^{1,1}_{;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q),$$

e.g.,

$$\mathcal{H}_{;0,}(Q) = \left\{ u \in \mathcal{H}(Q) : \exists (u_n)_{n \in \mathbb{N}} \subset H^{1,1}_{;0,}(Q) \text{ with } \lim_{n \to \infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0 \right\}.$$

Note that $H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0;0,}(Q)$ and $H^{1,1}_{;0,}(Q) \subset \mathcal{H}_{;0,}(Q)$, see [31, Lemma 3.5] for the first inclusion.

## 2.3 Transformation operator $\mathcal{H}_T$

For $u \in L^2(0,T)$ we consider the Fourier series

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad u_k = \frac{2}{T}\int_0^T u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt.$$

As in [30] we introduce the transformation operator $\mathcal{H}_T$ as

$$\mathcal{H}_T u(t) := \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right),$$

and it's inverse, i.e., for $v \in L^2(0,T)$,

$$\mathcal{H}_T^{-1} v(t) := \sum_{k=0}^{\infty} \overline{v}_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \overline{v}_k = \frac{2}{T}\int_0^T v(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt.$$

By construction we have $\mathcal{H}_T : H^1_{0,}(0,T) \to H^1_{,0}(0,T)$, and $\mathcal{H}_T^{-1} : H^1_{,0}(0,T) \to H^1_{0,}(0,T)$. In the following, we summarize some additional properties fulfilled by the operators $\mathcal{H}_T$ and $\mathcal{H}_T^{-1}$, see [30, 34].

**Proposition 2.1.**

1. *For any* $u, v \in L^2(0,T)$

$$\langle \mathcal{H}_T u, v\rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} v\rangle_{L^2(0,T)}.$$

2. *For all* $u \in H^1_{0,}(0,T)$

$$\partial_t \mathcal{H}_T u = -\mathcal{H}_T^{-1} \partial_t u.$$

3. $\mathcal{H}_T$ *and* $\mathcal{H}_T^{-1}$ *are norm preserving, i.e.,*

$$\|\mathcal{H}_T w\|_{L^2(0,T)} = \|w\|_{L^2(0,T)}, \quad \|\mathcal{H}_T^{-1} w\|_{L^2(0,T)} = \|w\|_{L^2(0,T)} \quad \forall w \in L^2(0,T).$$

4. *For all* $w \in L^2(Q)$

$$\langle w, \mathcal{H}_T w\rangle_{L^2(0,T)} \geq 0.$$

We conclude this subsection by extending the modified Hilbert transformation $\mathcal{H}_T$ to our functional spaces. For $u \in L^2(Q)$ we first have the decomposition

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} u_{i,k} \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \varphi_i(x),$$
$$u_{i,k} = \frac{2}{T} \int_Q u(x,t) \, \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \, \varphi_i(x) \, dx,$$

where $\varphi_i$ are the Neumann eigenfunctions of the Laplacian, i.e.,

$$-\Delta\varphi_i = \lambda_i \varphi_i \text{ in } \Omega, \quad \partial_{n_x}\varphi_i = 0 \quad \text{on } \Gamma, \quad \|\varphi_i\|_{L^2(\Omega)} = 1, \quad 0 = \lambda_0 < \lambda_i \quad \forall i \in \mathbb{N}.$$

They are an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1(\Omega)$, e.g., [19, Chapt. 2]. With this we define

$$\mathcal{H}_T u(x,t) := \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} u_{i,k} \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \varphi_i(x), \quad (x,t) \in Q,$$

with $\mathcal{H}_T : H^{1,1}_{;0,}(Q) \to H^{1,1}_{;,0}(Q)$. Analogously, $\mathcal{H}_T^{-1} : H^{1,1}_{;,0}(Q) \to H^{1,1}_{;0,}(Q)$.

**Remark 2.2.** *The time-reversal map* $\kappa_T$*, defined as [8, Eq. (2.36)]*

$$\kappa_T \, w(x,t) := w(x, T-t) \quad \textit{for } (x,t) \in Q, w \in H^1(Q), \tag{2.2}$$

*is often used instead of the transformation operator* $\mathcal{H}_T : H^{1,1}_{;0,}(Q) \to H^{1,1}_{;,0}(Q)$.

## 2.4 Fundamental solution and retarded potentials

Let us briefly present the *boundary layer potentials* for the wave equation, often called *retarded potentials*. We refer the reader to [9] and [16] for further details. First, we introduce the fundamental solution of the wave equation,

$$G(x,t) = \begin{cases} \dfrac{1}{2}\,\mathsf{H}(t-|x|), & n=1, \\[2ex] \dfrac{1}{2\pi}\,\dfrac{\mathsf{H}(t-|x|)}{\sqrt{t^2-|x|^2}}, & n=2, \\[2ex] \dfrac{1}{4\pi}\,\dfrac{\delta(t-|x|)}{|x|}, & n=3, \end{cases} \tag{2.3}$$

with $\delta$ the Dirac distribution, and $\mathsf{H}$ the Heaviside step function. Let $\mathscr{S}$ be the *single layer potential* and $\mathscr{D}$ the *double layer potential*, i.e., for $(x,t) \in Q$ and regular enough densities $w$ and $z$, respectively,

$$(\mathscr{S}w)(x,t) := \int_0^t \int_\Gamma G(x-y, t-\tau)\, w(y,\tau)\, ds_y\, d\tau, \tag{2.4}$$

$$(\mathscr{D}z)(x,t) := \int_0^t \int_\Gamma \partial_{n_y} G(x-y, t-\tau)\, z(y,\tau)\, ds_y\, d\tau. \tag{2.5}$$

Concretely, for $n = 3$, these are

$$(\mathscr{S}w)(x,t) := \frac{1}{4\pi} \int_\Gamma \frac{w(y, t-|x-y|)}{|x-y|} ds_y, \tag{2.6}$$

$$(\mathscr{D}z)(x,t) := \frac{1}{4\pi} \int_\Gamma \left[ \partial_{n_y} \frac{z(y, t-|x-y|)}{|x-y|} - \frac{\partial_{n_y} |x-y|}{|x-y|} \partial_t z(y, t-|x-y|) \right] ds_y. \tag{2.7}$$

The fact that the time argument is the retarded time $\tau = t - |x-y|$ motivates that $\mathscr{S}$ and $\mathscr{D}$ are usually called retarded potentials.

# 3 Green's Formula, Trace Spaces and Trace Operators

We introduce the lateral interior trace operator $\gamma_\Sigma^i : u \mapsto u_{|\Sigma}$ as continuous extension of the trace map defined in the pointwise sense for smooth functions. As in [22, Lemma 4.1] we can write a space-time Green's formula for $\varphi \in C^2(Q)$ and $\psi \in C^1(Q)$ as

$$\Phi(\varphi,\psi) = \int_0^T \int_\Omega \Box\varphi\, \psi\, dx\, dt + \int_0^T \int_\Gamma \partial_{n_x}\varphi\, \gamma_\Sigma^i \psi\, ds_x\, dt - \int_\Omega \Big[\partial_t \varphi\, \psi\Big]_{t=0}^T dx,$$

where

$$\Phi(\varphi,\psi) := -\int_0^T \int_\Omega \partial_t\varphi\, \partial_t\psi\, dx\, dt + \int_0^T \int_\Omega \nabla_x\varphi \cdot \nabla_x\psi\, dx\, dt\,. \tag{3.1}$$

In particular, for $\varphi \in C^2(Q)$ with $\partial_t\varphi(x,t)_{|t=0} = 0$ for $x \in \Omega$ and for $\psi \in C^1(Q)$ with $\psi(x,T) = 0$ for $x \in \Omega$, this gives Green's first formula

$$\Phi(\varphi,\psi) = \int_0^T \int_\Omega \Box\varphi\, \psi\, dx\, dt + \int_0^T \int_\Gamma \partial_{n_x}\varphi\, \gamma_\Sigma^i \psi\, ds_x\, dt. \tag{3.2}$$

## 3.1 Traces on $H^{1,1}_{;0,}(Q)$, $H^{1,1}_{;,0}(Q)$, and $\mathcal{H}_{;0,}(Q)$

Following [21, Theorem 2.1, Chapt. 4 and p. 19] we get that the interior trace map $\gamma_\Sigma^i$ is continuous and surjective from $H^1(Q)$ to $H^{1/2}(\Sigma)$. In addition, let $\mathcal{E}_\Sigma : H^{1/2}(\Sigma) \to H^1(Q)$ be a continuous right inverse.

Let us introduce the spaces

$$H_{0,}^{1/2}(\Sigma) := L^2(0,T;H^{1/2}(\Gamma)) \cap H_{0,}^{1/2}(0,T;L^2(\Gamma)),$$
$$H_{,0}^{1/2}(\Sigma) := L^2(0,T;H^{1/2}(\Gamma)) \cap H_{,0}^{1/2}(0,T;L^2(\Gamma)),$$

with $H_{0,}^{1/2}(0,T;L^2(\Gamma))$ and $H_{,0}^{1/2}(0,T;L^2(\Gamma))$ defined by interpolation as

$$H_{0,}^{1/2}(0,T;L^2(\Gamma)) := [H_{0,}^{1}(0,T;L^2(\Gamma), L^2(0,T;L^2(\Gamma)]_{1/2},$$
$$H_{,0}^{1/2}(0,T;L^2(\Gamma)) := [H_{,0}^{1}(0,T;L^2(\Gamma), L^2(0,T;L^2(\Gamma)]_{1/2}.$$

Then, we have the following result, which is stated in [16] without a proof. Here we provide one for completeness.

