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A continuous finite element framework for the pressure Poisson equation allowing non-Newtonian and compressible flow behaviour

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Abstract

Computing pressure fields from given flow velocities is a task arising in various engineering, biomedical and scientific computing applications. The so-called pressure Poisson equation (PPE) derived from the balance of linear momentum provides an attractive framework for such a task. However, the PPE increases the regularity requirements on the pressure and velocity spaces, thereby imposing theoretical and practical challenges for its application. In order to stay within a Lagrangian finite element framework, it is common practice to completely neglect the influence of viscosity and compressibility when computing the pressure, which limits the practical applicability of the pressure Poisson method. In this context, we present a mixed finite element framework which enables the use of this popular technique with generalised Newtonian fluids (e.g., blood) and compressible flows, while allowing standard finite element spaces to be employed for the given data and unknowns. This is accomplished through the use of appropriate vector calculus identities and simple projections of certain flow quantities. In the compressible case, the mixed formulation also includes an additional equation for retrieving the density field from the given velocities so that the pressure can be accurately determined. The potential of this new approach is showcased through numerical examples.

1. Introduction

Certain physical flow quantities such as velocity and temperature are quantifiable through visualization techniques. In the clinical environment, for instance, modern imaging methods allow blood velocity to be sampled in space and time [1], thereby providing a full kinematic description of the blood flow in large vessels. Measuring local blood pressure non-invasively, on the other hand, is hardly a viable task. This has constantly motivated the development of numerical techniques for retrieving pressure from measured velocity fields. Although the Navier-Stokes momentum equation provides a direct relation between flow velocity and pressure gradient, its vector-valued nature makes it a not so convenient equation when the goal is to solve for pressure, a scalar quantity. The most intuitive approach is to integrate the pressure gradient tangentially along some path that connects a point with known pressure to the point of interest. This method is known to perform poorly in the presence of noisy velocity measurements, due to excessive error accumulation along the integration path [2]. A popular alternative is to work with

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the pressure Poisson equation. By taking the divergence of both sides of the momentum equation, a scalar Poisson equation is generated for the pressure. In different formulations and discretisations, the PPE has become a standard in pressure retrieval for engineering and biomedical applications [3, 4, 5, 6, 7, 8, 9, 10]. Other numerical applications of the PPE include stabilization of equal-order finite element methods [11] and the decoupling of pressure and velocity for efficient time-stepping [12, 13].

Despite its apparent simplicity, the PPE brings an important numerical challenge: the additional regularity requirements induced by the divergence operator. Even in weak form, second-order differentiation of the velocity field is required in order to account for viscous effects. For this reason, C^1 interpolation of the velocity would be needed[8], a not-at-all trivial task in complex geometries. Thus, it is common practice to neglect the viscous term completely [14], which is generally reasonable in large arteries[15] but might lead to inaccurate estimates in slow-flow regions such as aneurysms[16]. Nonetheless, alternatives to the PPE for pressure computation normally introduce additional variables that increase the size of the problem. For instance, the method proposed by Švihlová et al.[1] introduces an artificial vector-valued variable to create a Stokes-like system, thereby increasing the total number of degrees of freedom by a factor of at least four (comparison between different techniques for pressure retrieval are available in the literature[1, 6, 14]). In this context, there is great appeal in constructing a new formulation for the PPE which properly accounts for viscous effects (including the non-Newtonian behaviour typically observed in blood), while allowing the use of standard C^0 finite element spaces for both the (unknown) pressure and the (given) velocities.

A major breakthrough towards allowing the use of C^0 finite elements for the PPE came from Johnston and Liu[12], who reformulated the second-order viscous term in the variational formulation as a first-order boundary integral. This approach has since been used successfully in decoupling pressure and velocity for efficient transient Navier-Stokes solvers[13, 17, 18, 19], but never in the context of pressure retrieval from given velocities. Moreover, their formulation does not allow for shear-dependent viscosity, which is a phenomenon observed in hemodynamic and polymeric flows. In the present work, we show how the additional terms induced by such variable viscosities can be handled so as to avoid extra regularity requirements. We further construct a general finite element framework, independent of flow regime, by extending the formulation to compressible flows. When incompressibility is dropped, we maintain standard regularity requirements by constructing a mixed formulation where the density and the velocity divergence are introduced as additional variables. In that case, we show how the density field can also be obtained from the velocity, without the need for invoking thermodynamic equations of state.