**Lemma 3.1.** *The interior trace map $\gamma_\Sigma^i$ is continuous and surjective from $H_{;0,}^{1,1}(Q)$ to $H_{0,}^{1/2}(\Sigma)$.*

*Proof.* We adapt the proof of [21, Theorem 2.1, Chapt. 4] to $H_{;0,}^{1,1}(Q)$ (instead of $H^1(Q)$). Recall that

$$u \in H_{;0,}^{1,1}(Q) = L^2(0,T;H^1(\Omega)) \cap H_{0,}^1(0,T;L^2(\Omega)).$$

Without loss of generality, we can take $\Omega = \{x \in \mathbb{R}^n \,:\, x_n > 0\}$ and $\Gamma = \{x \in \mathbb{R}^n \,:\, x_n = 0\}$. Then, by using the notation $x = \{x', x_n\}$, with $x' = \{x_1, \ldots, x_{n-1}\}$, we can write:

$$\begin{aligned} u \in H_{;0,}^{1}(Q) \;\Leftrightarrow\; & u \in L^2(\mathbb{R}_{+,x_n}; L^2(0,T;H^1(\mathbb{R}_{x'}^{n-1})) \cap L^2(0,T;L^2(\mathbb{R}_{x'}^{n-1})), \\ & u \in L^2(\mathbb{R}_{+,x_n}; L^2(0,T;L^2(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^1(0,T;L^2(\mathbb{R}_{x'}^{n-1})). \end{aligned}$$

Then, we can apply Theorem 4.2 from [20, Chapt. 1] with

$$X = L^2(0,T;H^1(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^1(0,T;L^2(\mathbb{R}_{x'}^{n-1})), \quad Y = L^2(0,T;L^2(\mathbb{R}_{x'}^{n-1})),$$

to get that $u(x',0,t) \in [X,Y]_{1/2}$. Now, let us point out that Theorem 13.1 in [20, Chapt. 1] gives

$$\begin{aligned} [X,Y]_{1/2} &= [L^2(0,T;H^1(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^1(0,T;L^2(\mathbb{R}_{x'}^{n-1})), Y]_{1/2} \\ &= [L^2(0,T;H^1(\mathbb{R}_{x'}^{n-1})), Y]_{1/2} \cap [H_{0,}^1(0,T;L^2(\mathbb{R}_{x'}^{n-1})), Y]_{1/2}. \end{aligned}$$

Consequently, by interpolation we get

$$[X,Y]_{1/2} = L^2(0,T;H^{1/2}(\mathbb{R}_{x'}^{n-1})) \cap H_{0,}^{1/2}(0,T;L^2(\mathbb{R}_{x'}^{n-1})),$$

which corresponds to $H_{0,}^{1/2}(\mathbb{R}_{x'}^{n-1} \times [0,T])$. Hence, we conclude that $\gamma_\Sigma^i u \in H_{0,}^{1/2}(\Sigma)$. Surjectivity also follows from Theorem 4.2 in [20, Chapt. 1]. $\square$

By similar arguments, one can also prove:

**Lemma 3.2.** *The interior trace map $\gamma_\Sigma^i$ is continuous and surjective from $H_{;,0}^{1,1}(Q)$ to $H_{,0}^{1/2}(\Sigma)$.*

Finally, we define the lateral trace space

$$\mathcal{H}_{0,}(\Sigma) := \left\{ v = \gamma_\Sigma^i V \quad \text{for all } V \in \mathcal{H}_{;0,}(Q) \right\}$$

with the norm

$$\|v\|_{\mathcal{H}_{0,}(\Sigma)} := \inf_{V \in \mathcal{H}_{;0,}(Q):\gamma_\Sigma^i V = v} \|V\|_{\mathcal{H}(Q)}\,.$$

**Remark 3.3.** *By the definition of $\mathcal{H}_{0,}(\Sigma)$ and using the linearity of $\gamma^i_\Sigma$, we have that for any $v \in \mathcal{H}_{0,}(\Sigma)$ there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset H^{1/2}_{0,}(\Sigma)$ such that $\lim\limits_{n\to\infty} \|v - v_n\|_{\mathcal{H}_{0,}(\Sigma)} = 0$.*

**Remark 3.4.** *The trace spaces investigated in this paper are closely related to the spaces used in the classical time dependent BEM approach for the wave equation, introduced by Bamberger and Ha–Duong [4]. Indeed, as pointed out in [16, Remark 2], $H^{1/2}_{0,}(\Sigma)$ agrees with*

$$H^{1/2,1/2}_{\sigma,\Gamma} := \left\{ u \in LT(\sigma, H^{1/2}(\Gamma))\,;\, \int_{\mathbb{R}+i\sigma} |\hat{u}|_{1/2,\omega,\Gamma}\, d\omega < \infty \right\}$$

*when $\sigma = 0$. Additionally, $\left(H^{1/2}_{0,}(\Sigma)\right)'$ corresponds to*

$$H^{-1/2,-1/2}_{\sigma,\Gamma} := \left\{ u \in LT(\sigma, H^{-1/2}(\Gamma))\,;\, \int_{\mathbb{R}+i\sigma} |\hat{u}|_{-1/2,\omega,\Gamma}\, d\omega < \infty \right\}$$

*when $\sigma = 0$. Remarkably, $\sigma$ is taken to be zero for practical computations and numerical experiments, yet the classical time dependent BEM does not cover this case. We refer to [16] for the detailed definitions and a more comprehensive discussion.*

# 4 Initial boundary value problems

## 4.1 Homogeneous Dirichlet data

Instead of (2.1), let us first consider the Dirichlet initial boundary value problem with zero boundary conditions,

$$\begin{array}{rcll} \Box u(x,t) &=& f(x,t) & \text{for } (x,t) \in Q, \\ u(x,t) &=& 0 & \text{for } (x,t) \in \Sigma, \\ u(x,0) = \partial_t u(x,t)_{|t=0} &=& 0 & \text{for } x \in \Omega. \end{array} \tag{4.1}$$

A possible variational formulation of (4.1) is to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\int_0^T \int_\Omega \partial_t u\, \partial_t v\, dx\, dt + \int_0^T \int_\Omega \nabla_x u \cdot \nabla_x v\, dx\, dt = \int_0^T \int_\Omega f\, v\, dx\, dt \tag{4.2}$$

is satisfied for all $v \in H^{1,1}_{0;,0}(Q)$. When assuming $f \in L^2(Q)$ we are able to construct a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation (4.1), satisfying the stability estimate [30, Theorem 5.1], see also [19, Chapt. IV, Theorem 3.1],

$$\|u\|_{H^{1,1}_{0;0,}(Q)} \leq \frac{1}{\sqrt{2}}\, T\, \|f\|_{L^2(Q)}.$$

While the variational formulation (4.2) is well posed also for $f \in [H^{1,1}_{0;,0}(Q)]'$, it is not possible to prove a related inf-sup condition to ensure the existence of a unique solution $u \in H^{1,1}_{0;0,}(Q)$, see [34, Theorem 4.2.24]. However, by definition we have the inf-sup condition

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \quad \text{for all } u \in \mathcal{H}_{0;0,}(Q), \tag{4.3}$$

and therefore we conclude unique solvability of the variational formulation to find $u \in \mathcal{H}_{0;0,}(Q)$ such that

$$\langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-} = \langle f, v\rangle_Q \tag{4.4}$$

is satisfied for all $v \in H^{1,1}_{0;,0}(Q)$, see [31, Theorem 3.9]. Moreover, for the solution $u$ it holds

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, v\rangle_Q|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \leq \|f\|_{[H^{1,1}_{0;,0}(Q)]'}.$$

In fact, (4.4) is the variational formulation of the operator equation $\mathcal{E}'\Box\widetilde{u} = f$ in $[H^{1,1}_{0;,0}(Q)]'$, i.e.,

$$f_u(v) := \langle \mathcal{E}'\Box\widetilde{u}, v\rangle_Q = \langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-} \quad \text{for } v \in H^{1,1}_{0;,0}(Q) \subset H^{1,1}_{;,0}(Q)$$

is a continuous linear functional with norm

$$\|f_u\|_{[H^{1,1}_{;,0}(Q)]'} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|f_u(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{;,0}(Q)}} = \|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'}.$$

Recall that for $u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0;0,}(Q)$ we have

$$f_u(v) = \langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-} = -\langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} \quad \text{for all } v \in H^{1,1}_{0;,0}(Q).$$

Using the Hahn–Banach theorem, e.g., [33, Chapt. IV., Sect. 5], [6, Theorem 5.9-1], there exists a linear continuous functional $\widetilde{f_u} : H^{1,1}_{;,0}(Q) \to \mathbb{R}$ satisfying

$$\widetilde{f_u}(v) = f_u(v) \quad \text{for all } v \in H^{1,1}_{0;,0}(Q), \tag{4.5}$$
$$\|\widetilde{f_u}\|_{[H^{1,1}_{;,0}(Q)]'} = \|f_u\|_{[H^{1,1}_{;,0}(Q)]'} = \|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'}.$$

Indeed, for $u \in H^{1,1}_{0;0,}(Q)$, we have the explicit representation

$$\widetilde{f_u}(v) := \langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-} = -\langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} \quad \forall\, v \in H^{1,1}_{;,0}(Q). \tag{4.6}$$

In the following we assume $f \in [H^{1,1}_{;,0}(Q)]'$, and we consider the variational formulation to find $\lambda_i \in [H^{1/2}_{,0}(\Sigma)]'$ such that

$$\langle (\gamma^i_\Sigma)'\lambda_i, v\rangle_Q = \langle \lambda_i, \gamma^i_\Sigma v\rangle_\Sigma = \widetilde{f_u}(v) - \langle f, v\rangle_Q \quad \text{for all } v \in H^{1,1}_{;,0}(Q). \tag{4.7}$$

For $v \in H^{1,1}_{0;,0}(Q) \subset H^{1,1}_{;,0}(Q)$, it holds

$$\widetilde{f_u}(v) - \langle f, v\rangle_Q = f_u(v) - \langle f, v\rangle_Q = \langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-} - \langle f, v\rangle_Q = 0,$$

i.e.,

$$\widetilde{f_u} - f \in (\ker\gamma^i_\Sigma)^0 = (H^{1,1}_{0;,0}(Q))^0 := \left\{ g \in [H^{1,1}_{;,0}(Q)]' : \langle g, v\rangle_Q = 0 \quad \forall\, v \in H^{1,1}_{0;,0}(Q) \right\}.$$

By the closed range theorem, we obtain

$$\widetilde{f_u} - f \in \mathrm{Im}_{[H^{1/2}_{,0}(\Sigma)]'}(\gamma^i_\Sigma)',$$

which ensures existence of a solution $\lambda_i \in [H^{1/2}_{,0}(\Sigma)]'$ of the variational formulation (4.7). Since the norm in $[H^{1/2}_{,0}(\Sigma)]'$ is defined by duality, this immediately implies the inf-sup condition

$$\|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'} = \sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\langle \lambda, \gamma^i_\Sigma v\rangle_\Sigma|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \quad \text{for all } \lambda \in [H^{1/2}_{,0}(\Sigma)]',$$

and therefore uniqueness of $\lambda_i \in [H^{1/2}_{,0}(\Sigma)]'$. Moreover, this also gives

$$\begin{aligned}\|\lambda_i\|_{[H^{1/2}_{,0}(\Sigma)]'} &= \sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\langle \lambda_i, \gamma_\Sigma v\rangle_\Sigma|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \\ &= \sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f}_u(v) - \langle f, v\rangle_Q|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \leq 2\,\|f\|_{[H^{1,1}_{;,0}(Q)]'},\end{aligned}$$

where we used

$$\begin{aligned}\sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f}_u(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}} &= \|\widetilde{f}_u\|_{[H^{1,1}_{;,0}(Q)]'} \\ &= \|f_u\|_{[H^{1,1}_{;,0}(Q)]'} = \|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} \leq \|f\|_{[H^{1,1}_{;,0}(Q)]'}.\end{aligned}$$

We now rewrite the variational formulation (4.7) as

$$\widetilde{f}_u(v) = \langle f, v\rangle_Q + \langle \lambda_i, \gamma^i_\Sigma v\rangle_\Sigma, \quad v \in H^{1,1}_{;,0}(Q).$$

In particular, for $u \in H^{1,1}_{0;0,}(Q)$, and using (4.6), this gives

$$-\langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} = \langle f, v\rangle_Q + \langle \lambda_i, \gamma^i_\Sigma v\rangle_\Sigma, \quad v \in H^{1,1}_{;,0}(Q).$$

i.e.,

$$\Phi(u, v) = \langle f, v\rangle_Q + \langle \lambda_i, \gamma^i_\Sigma v\rangle_\Sigma.$$

When comparing this with Green's first formula (3.2) for suitable chosen functions, we observe that $\lambda_i$ corresponds to the spatial normal derivative of $u$. Hence, also in the general case we shall write $\gamma^i_N u := \partial_{n_x} u = \lambda_i$ and call this distribution the interior spatial normal derivative of $u \in \mathcal{H}_{0;0,}(Q)$, i.e.,

$$\gamma^i_N : \mathcal{H}_{0;0,}(Q) \to [H^{1/2}_{,0}(\Sigma)]'.$$

In a similar way, we also define

$$\gamma^i_N : \mathcal{H}_{0;,0}(Q) \to [H^{1/2}_{0,}(\Sigma)]'.$$

For a related approach in the case of an elliptic equation, see also [22, pp. 116–117].

## 4.2 Inhomogeneous Dirichlet data

Next we consider the Dirichlet boundary value problem (2.1). For $g \in \mathcal{H}_{0,}(\Sigma)$ there exists, by definition, an extension $u_g = \mathcal{E}_\Sigma g \in \mathcal{H}_{;0,}(Q)$, and the zero extension $\widetilde{u}_g \in L^2(Q_-)$. Thus, it remains to find $u_0 := u - u_g \in \mathcal{H}_{0;0,}(Q)$ satisfying

$$\langle \Box\widetilde{u}_0, \mathcal{E}v\rangle_{Q_-} = \langle f, v\rangle_Q - \langle \Box\widetilde{u}_g, \mathcal{E}v\rangle_{Q_-} \quad \text{for all } v \in H^{1,1}_{0;,0}(Q).$$

Note that $u_g \in \mathcal{H}_{;0,}(Q) \subset \mathcal{H}(Q)$ involves $\Box\widetilde{u}_g \in [H^1_0(Q_-)]'$. For the solution $u_0$ we obtain

$$\begin{aligned}\|\Box\widetilde{u}_0\|_{[H^1_0(Q_-)]'} &= \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box\widetilde{u}_0, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{;,0}(Q)}} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, v\rangle_Q - \langle \Box\widetilde{u}_g, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \\ &\leq \|f\|_{[H^{1,1}_{0;,0}(Q)]'} + \|\mathcal{E}\|_{H^{1,1}_{0;,0}(Q), H^1_0(Q_-)} \|g\|_{\mathcal{H}_{0,}(\Sigma)},\end{aligned}$$

where we have used

$$\|\Box\widetilde{u}_g\|_{[H^1_0(Q_-)]'} \leq \sqrt{\|u_g\|^2_{L^2(Q)} + \|\Box\widetilde{u}_g\|^2_{[H^1_0(Q_-)]'}} = \|u_g\|_{\mathcal{H}(Q)} = \|g\|_{\mathcal{H}_{0,}(\Sigma)}.$$

As before, we can determine $\lambda_i \in [H^{1/2}_{,0}(\Sigma)]'$ as unique solution of the variational formulation (4.7), and where $\gamma^i_N u := \lambda_i$ is again the spatial normal derivative of the solution $u$ of the Dirichlet boundary value problem (2.1), satisfying

$$\|\lambda_i\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq 2\,\|f\|_{[H^{1,1}_{0;,0}(Q)]'} + \|\mathcal{E}\|_{H^{1,1}_{0;,0}(Q),H^1_0(Q_-)}\|g\|_{\mathcal{H}_{0,}(\Sigma)}\,. \tag{4.8}$$

Specially, for $f \equiv 0$, this describes the interior Dirichlet to Neumann map $g \mapsto \lambda_i = \gamma^i_N u$, where $u$ is the solution of the homogeneous wave equation with zero initial data. This can be written as $\lambda_i = \mathsf{S}_i\, g$, where $\mathsf{S}_i : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]'$ is the so-called Steklov-Poincaré operator, and from (4.8) we immediately conclude

$$\|\,\mathsf{S}_i\, g\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq c^{S_i}_2\, \|g\|_{\mathcal{H}_{0,}(\Sigma)} \quad \text{for all } g \in \mathcal{H}_{0,}(\Sigma), \tag{4.9}$$

with $c^{S_i}_2 := \|\mathcal{E}\|_{H^{1,1}_{0;,0}(Q),H^1_0(Q_-)}$.