The rest of this paper is organised as follows. We begin the formulation statement by devising a new pressure Poisson equation from the balance of linear momentum for general fluid flows. After a detailed step-by-step derivation for the variational problems is presented, we comment specifically on the compressible Newtonian case and the incompressible quasi-Newtonian case. Numerical aspects are discussed, along with numerical examples which showcase the accuracy of the proposed approaches.

2. Formulation

2.1. The pressure Poisson equation

We will focus on stationary flows for simplicity of presentation, but the extension to time-dependent problems is straightforward. The two laws that will be used throughout the paper are the balance of linear momentum and the conservation of mass. In differential form they can be stated as, respectively,

$$(\rho\nabla\mathbf{u})\mathbf{u} - \nabla\cdot\mathbb{S} + \nabla p = \mathbf{b}, \tag{1}$$

$$\nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

where $\mathbf{b}$ is a given body force, $\mathbf{u}$ is the flow velocity, p is the pressure, ρ is the density and $\mathbb{S}$ is the viscous stress tensor. We start with a general setting where both variable viscosity and compressibility are allowed. In this case, the viscous tensor can be written as

$$\mathbb{S} = \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\mathbb{I}\right],$$

in which μ is the dynamic viscosity and $\mathbb{I}$ is the $d\times d$ identity tensor, $d = 2, 3$ being the number of spatial dimensions. Thus,

$$\nabla\cdot\mathbb{S} = \mu\left[\Delta\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}) - \frac{2}{3}\nabla(\nabla\cdot\mathbf{u})\right] + \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\mathbb{I}\right]\nabla\mu, \tag{3}$$

but $\Delta\mathbf{u} \equiv \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})$. Hence:

$$\nabla\cdot\mathbb{S} = \mu\left[\frac{4}{3}\nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})\right] + \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right]\nabla\mu - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\nabla\mu. \tag{4}$$

For now, the viscosity and density fields are assumed to be known. As we will show later, the density can be determined from the velocity, and so can the viscosity in many practical cases. Thus, we can write the momentum equation (1) as $\nabla p = \mathbf{f}(\mathbf{u})$, with

$$\mathbf{f} := \mathbf{b} - (\rho\nabla\mathbf{u})\mathbf{u} + \mu\left[\frac{4}{3}\nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})\right] + \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right]\nabla\mu - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\nabla\mu. \tag{5}$$

Now, let us consider a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$. We can generate a Poisson equation for the pressure by applying $(-\nabla\cdot)$ to both sides of Eq. (5). Similarly, appropriate Neumann boundary conditions for the pressure can be obtained by dotting both sides of Eq. (5) by the unit normal vector $\mathbf{n}$[20, 21]. This gives us the boundary value problem

$$-\Delta p = -\nabla\cdot\mathbf{f}(\mathbf{u}) \ \text{ in } \Omega, \tag{6}$$

$$\frac{\partial p}{\partial n} = \mathbf{n}\cdot\mathbf{f}(\mathbf{u}) \ \text{ on } \Gamma := \partial\Omega. \tag{7}$$

A scaling condition is needed to close the problem. It suffices to prescribe the value of the pressure at any point in the domain, or a global constraint such as $\int_\Omega p\,\mathrm{d}\Omega = \varrho$, for some $\varrho\in\mathbb{R}$. In the following derivations, zero mean pressure will be assumed ($\varrho = 0$), but in Section 2.4 we comment on appropriate scalings for different situations.

We are now in position to start devising a variational formulation for the PPE. Testing with a function $q\in H^1(\Omega)$ and using Green's first formula leads to

$$(\nabla q, \nabla p) - \left\langle q, \frac{\partial p}{\partial n}\right\rangle_\Gamma = -(q, \nabla\cdot\mathbf{f}), \tag{8}$$

with $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_\Gamma$ denoting, respectively, the $L^2(\Omega)$ inner product and duality paring. Integrating the right-hand side of Eq. (8) by parts and substituting the Neumann boundary condition leads to the weak form

$$(\nabla q, \nabla p) = \left(\nabla q, \mathbf{b} - (\rho\nabla\mathbf{u})\mathbf{u} - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\nabla\mu\right) + \left(\nabla q, \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right]\nabla\mu\right) + \frac{4}{3}\left(\nabla q, \mu\nabla\left(\nabla\cdot\mathbf{u}\right)\right) - \left(\mu\nabla q, \nabla\times\left(\nabla\times\mathbf{u}\right)\right).$$