As before, we can write the variational formulation (4.7) as

$$\widetilde{f_u}(v) = \langle f, v\rangle_Q + \langle \lambda_i, \gamma^i_\Sigma v\rangle_\Sigma, \quad v \in H^{1,1}_{;,0}(Q).$$

Now, for $u \in \mathcal{H}_{;0,}(Q)$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^{1,1}_{;,0}(Q)$ with

$$\lim_{n\to\infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0.$$

Hence we can write

$$\widetilde{f_u}(v) = \lim_{n\to\infty} \widetilde{f_{u_n}}(v) = \lim_{n\to\infty} \Big[ -\langle \partial_t u_n, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x v\rangle_{L^2(Q)}\Big].$$

In particular, for $v \in H^{1,1}_{;,0}(Q) \cap H^2(Q)$, we can apply integration by parts to obtain

$$\begin{aligned}\widetilde{f_u}(v) &= \lim_{n\to\infty} \Big[\langle u_n, \Box v\rangle_{L^2(Q)} + \langle \gamma^i_\Sigma u_n, \gamma^i_N v\rangle_\Sigma - \langle u_n(T), \partial_t v(T)\rangle_{L^2(\Omega)}\Big]\\ &= \langle u, \Box v\rangle_{L^2(Q)} + \langle \gamma^i_\Sigma u, \gamma^i_N v\rangle_\Sigma - \langle u(T), \partial_t v(T)\rangle_{L^2(\Omega)}\,.\end{aligned}$$

With this we finally obtain Green's second formula for the solution $u \in \mathcal{H}_{;0,}(Q)$ of (2.1) and $v \in H^{1,1}_{;,0}(Q) \cap H^2(Q)$,

$$\langle u, \Box v\rangle_{L^2(Q)} + \langle \gamma^i_\Sigma u, \gamma^i_N v\rangle_\Sigma - \langle u(T), \partial_t v(T)\rangle_{L^2(\Omega)} = \langle f, v\rangle_Q + \langle \gamma^i_N u, \gamma^i_\Sigma v\rangle_\Sigma. \tag{4.10}$$

## 4.3 The Neumann boundary value problem

We now consider (as in (4.7)) the variational problem to find $u \in \mathcal{H}_{;0,}(Q)$ such that

$$\widetilde{f_u}(v) = \langle f, v\rangle_Q + \langle \lambda, \gamma^i_\Sigma v\rangle_\Sigma \quad \text{for all } v \in H^{1,1}_{;,0}(Q), \tag{4.11}$$

when $\lambda \in [H^{1/2}_{,0}(\Sigma)]'$ is given. This is the generalized variational formulation of the Neumann boundary value problem

$$\Box u = f \quad \text{in } Q, \quad \gamma^i_N u = \lambda \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0. \tag{4.12}$$

**Lemma 4.1.** *For all $u \in \mathcal{H}_{;0,}(Q)$ there holds the inf-sup stability condition*

$$\frac{\sqrt{2}}{\sqrt{2+T^2}}\, \|u\|_{\mathcal{H}(Q)} \leq \sup_{0 \neq v \in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f}_u(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}}\,. \tag{4.13}$$

*Proof.* Using (4.5) and the norm definition by duality, we first have

$$\|\Box \widetilde{u}\|_{[H^1_0(Q_-)]'} = \|\widetilde{f}_u\|_{[H^{1,1}_{;,0}(Q)]'} = \sup_{0 \neq v \in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f}_u(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}}\,.$$

Now, for $0 \neq u \in \mathcal{H}_{;0,}(Q)$, there exists a non-trivial sequence $(u_n)_{n\in\mathbb{N}} \subset H^{1,1}_{;0,}(Q)$, $u_n \not\equiv 0$, with

$$\lim_{n\to\infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0.$$

For each $u_n \in H^{1,1}_{;0,}(Q)$ we can write, as in (4.6),

$$\widetilde{f}_{u_n}(v) = -\langle \partial_t u_n, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x v\rangle_{L^2(Q)} \quad \text{for all } v \in H^{1,1}_{;,0}(Q),$$

and we define $w_n \in H^{1,1}_{;,0}(Q)$ as the unique solution of the variational formulation

$$-\langle \partial_t v, \partial_t w_n\rangle_{L^2(Q)} + \langle \nabla_x v, \nabla_x w_n\rangle_{L^2(Q)} = \langle u_n, v\rangle_{L^2(Q)} \quad \text{for all } v \in H^{1,1}_{;0,}(Q).$$

This variational formulation corresponds to a Neumann boundary value problem for the wave equation with a volume source $u_n \in L^2(Q)$, and zero conditions at the terminal time $t = T$. As for the related Dirichlet problem we conclude the bound

$$\|w_n\|_{H^{1,1}_{;,0}(Q)} \leq \frac{1}{\sqrt{2}}\, T\, \|u_n\|_{L^2(Q)}.$$

In particular, for the test function $v = u_n$, the variational formulation gives

$$-\langle \partial_t u_n, \partial_t w_n\rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x w_n\rangle_{L^2(Q)} = \|u_n\|^2_{L^2(Q)}.$$

With this, we now conclude

$$\begin{aligned}
\|\widetilde{f}_{u_n}\|_{[H^{1,1}_{;,0}(Q)]'} &= \sup_{0 \neq v \in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f}_{u_n}(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \geq \frac{|\widetilde{f}_{u_n}(w_n)|}{\|w_n\|_{H^{1,1}_{;,0}(Q)}} \\
&= \frac{|-\langle \partial_t u_n, \partial_t w_n\rangle_{L^2(Q)} + \langle \nabla_x u_n, \nabla_x w_n\rangle_{L^2(Q)}|}{\|w_n\|_{H^{1,1}_{;,0}(Q)}} = \frac{\|u_n\|^2_{L^2(Q)}}{\|w_n\|_{H^{1,1}_{;,0}(Q)}} \geq \frac{\sqrt{2}}{T}\, \|u_n\|_{L^2(Q)}\,.
\end{aligned}$$

Completion for $n \to \infty$ now gives

$$\|\widetilde{f}_u\|_{[H^{1,1}_{;,0}(Q)]'} \geq \frac{\sqrt{2}}{T}\, \|u\|_{L^2(Q)}\,.$$

Hence, we can write, for some $\alpha \in (0,1)$,

$$\begin{aligned}
\|\widetilde{f}_u\|^2_{[H^{1,1}_{;,0}(Q)]'} &= \alpha\, \|\widetilde{f}_u\|^2_{[H^{1,1}_{;,0}(Q)]'} + (1-\alpha)\, \|\widetilde{f}_u\|^2_{[H^{1,1}_{;,0}(Q)]'} \\
&\geq \alpha\, \frac{2}{T^2}\, \|u\|^2_{L^2(Q)} + (1-\alpha)\, \|\Box \widetilde{u}\|^2_{[H^1_0(Q_-)]'} \\
&= (1-\overline{\alpha})\Big(\|u\|^2_{L^2(Q)} + \|\Box \widetilde{u}\|^2_{[H^1_0(Q_-)]'}\Big) = (1-\overline{\alpha})\, \|u\|^2_{\mathcal{H}(Q)},
\end{aligned}$$

when

$$1 - \overline{\alpha} = \overline{\alpha}\,\frac{2}{T^2}$$

is satisfied, i.e.,

$$\overline{\alpha} = \frac{T^2}{2+T^2}, \quad 1 - \overline{\alpha} = \frac{2}{2+T^2}.$$

This concludes the proof. $\square$

**Lemma 4.2.** *For all $0 \neq v \in H^{1,1}_{;,0}(Q)$, there exists a function $u_v \in \mathcal{H}_{;0,}(Q)$ such that*

$$\widetilde{f_{u_v}}(v) > 0. \tag{4.14}$$

*Proof.* For $0 \neq v \in H^{1,1}_{;,0}(Q)$, there exists a unique solution $u_v \in H^{1,1}_{;0,}(Q) \subset \mathcal{H}_{;0,}(Q)$, satisfying

$$-\langle \partial_t u_v, \partial_t w\rangle_{L^2(Q)} + \langle \nabla_x u_v, \nabla_x w\rangle_{L^2(Q)} = \langle v, w\rangle_{L^2(Q)} \quad \text{for all } w \in H^{1,1}_{;,0}(Q),$$

and, for $w = v$, we obtain

$$\widetilde{f_{u_v}}(v) = -\langle \partial_t u_v, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u_v, \nabla_x v\rangle_{L^2(Q)} = \|v\|^2_{L^2(Q)} > 0.$$

$\square$

The inf-sup condition (4.13) and the surjectivity condition (4.14) ensure unique solvability of the variational formulation (4.11), i.e., for the unique solution $u \in \mathcal{H}_{;0,}(Q)$ we obtain

$$\begin{aligned}
\frac{\sqrt{2}}{\sqrt{2+T^2}}\,\|u\|_{\mathcal{H}(Q)} \quad &\leq \quad \sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f_u}(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \\
&= \quad \sup_{0\neq v\in H^{1,1}_{;,0}(Q)} \frac{|\langle f, v\rangle_Q + \langle \lambda, \gamma^i_\Sigma v\rangle_\Sigma|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \;\leq\; \|f\|_{[H^{1,1}_{;,0}(Q)]'} + \|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'},
\end{aligned}$$

and when taking the lateral trace this gives

$$\|\gamma^i_\Sigma u\|_{\mathcal{H}_{0,}(\Sigma)} \leq \|u\|_{\mathcal{H}(Q)} \leq \frac{1}{\sqrt{2}}\sqrt{2+T^2}\,\Big[\|f\|_{[H^{1,1}_{;,0}(Q)]'} + \|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'}\Big]. \tag{4.15}$$

In particular, for $f \equiv 0$, this defines the interior Neumann to Dirichlet map $\lambda \mapsto \gamma^i_\Sigma u$ which can be written as $\gamma^i_\Sigma u = \mathsf{S}_i^{-1}\,\lambda$ when using the inverse of the Steklov–Poincaré operator $\mathsf{S}_i$. From (4.15) we then conclude

$$\|\,\mathsf{S}_i^{-1}\,\lambda\|_{\mathcal{H}_{0,}(\Sigma)} \leq c_2^{S_i^{-1}}\,\|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'} \quad \text{for all } \lambda \in [H^{1/2}_{,0}(\Sigma)]', \quad c_2^{S_i^{-1}} := \frac{1}{\sqrt{2}}\sqrt{2+T^2}\,. \tag{4.16}$$

Now, using (4.9) and duality this gives

$$\|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'} = \|\,\mathsf{S}_i\,\gamma^i_\Sigma u\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq c_2^{S_i}\,\|\gamma^i_\Sigma u\|_{\mathcal{H}_{0,}(\Sigma)} = c_2^{S_i} \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle\gamma^i_\Sigma u, \mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}},$$

i.e., the inf-sup stability condition

$$\frac{1}{c_2^{S_i}}\,\|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle\mathsf{S}_i^{-1}\,\lambda, \mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}} \quad \text{for all } \lambda \in [H^{1/2}_{,0}(\Sigma)]'. \tag{4.17}$$

Furthermore, using (4.16) for $g_i := \gamma_\Sigma^i u$ and duality we obtain

$$\begin{aligned} \|g_i\|_{\mathcal{H}_{0,}(\Sigma)} &= \|\gamma_\Sigma^i u\|_{\mathcal{H}_{0,}(\Sigma)} = \| \mathsf{S}_i^{-1}\, \lambda\|_{\mathcal{H}_{0,}(\Sigma)} \\ &\le c_2^{S_i^{-1}} \, \|\lambda\|_{[H^{1/2}_{,0}(\Sigma)]'} = c_2^{S_i^{-1}} \sup_{0\ne v\in H^{1/2}_{,0}(\Sigma)} \frac{|\langle \lambda, v\rangle_\Sigma|}{\|v\|_{H^{1/2}_{,0}(\Sigma)}}, \end{aligned}$$

i.e., the inf-sup condition

$$\frac{1}{c_2^{S_i^{-1}}}\, \|g_i\|_{\mathcal{H}_{0,}(\Sigma)} \le \sup_{0\ne v\in H^{1/2}_{,0}(\Sigma)} \frac{|\langle \mathsf{S}_i\, g_i, v\rangle_\Sigma|}{\|v\|_{H^{1/2}_{,0}(\Sigma)}} \quad \text{for all } g_i \in \mathcal{H}_{0,}(\Sigma). \tag{4.18}$$

## 4.4 Adjoint problems

Related to the variational problem (4.11) we now consider the adjoint problem to find $w \in H^{1,1}_{;,0}(Q)$ such that

$$\widetilde{f_u}(w) = \langle f, u\rangle_Q + \langle g, \gamma_\Sigma^i u\rangle_\Sigma \tag{4.19}$$

is satisfied for all $u \in \mathcal{H}_{;0,}(Q)$. For $w \in H^{1,1}_{;,0}(Q)$, let $u_w \in \mathcal{H}_{;0,}(Q)$ be the unique solution of the variational problem

$$\widetilde{f_{u_w}}(v) = \langle \partial_t w, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x w, \nabla_x v\rangle_{L^2(Q)} \quad \text{for all } v \in H^{1,1}_{;,0}(Q).$$

For $v = w$, this gives

$$\widetilde{f_{u_w}}(w) = \|w\|^2_{H^{1,1}_{;,0}(Q)}.$$

Moreover, using the inf-sup stability condition (4.13), we obtain

$$\begin{aligned} \frac{\sqrt{2}}{\sqrt{2+T^2}}\, \|u_w\|_{\mathcal{H}(Q)} &\le \sup_{0\ne v\in H^{1,1}_{;,0}(Q)} \frac{|\widetilde{f_{u_w}}(v)|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \\ &= \sup_{0\ne v\in H^{1,1}_{;,0}(Q)} \frac{|\langle \partial_t w, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x w, \nabla_x v\rangle_{L^2(Q)}|}{\|v\|_{H^{1,1}_{;,0}(Q)}} \le \|w\|_{H^{1,1}_{;,0}(Q)}, \end{aligned}$$

and thus it follows that

$$\widetilde{f_{u_w}}(w) = \|w\|^2_{H^{1,1}_{;,0}(Q)} \ge \frac{\sqrt{2}}{\sqrt{2+T^2}}\, \|u_w\|_{\mathcal{H}(Q)} \|w\|_{H^{1,1}_{;,0}(Q)}.$$

In other words, we have

$$\frac{\sqrt{2}}{\sqrt{2+T^2}}\, \|w\|_{H^{1,1}_{;,0}(Q)} \le \sup_{0\ne u\in \mathcal{H}_{;0,}(Q)} \frac{|\widetilde{f_u}(w)|}{\|u\|_{\mathcal{H}(Q)}} \quad \text{for all } w \in H^{1,1}_{;,0}(Q).$$

Since the inf-sup condition (4.13) also implies surjectivity, unique solvability of the variational formulation (4.19) follows. In fact, for $f \in [\mathcal{H}_{;0,}(Q)]'$ and $g \in [\mathcal{H}_{0,}(\Sigma)]'$ we have $w \in H^{1,1}_{;,0}(Q)$ as the weak solution of the adjoint Neumann problem for the wave equation

$$\Box w = f \quad \text{in } Q, \quad \gamma_N^i w = g \quad \text{on } \Sigma, \quad w = \partial_t w = 0 \quad \text{on } \Sigma_T. \tag{4.20}$$

# 5 Boundary Integral Equations

## 5.1 Representation formula

Let $u \in \mathcal{H}_{;0,}(Q)$ be a solution of the generalized wave equation $\mathcal{E}'\Box u = f$ in $[H^{1,1}_{;,0}(Q)]'$. For $(x,t) \in Q$ and $v(y,\tau) = \kappa_t G(x-y,\tau)$, with $G(\cdot,\cdot)$ being the fundamental solution introduced in (2.3) and $\kappa_t$ the time-reversal map from (2.2), formula (4.10) becomes the representation formula

$$u(x,t) = \int_0^t \int_\Omega f(y,\tau)\, G(x-y,t-\tau)\, dy\, d\tau + \left\langle \gamma_N^i u\,,\, \gamma_\Sigma^i v \right\rangle_\Sigma - \left\langle \gamma_\Sigma^i u\,,\, \gamma_N^i v \right\rangle_\Sigma .$$

In particular, for $f \equiv 0$, we conclude the following representation formula

$$u(x,t) = (\mathscr{S}\gamma_N^i u)(x,t) - (\mathscr{D}\gamma_\Sigma^i u)(x,t), \quad (x,t) \in Q, \tag{5.1}$$

with the single and double layer potentials $\mathscr{S}$ and $\mathscr{D}$, defined as in (2.4) and (2.5), respectively.

## 5.2 Single layer potential

We first recall the definition (2.4) of the single layer potential

$$u_w(x,t) = (\mathscr{S}w)(x,t) = \int_0^t \int_\Gamma G(x-y,t-\tau)\, w(y,\tau)\, ds_y\, d\tau, \quad (x,t) \in Q.$$

**Proposition 5.1.** *For the single layer potential we have*

$$\mathscr{S} : [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{;0,}(Q).$$

*Proof.* For $u_w = \mathscr{S}w$ and a suitable $\psi$, we can write the duality pairing as extension of the inner product in $L^2(Q)$ as

$$\begin{aligned}
\langle u_w, \psi\rangle_Q &= \int_0^T \int_\Omega u_w(x,t)\, \psi(x,t)\, dx\, dt \\
&= \int_0^T \int_\Omega \int_0^t \int_\Gamma G(x-y,t-\tau)\, w(y,\tau)\, ds_y\, d\tau\, \psi(x,t)\, dx\, dt \\
&= \int_0^T \int_\Gamma w(y,\tau) \int_\tau^T G(x-y,t-\tau)\, \psi(x,t)\, dx\, dt\, ds_y\, d\tau \\
&= \int_0^T \int_\Gamma w(y,\tau)\, \varphi_\psi(y,\tau)\, ds_y\, d\tau \\
&= \langle w, \gamma_\Sigma^i \varphi_\psi\rangle_\Sigma,
\end{aligned}$$

where

$$\varphi_\psi(y,\tau) = \int_\tau^T G(x-y,t-\tau)\, \psi(x,t)\, dx\, dt, \quad (y,\tau) \in Q,$$

is a solution of the adjoint problem (4.20). Hence, for $\psi \in [\mathcal{H}_{;0,}(Q)]'$, we obtain $\varphi_\psi \in H^{1,1}_{;,0}(Q)$, and therefore $\gamma_\Sigma^i \varphi_\psi \in H^{1/2}_{,0}(\Sigma)$. From this, we conclude that $u_w \in \mathcal{H}_{;0,}(Q)$ when $w \in [H^{1/2}_{,0}(\Sigma)]'$ is given. □

As a corollary of the previous result, we can define the single layer boundary integral operator

$$\mathsf{V} := \gamma_\Sigma^i \mathscr{S} : [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{0,}(\Sigma), \tag{5.2}$$

and the normal derivative of the single layer potential,

$$\gamma_N^i \mathscr{S} : [H^{1/2}_{,0}(\Sigma)]' \to [H^{1/2}_{,0}(\Sigma)]'. \tag{5.3}$$