Note that we have split the right-hand side into four contributions. From a Physics standpoint, all of them could be computed as soon as the density and viscosity are determined. However, in the context of finite element spaces, some of them impose numerical challenges that must be addressed carefully, since they directly or indirectly require second-order differentiation. The first term is straightforward for most fluids and flow regimes. The second one requires a specific technique in the case of non-Newtonian fluids such as blood, and will be tackled in Section 2.3. The third one only requires special attention for compressible flows, and will be dealt with in Section 2.2. The last term requires attention even for the incompressible Newtonian case, so it will be addressed first. We begin by using integration by parts to write

$$-\left(\mu\nabla q, \nabla\times\left(\nabla\times\mathbf{u}\right)\right) = \langle\mathbf{n}\times\nabla q, \nabla\times\mathbf{u}\rangle_\Gamma - \left(\nabla\times\left(\mu\nabla q\right), \nabla\times\mathbf{u}\right),$$

but

$$\begin{aligned}\left(\nabla\times\left(\mu\nabla q\right), \nabla\times\mathbf{u}\right) &= \left(\mu\nabla\times\left(\nabla q\right) + \nabla\mu\times\nabla q, \nabla\times\mathbf{u}\right)\\ &= \left(\nabla q, \left(\nabla\times\mathbf{u}\right)\times\nabla\mu\right) = \left(\nabla q, \left[\nabla\mathbf{u} - (\nabla\mathbf{u})^\top\right]\nabla\mu\right).\end{aligned}$$

Therefore, the weak form of the PPE simplifies to

$$\begin{aligned}(\nabla q, \nabla p) = \left(\nabla q, \mathbf{b} - (\rho\nabla\mathbf{u})\mathbf{u} - \left(\frac{2}{3}\nabla\cdot\mathbf{u}\right)\nabla\mu\right) + \\ \frac{4}{3}\left(\nabla q, \mu\nabla\left(\nabla\cdot\mathbf{u}\right)\right) + 2\left(\nabla q, (\nabla\mathbf{u})^\top\nabla\mu\right) + \langle\mathbf{n}\times\nabla q, \mu\nabla\times\mathbf{u}\rangle_\Gamma.\end{aligned} \tag{9}$$

What we have now is a general variational framework for the PPE, allowing generalised Newtonian behaviour and compressibility. In the incompressible Newtonian case, i.e., $\nabla\mu$ and $\nabla\cdot\mathbf{u}$ both zero, the variational formulation reads simply: Given $\mathbf{b}\in X'$ and $\mathbf{u}\in\tilde{X}$, find $p\in Y$ such that for all $q\in\tilde{Y}$

$$(\nabla q, \nabla p) = (\nabla q, \mathbf{b} - (\rho\nabla\mathbf{u})\mathbf{u}) + \langle\mathbf{n}\times\nabla q, \mu\nabla\times\mathbf{u}\rangle_\Gamma, \tag{10}$$

in which $X = [H^1(\Omega)]^d$, $Y = H^1_*(\Omega) := \{q\in H^1(\Omega):\ (q,1)=0\}$ and

$$\begin{aligned}\tilde{X} &= \left\{\mathbf{w}\in X:\ (\nabla\times\mathbf{w})\,|_\Gamma\in L^2(\Gamma)\right\},\\ \tilde{Y} &= \left\{q\in H^1(\Omega):\ \mathbf{n}\times\nabla q|_\Gamma\in L^2(\Gamma)\right\}.\end{aligned}$$

Notice that we are left with only first-order derivatives, which allows us to use standard C^0 finite element spaces for both the unknown pressure and the given velocity data [12]. However, when compressibility or shear-dependent viscosity are allowed, a mixed framework is needed, as we will show next.

2.2. Compressible flows

The compressible case brings two issues, one of mathematical nature and another one from a physical standpoint. The former regards the need for higher-order differentiation of the velocity field, due to the presence of the grad-div term in Eq. (9). Handling that within a C^0 framework is possible using a simple projection step, which will be described later. The second issue is related to the computation of the density field, which for compressible flows is a variable, rather than a given parameter. Fortunately, we can use the conservation of mass (2) to obtain ρ in terms of the given velocities. This gives us a hyperbolic problem which is well posed if the density is known at the inlet, i.e., one must solve[22]

$$\mathbf{u} \cdot \nabla \rho + (\nabla \cdot \mathbf{u})\, \rho = 0 \text{ in } \Omega \tag{11}$$

$$\rho = \rho_{in} \text{ on } \Gamma_{in}, \tag{12}$$

where ρ_{in} is a strictly positive function and Γ_{in} is the portion of the boundary where $\mathbf{u} \cdot \mathbf{n} < 0$. The corresponding variational formulation is: Given $\mathbf{u} \in H(\text{div}, \Omega)$, find $\rho \in H^1(\Omega)$, $\rho = \rho_{in}$ on Γ_{in}, such that for all $r \in L^2(\Omega)$,

$$(r, \mathbf{u} \cdot \nabla \rho + (\nabla \cdot \mathbf{u})\, \rho) = 0. \tag{13}$$