## 5.3 Double layer potential

We first recall the definition (2.5) of the double layer potential

$$u_z(x,t) = (\mathscr{D}z)(x,t) = \int_0^t \int_\Gamma \partial_{n_y} G(x-y,t-\tau)\, z(y,\tau)\, ds_y\, d\tau, \quad (x,t) \in Q.$$

**Proposition 5.2.** *For the double layer potential we have*

$$\mathscr{D} : \mathcal{H}_{0,}(\Sigma) \to \mathcal{H}_{;0,}(Q).$$

*Proof.* For $u_z = \mathscr{D}z$ and a suitable $\psi$ we can write the duality pairing as extension of the inner product in $L^2(Q)$ as

$$\begin{aligned}
\langle u_z, \psi\rangle_Q &= \int_0^T \int_\Omega u_z(x,t)\, \psi(x,t)\, dx\, dt \\
&= \int_0^T \int_\Omega \int_0^t \int_\Gamma \partial_{n_y} G(x-y,t-\tau)\, z(y,\tau)\, ds_y\, d\tau\, \psi(x,t)\, dx\, dt \\
&= \int_0^T \int_\Gamma z(y,\tau)\, \partial_{n_y} \int_\tau^T \quad G(x-y,t-\tau)\, \psi(x,t)\, dx\, dt\, ds_y\, d\tau \\
&= \int_0^T \int_\Gamma z(y,\tau)\, \partial_{n_y} \varphi_\psi(y,\tau)\, ds_y\, d\tau \\
&= \langle z, \gamma_N^i \varphi_\psi\rangle_\Sigma,
\end{aligned}$$

where

$$\varphi_\psi(y,\tau) = \int_\tau^T \int_\Gamma G(x-y,t-\tau)\, \psi(x,t)\, dx\, dt, \quad (y,\tau) \in Q,$$

is a solution of the adjoint problem (4.20). Hence, for $\psi \in [\mathcal{H}_{;0,}(Q)]'$ we obtain $\varphi_\psi \in H^{1,1}_{;,0}(Q)$, and $\gamma_N^i \varphi_\psi \in [\mathcal{H}_{0,}(\Sigma)]'$. From this we conclude $u_z \in \mathcal{H}_{;0,}(Q)$ when $z \in \mathcal{H}_{0,}(\Sigma)$ is given. □

With the previous result we are in a position to consider the lateral trace of the double layer potential

$$\gamma_\Sigma^i \mathscr{D} : \mathcal{H}_{0,}(\Sigma) \to \mathcal{H}_{0,}(\Sigma), \tag{5.4}$$

and the so-called hypersingular boundary integral operator as normal derivative of the double layer potential,

$$\mathsf{W} := -\gamma_N^i \mathscr{D} : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]'. \tag{5.5}$$

## 5.4 Boundary integral operators and Calderón identities

Without loss of generality, let us consider the complementary domains

$$\Omega^c := B_R \setminus \overline{\Omega} \quad \text{and} \quad Q^c := \Omega^c \times (0,T),$$

with $B_R := \{\mathbf{x} \in \mathbb{R}^n \ : \ |\mathbf{x}| < R\}$ is a sufficiently large ball containing $\Gamma$. With this, we define the exterior traces $\gamma_\Sigma^e$ and $\gamma_N^e$ following the same ideas from Subsection 3.1, but using $Q^c$ instead of $Q$.

**Remark 5.3.** *Clearly, the mappings*

$$\begin{aligned}
\gamma_\Sigma^e &: H^{1,1}_{;0,}(Q^c) \to H^{1/2}_{0,}(\Sigma), & \gamma_\Sigma^e &: H^{1,1}_{;,0}(Q^c) \to H^{1/2}_{,0}(\Sigma), \\
\gamma_\Sigma^e &: \mathcal{H}_{;0,}(Q^c) \to \mathcal{H}_{0,}(\Sigma), & \gamma_\Sigma^e &: \mathcal{H}_{;,0}(Q^c) \to \mathcal{H}_{,0}(\Sigma),
\end{aligned}$$

*are continuous and surjective, while*

$$\gamma_N^e : H^{1,1}_{;0,}(Q^c) \to [H^{1/2}_{,0}(\Sigma)]', \qquad \gamma_N^e : H^{1,1}_{;,0}(Q^c) \to [H^{1/2}_{0,}(\Sigma)]',$$
$$\gamma_N^e : \mathcal{H}_{;0,}(Q^c) \to [H^{1/2}_{,0}(\Sigma)]', \qquad \gamma_N^e : \mathcal{H}_{;,0}(Q^c) \to [H^{1/2}_{0,}(\Sigma)]',$$

*are continuous. Moreover, Green's formulae and other properties of the interior trace operators $\gamma_\Sigma^i$ and $\gamma_N^i$ also apply to these exterior traces in their corresponding spaces. Indeed, following Propositions 5.1 and 5.2, we have the continuity of the mappings*

$$\mathscr{S} : [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{;0,}(Q^c), \qquad \mathscr{D} : \mathcal{H}_{0,}(\Sigma) \quad \to \mathcal{H}_{;0,}(Q^c).$$

We define the jumps across $\Sigma$ by

$$[\gamma_\Sigma u] := \gamma_\Sigma^e u - \gamma_\Sigma^i u, \qquad [\gamma_N u] := \gamma_N^e u - \gamma_N^i u,$$

which clearly do not depend on the choice of $B_R$. Now we can state the following result:

**Proposition 5.4.** *The following jump relations hold for all $w \in [H^{1/2}_{,0}(\Sigma)]'$ and $z \in \mathcal{H}_{0,}(\Sigma)$,*

$$[\gamma_\Sigma \mathscr{S} w] = 0, \qquad [\gamma_N \mathscr{S} w] = -w, \qquad [\gamma_\Sigma \mathscr{D} z] = z, \qquad [\gamma_N \mathscr{D} z] = 0.$$

*Proof.* The jump relations are known to hold when $w$ and $z$ are smooth, e.g., [10, Sect. 2.2.1], and [25, Sect. 1.3]. We extend them to $(w, z) \in [H^{1/2}_{,0}(\Sigma)]' \times \mathcal{H}_{0,}(\Sigma)$ by using that the combined trace map $(\gamma_\Sigma, \gamma_N) : u \mapsto (\gamma_\Sigma u, \gamma_N u)$ maps $C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)\big|_{\overline{Q}}$ onto a dense subspace of $[H^{1/2}_{,0}(\Sigma)]' \times H^{1/2}_{0,}(\Sigma)$ (cf. [7, Lemma 3.5]), and that $H^{1/2}_{0,}(\Sigma)$ is dense in $\mathcal{H}_{0,}(\Sigma)$. □

We can now define the boundary integral operators as follows:

**Definition 5.5.**

$$\begin{aligned}
\mathsf{V}\, w &:= \gamma_\Sigma^i \mathscr{S} w = \gamma_\Sigma^e \mathscr{S} w,\\
\mathsf{K}\, z &:= \frac{1}{2}\left(\gamma_\Sigma^i \mathscr{D} z + \gamma_\Sigma^e \mathscr{D} z\right),\\
\mathsf{K}'\, w &:= \frac{1}{2}\left(\gamma_N^i \mathscr{S} w + \gamma_N^e \mathscr{S} w\right),\\
\mathsf{W}\, z &:= -\gamma_N^i \mathscr{D} z = -\gamma_N^e \mathscr{D} z\,.
\end{aligned}$$

From this definition and (5.2), (5.3), (5.4), (5.5), we obtain:

**Theorem 5.6.** *The boundary integral operators introduced in Definition 5.5 are continuous in the following spaces:*

$$\begin{aligned}
\mathsf{V} &: [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{0,}(\Sigma),\\
\mathsf{K} &: \mathcal{H}_{0,}(\Sigma) \quad \to \mathcal{H}_{0,}(\Sigma),\\
\mathsf{K}' &: [H^{1/2}_{,0}(\Sigma)]' \to [H^{1/2}_{,0}(\Sigma)]',\\
\mathsf{W} &: \mathcal{H}_{0,}(\Sigma) \quad \to [H^{1/2}_{,0}(\Sigma)]'.
\end{aligned}$$

Next, we take traces on the representation formula (5.1) and get

$$\begin{aligned}
\gamma_\Sigma^i u &= (\tfrac{1}{2}\,\mathsf{I} - \mathsf{K})\gamma_\Sigma^i u + \mathsf{V}\,\gamma_N^i u,\\
\gamma_N^i u &= \mathsf{W}\,\gamma_\Sigma^i u + (\tfrac{1}{2}\,\mathsf{I} + \mathsf{K}')\gamma_N^i u.
\end{aligned}$$

As usual, we can rewrite this as

$$\begin{pmatrix} \gamma_\Sigma^i u \\ \gamma_N^i u \end{pmatrix} = \underbrace{\begin{pmatrix} (\frac{1}{2}\,\mathsf{I}-\mathsf{K}) & \mathsf{V} \\ \mathsf{W} & (\frac{1}{2}\,\mathsf{I}+\mathsf{K}') \end{pmatrix}}_{=:\mathsf{C}_Q^i} \begin{pmatrix} \gamma_\Sigma^i u \\ \gamma_N^i u \end{pmatrix} \tag{5.6}$$

with the interior Calderon projection $\mathsf{C}_Q^i$.

Using standard arguments (see for example [25, Sect. 1.4]), we can now prove

$$\begin{pmatrix} z \\ w \end{pmatrix} = \mathsf{C}_Q^i \begin{pmatrix} z \\ w \end{pmatrix}, \quad \forall w \in [H^{1/2}_{,0}(\Sigma)]',\, z \in \mathcal{H}_{0,}(\Sigma).$$

Furthermore, this gives $(\mathsf{C}_Q^i)^2 = \mathsf{C}_Q^i$, from which we get

$$\mathsf{V}\,\mathsf{W} = (\frac{1}{2}\,\mathsf{I}-\mathsf{K})(\frac{1}{2}\,\mathsf{I}+\mathsf{K}), \quad \mathsf{W}\,\mathsf{V} = (\frac{1}{2}\,\mathsf{I}-\mathsf{K}')(\frac{1}{2}\,\mathsf{I}+\mathsf{K}'), \quad \mathsf{V}\,\mathsf{K}' = \mathsf{K}\,\mathsf{V}, \quad \mathsf{K}'\,\mathsf{W} = \mathsf{W}\,\mathsf{K}.$$

## 5.5 Coercivity of boundary integral operators

In this subsection, we are going to prove coercivity properties of boundary integral operators, i.e., of the single layer boundary integral operator $\mathsf{V}$ and the hypersingular boundary integral operator $\mathsf{W}$, which ensure unique solvability of related boundary integral equations.

**Theorem 5.7.** *The single layer boundary integral operator* $\mathsf{V} : [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{0,}(\Sigma)$ *satisfies the inf-sup stability condition*

$$c_1^V\, \|w\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle \mathsf{V}\,w,\mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}} \quad \textit{for all } w \in [H^{1/2}_{,0}(\Sigma)]'. \tag{5.7}$$

*Proof.* For $w \in [H^{1/2}_{,0}(\Sigma)]'$ we consider the single layer potential $u = \mathscr{S}w$ which defines a solution $u \in \mathcal{H}_{0,}(Q)$ of the homogeneous wave equation. When taking the lateral trace of $u$ this gives $g = \gamma_\Sigma^i u = \mathsf{V}\,w \in \mathcal{H}_{0,}(\Sigma)$. In fact, $u$ is the unique solution of the Dirichlet boundary value problem

$$\Box u = 0 \quad \text{in } Q, \quad \gamma_\Sigma^i u = g \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0.$$

When using the interior Steklov–Poincaré operator $\mathsf{S}_i$ we can determine the related interior Neumann trace

$$\lambda_i = \gamma_N^i u = \mathsf{S}_i\, g \in [H^{1/2}_{,0}(\Sigma)]'.$$

Since the Steklov–Poincaré operator $\mathsf{S}_i$ is invertible, this gives $g = \mathsf{S}_i^{-1}\,\lambda_i$, i.e., $g = \gamma_\Sigma^i u$ is the lateral trace of the solution of the Neumann boundary value problem

$$\Box u = 0 \quad \text{in } Q, \quad \gamma_N^i u = \lambda_i \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0.$$

From the inf-sup stability condition (4.17) of the inverse interior Steklov-Poincaré operator $\mathsf{S}_i^{-1}$ we now conclude

$$\frac{1}{c_2^{S_i}}\, \|\lambda_i\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle \mathsf{S}_i^{-1}\,\lambda_i,\mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}} = \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle \mathsf{V}\,w,\mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}}$$

For the exterior problem we can derive a related estimate, i.e.,

$$\frac{1}{c_2^{S_e}}\, \|\lambda_e\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq \sup_{0\neq\mu\in[\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle \mathsf{V}\,w,\mu\rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}},$$

where $\lambda_e$ is the exterior Neumann trace of the single layer potential $u = \mathscr{S} w$. Now, and using the jump relation of the adjoint double layer potential, this gives

$$\begin{aligned}
\|w\|_{[H^{1/2}_{,0}(\Sigma)]'} &= \|\lambda_i - \lambda_e\|_{[H^{1/2}_{,0}(\Sigma)]'} \\
&\leq \|\lambda_i\|_{[H^{1/2}_{,0}(\Sigma)]'} + \|\lambda_e\|_{[H^{1/2}_{,0}(\Sigma)]'} \leq (c_2^{S_i} + c_2^{S_e}) \sup_{0 \neq \mu \in [\mathcal{H}_{0,}(\Sigma)]'} \frac{|\langle \mathsf{V}\, w, \mu \rangle_\Sigma|}{\|\mu\|_{[\mathcal{H}_{0,}(\Sigma)]'}} ,
\end{aligned}$$

which implies the desired inf-sup condition. □

While the inf-sup stability condition (5.7) ensures uniqueness of a solution of a related boundary integral equation, the following result will provide solvability.

**Lemma 5.8.** *For any* $0 \neq \mu \in [\mathcal{H}_{0,}(\Sigma)]'$ *there exists a* $w_\mu \in [H^{1/2}_{,0}(\Sigma)]'$ *such that*

$$\langle \mathsf{V}\, w_\mu, \mu \rangle_\Sigma > 0$$

*is satisfied.*

*Proof.* For given $0 \neq \mu \in [\mathcal{H}_{0,}(\Sigma)]'$ we define the adjoint single layer potential $u_\mu \in H^{1,1}_{;,0}(Q)$ by

$$u_\mu(y,\tau) = \int_\tau^T \int_\Gamma G(x-y, t-\tau)\, \mu(x,t)\, ds_x\, dt \quad \text{for } (y,\tau) \in Q.$$

For the lateral trace $\gamma_\Sigma^i u_\mu \in H^{1/2}_{,0}(\Sigma)$ and arbitrary $w \in [H^{1/2}_{,0}(\Sigma)]'$ we then have

$$\begin{aligned}
\langle w, \gamma_\Sigma^i u_\mu \rangle_\Sigma &= \int_0^T \int_\Gamma w(y,\tau) \int_\tau^T \int_\Gamma G(x-y,t-\tau)\, \mu(x,t)\, ds_x\, dt\, ds_y\, d\tau \\
&= \int_0^T \int_\Gamma \int_0^t \int_\Gamma G(x-y,t-\tau)\, w(y,\tau)\, ds_y\, d\tau\, \mu(x,t)\, ds_x\, dt \;=\; \langle \mathsf{V}\, w, \mu \rangle_\Sigma \,.
\end{aligned}$$

Moreover, we compute

$$U_\mu(x,t) := \int_0^t u_\mu(x,s)\, ds \quad \text{for } (x,t) \in Q,$$

with the lateral trace $g_\mu := \gamma_\Sigma^i U_\mu \in H^{1/2}_{0,}(\Sigma) \subset \mathcal{H}_{0,}(\Sigma)$. Hence, there exists a unique solution $v_\mu \in \mathcal{H}_{;0,}(Q)$ of the Dirichlet problem for the wave equation,

$$\Box v_\mu = 0 \quad \text{in } Q, \qquad v_\mu = g_\mu \quad \text{on } \Sigma, \qquad v_\mu = \partial_t v_\mu = 0 \quad \text{on } \Sigma_0.$$

We then conclude

$$\begin{aligned}
&\int_0^t \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x,s)\, \partial_s v_\mu(x,s)\, ds_x\, ds \\
&\quad = \int_0^t \int_\Omega \Big[ \partial_{ss} v_\mu(x,s)\, \partial_s v_\mu(x,s) + \nabla_x v_\mu(x,s) \cdot \nabla_x \partial_s v_\mu(x,s) \Big]\, dx\, ds \\
&\quad = \frac{1}{2} \int_0^t \frac{d}{ds} \int_\Omega \Big[ [\partial_s v_\mu(x,s)]^2 + [\nabla_x v_\mu(x,s)]^2 \Big]\, dx\, ds \\
&\quad = \frac{1}{2}\, \|\partial_t v_\mu(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\, \|\nabla_x v_\mu(t)\|^2_{L^2(\Omega)} \geq 0 \quad \text{for all } t \in (0,T].
\end{aligned}$$

In the case

$$\frac{1}{2}\, \|\partial_t v_\mu(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\, \|\nabla_x v_\mu(t)\|^2_{L^2(\Omega)} = 0 \quad \text{for all } t \in (0,T],$$

and together with the zero initial conditions, we would conclude $v_\mu \equiv 0$ in $Q$, which then implies $g_\mu \equiv 0$ on $\Sigma$, and thus $u_\mu \equiv 0$. But this contradicts $\mu \not\equiv 0$. Therefore we have

$$\int_0^T \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x,t)\, \partial_t v_\mu(x,t)\, ds_x\, dt = \frac{1}{2}\, \|\partial_t v_\mu(T)\|^2_{L^2(\Omega)} + \frac{1}{2}\, \|\nabla_x v_\mu(T)\|^2_{L^2(\Omega)} > 0,$$

and with

$$\partial_t U_\mu = u_\mu \quad \text{in } Q, \quad \partial_t v_\mu = \partial_t g_\mu \quad \text{on } \Sigma, \quad g_\mu = \gamma^i_\Sigma U_\mu, \quad w_\mu := \gamma^i_N v_\mu \in [H^{1/2}_{,0}(\Sigma)]'$$

we finally conclude

$$\langle \mathsf{V}\, w_\mu, \mu\rangle_\Sigma = \langle w_\mu, \gamma^i_\Sigma u_\mu\rangle_\Sigma = \int_0^T \int_\Gamma \frac{\partial}{\partial n_x} v_\mu(x,t)\, \partial_t v_\mu(x,t)\, ds_x dt > 0.$$

□

The solution of the Dirichlet boundary value problem

$$\Box u = 0 \quad \text{in } Q, \quad u = g \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0$$

is given by the representation formula

$$u(x,t) = (\mathscr{S}\gamma^i_N u)(x,t) - (\mathscr{D} g)(x,t) \quad \text{for } (x,t) \in Q,$$

where we can determine the yet unknown Neumann datum $w = \gamma^i_N u \in [H^{1/2}_{,0}(\Sigma)]'$ as the unique solution of the first kind boundary integral equation

$$\mathsf{V}\, w = (\frac{1}{2}\,\mathsf{I} + \mathsf{K})g \quad \text{on } \Sigma, \tag{5.8}$$

i.e., of the variational formulation

$$\langle \mathsf{V}\, w, \mu\rangle_\Sigma = \langle(\frac{1}{2}\,\mathsf{I} + \mathsf{K})g, \mu\rangle_\Sigma \quad \text{for all } \mu \in [\mathcal{H}_{0,}(\Sigma)]'. \tag{5.9}$$

Solvability of the variational formulation (5.9) follows from Lemma 5.8, while uniqueness of the solution is a consequence of Theorem 5.7. Instead of the variational formulation (5.9), we may use the modified Hilbert transformation $\mathcal{H}_T$ as defined in subsection 2.3 to end up with an equivalent variational problem to find $w \in [H^{1/2}_{,0}(\Sigma)]'$ such that

$$\langle \mathcal{H}_T\, \mathsf{V}\, w, \mu\rangle_\Sigma = \langle \mathcal{H}_T(\frac{1}{2}\,\mathsf{I} + \mathsf{K})g, \mu\rangle_\Sigma \quad \text{for all } \mu \in [\mathcal{H}_{,0}(\Sigma)]'. \tag{5.10}$$

Due to the inclusion $H^{1/2}_{,0}(\Sigma) \subset \mathcal{H}_{,0}(\Sigma)$, we obviously have $[\mathcal{H}_{,0}(\Sigma)]' \subset [H^{1/2}_{,0}(\Sigma)]'$ which will allow for a Galerkin–Bubnov space-time boundary element discretization of (5.10).

**Remark 5.9.** *For a solution $u$ of the homogeneous wave equation with zero initial data but inhomogeneous Dirichlet boundary conditions and a suitable test function $v$ we can write Green's first formula as*

$$\int_0^T \int_\Omega \partial_{n_x} u\, v\, dx\, dt = \int_0^T \int_\Omega \left[\partial_{tt} u\, v + \nabla_x u \cdot \nabla_x v\right] dx\, dt\,.$$

*In particular, for $v = \partial_t u$, this results in the energy representation*

$$\begin{aligned} E(u) := &\int_0^T \int_\Omega \Big[ \partial_{tt} u \, \partial_t u + \nabla_x u \cdot \nabla_x \partial_t u \Big] \, dx \, dt \\ = & \, \frac{1}{2} \, \|\partial_t u(T)\|^2_{L^2(\Omega)} + \frac{1}{2} \, \|\nabla_x u(T)\|^2_{L^2(\Omega)} > 0 \, . \end{aligned}$$

*Note that this representation is the basis of the energetic BEM, see, e.g., [3]. Instead, when using the particular test function $v = \mathcal{H}_T u$ and Proposition 2.1 this gives*

$$\int_0^T \int_\Gamma \frac{\partial}{\partial n_x} u \, \mathcal{H}_T u \, ds_x dt = \int_0^T \int_\Omega \Big[ \mathcal{H}_T \partial_t u \, \partial_t u + \nabla_x u \cdot \mathcal{H}_T \nabla_x u \Big] \, dx \, dt \geq 0 \, .$$

*Specifically, for the single layer potential $u = \mathscr{S} w$ in $\mathbb{R}^{n+1} \backslash \Sigma$ we then conclude*

$$\langle w, \mathcal{H}_T \, \mathsf{V} \, w \rangle_\Sigma = \int_0^T \int_\Omega \Big[ \mathcal{H}_T \partial_t u \, \partial_t u + \nabla_x u \cdot \mathcal{H}_T \nabla_x u \Big] \, dx \, dt \geq 0 \, .$$

*In fact, when considering the spatially one-dimensional case $n = 1$ we can prove the following ellipticity estimate [28, 32]*

$$\langle w, \mathcal{H}_T \, \mathsf{V} \, w \rangle_\Sigma \geq c_1^V \, \|w\|^2_{[H^{1/2}_{,0}(\Sigma)]'} \quad \textit{for all } w \in [H^{1/2}_{,0}(\Sigma)]'.$$

Since the single layer boundary integral operator $\mathsf{V} : [H^{1/2}_{,0}(\Sigma)]' \to \mathcal{H}_{0,}(\Sigma)$ is invertible, we can write the solution of the boundary integral equation (5.8) as

$$w = \gamma_N^i u = \mathsf{V}^{-1}(\frac{1}{2} \, \mathsf{I} + \mathsf{K}) g = \mathsf{S}_i \, g,$$

representing the Dirichlet to Neumann map with the interior Steklov–Poincaré operator

$$\mathsf{S}_i = \mathsf{V}^{-1}(\frac{1}{2} \, \mathsf{I} + \mathsf{K}) : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]' \, .$$

Hence we find that

$$\mathsf{V} \, \mathsf{S}_i = \frac{1}{2} \, \mathsf{I} + \mathsf{K} : \mathcal{H}_{0,}(\Sigma) \to \mathcal{H}_{0,}(\Sigma)$$

is invertible. As we can formulate a related boundary integral equation also for the exterior Dirichlet boundary value problem,

$$\mathsf{V} \, \gamma_N^e u = (-\frac{1}{2} \, \mathsf{I} + \mathsf{K}) g \quad \text{on } \Sigma,$$

this gives that the exterior Steklov–Poincaré operator

$$\mathsf{S}_e = -\, \mathsf{V}^{-1}(\frac{1}{2} \, \mathsf{I} - \mathsf{K}) : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]'$$

is invertible, and so is

$$\frac{1}{2} \, \mathsf{I} - \mathsf{K} = -\, \mathsf{V} \, \mathsf{S}_e : \mathcal{H}_{0,}(\Sigma) \to \mathcal{H}_{0,}(\Sigma) \, .$$

Consequently

$$\mathsf{V} \, \mathsf{W} = (\frac{1}{2} \, \mathsf{I} - \mathsf{K})(\frac{1}{2} \, \mathsf{I} + \mathsf{K}) : \mathcal{H}_{0,}(\Sigma) \to \mathcal{H}_{0,}(\Sigma),$$

and thus

$$\mathsf{W} = \mathsf{V}^{-1}(\frac{1}{2} \, \mathsf{I} - \mathsf{K})(\frac{1}{2} \, \mathsf{I} + \mathsf{K}) : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]' \, .$$

This finally implies that the hypersingular boundary integral operator $\mathsf{W} : \mathcal{H}_{0,}(\Sigma) \to [H^{1/2}_{,0}(\Sigma)]'$ satisfies the inf-sup stability condition

$$c_1^{\mathsf{W}} \, \|v\|_{\mathcal{H}_{0,}(\Sigma)} \leq \sup_{0 \neq \eta \in H^{1/2}_{,0}(\Sigma)} \frac{|\langle \mathsf{W}\, v, \eta \rangle_\Sigma|}{\|\eta\|_{H^{1/2}_{,0}(\Sigma)}} \quad \text{for all } v \in \mathcal{H}_{0,}(\Sigma). \tag{5.11}$$

The solution of the Neumann boundary value problem

$$\Box u = 0 \quad \text{in } Q, \quad \partial_{n_x} u = \lambda \quad \text{on } \Sigma, \quad u = \partial_t u = 0 \quad \text{on } \Sigma_0$$

is given by the representation formula

$$u(x,t) = (\mathscr{S}\lambda)(x,t) - (\mathscr{D}z)(x,t) \quad \text{for } (x,t) \in Q,$$

where we can determine the yet unknown Dirichlet datum $z = \gamma_\Sigma^i u \in \mathcal{H}_{0,}(\Sigma)$ as the unique solution of the first kind boundary integral equation

$$\mathsf{W}\, z = (\frac{1}{2}\,\mathsf{I} - \mathsf{K})\lambda \quad \text{on } \Sigma.$$

Unique solvability follows as described above.

# 6 Conclusions

In this paper, we presented a new framework to describe the mapping properties of boundary integral operators for the wave equation. The results are similar as known for the boundary integral operators for elliptic partial differential equations, i.e., providing ellipticity and boundedness with respect to function spaces of the same Sobolev spaces. This will be the starting point to derive quasi-optimal error estimates for related boundary element methods which are not available so far, and which will be reported in forthcoming work. Other topics of interest include efficient implementations of the proposed scheme using the modified Hilbert transformation, a posteriori error estimates and adaptivity, an efficient solution of the resulting linear systems of algebraic equations, and the coupling with space-time finite element methods.

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