Since the velocity is assumed to be known, we end up with a linear problem, which on the discrete level will translate to a simple linear algebraic system. The density can then be fed into the right-hand side of the PPE.

Now we turn our attention to the issue of second-order differentiation in the PPE. In order to solve that, one can simply project the velocity divergence $\phi := \nabla \cdot \mathbf{u}$ onto a continuous space before inserting it into the PPE. On the discrete level, this corresponds to a scalar mass matrix problem. We finally formulate our mixed problem as: Given $\mathbf{b} \in X'$ and $\mathbf{u} \in \tilde{X}$, find $(p, \rho, \phi) \in Y \times W^2$, with $\rho = \rho_{in}$ on Γ_{in}, such that for all $(q, r, w) \in \tilde{Y} \times Z^2$,

$$(r, \mathbf{u} \cdot \nabla \rho + (\nabla \cdot \mathbf{u})\, \rho) = 0, \tag{14}$$

$$(w, \phi) = (w, \nabla \cdot \mathbf{u})\,, \tag{15}$$

$$\begin{aligned}(\nabla q, \nabla p) = (\nabla q, \mathbf{b}) - (\nabla q, (\rho \nabla \mathbf{u})\mathbf{u}) + \frac{2}{3}(\nabla q, 2\mu \nabla \phi - \phi \nabla \mu) + \\ 2\left(\nabla q, (\nabla \mathbf{u})^\top \nabla \mu\right) + \langle \mathbf{n} \times \nabla q, \mu \nabla \times \mathbf{u} \rangle_\Gamma\,,\end{aligned} \tag{16}$$

where $Z = L^2(\Omega)$, $W = H^1(\Omega)$ and the remaining spaces are as in the incompressible case. We have at last a mixed variational formulation that can be seen as three linear problems to be solved in sequence.

How to treat the viscosity is a problem-dependent matter. It of course depends on the density, since $\mu = \rho\nu$, with ν being the kinematic viscosity. For gas flows, variations of ν may be given according to local temperatures[23], or sometimes even neglected depending on the flow regime. For generalised Newtonian fluids such as blood, there are models that relate the viscosity to the velocity gradient. We shall focus on the latter case, which is especially relevant for biomedical applications.

2.3. Generalised Newtonian fluids

There are various types of materials and solid-liquid mixtures that behave as generalised Newtonian fluids. This so-called quasi-Newtonian behaviour is observed when the viscosity depends locally on the flow field itself. In hemodynamics, the viscosity is often modelled by a nonlinear dependence on the shear rate: $\mu = \mu\left(\nabla^s \mathbf{u}\right) = \eta\left(\dot{\gamma}\right)$, where

$$\nabla^s \mathbf{u} := \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right] \quad \text{and} \quad \dot{\gamma} := \sqrt{\frac{1}{2}\nabla^s \mathbf{u} : \nabla^s \mathbf{u}}\ .$$

In that case, the PPE would require second-order differentiation of the velocity, since the formulation contains the gradient of the viscosity field. Once again, we can stay in the C^0 FEM framework by using a simple L^2 projection onto $H^1(\Omega)$. Instead of plugging the expression $\mu = \mu\left(\nabla^s \mathbf{u}\right)$ directly into the PPE, we treat μ as an independent unknown, and enforce the rheological law weakly. Once the continuous viscosity field is computed from the given velocities, it can be inserted into the PPE. Assuming incompressibility, the mixed problem now reads: Given $\mathbf{b} \in X'$ and $\mathbf{u} \in \tilde{X}$, find $(p, \mu) \in Y \times W$ such that for all $(q, w) \in \tilde{Y} \times Z$,

$$(w, \mu) = \left(w, \eta\left(\dot{\gamma}\right)\right), \tag{17}$$

$$(\nabla q, \nabla p) = (\nabla q, \mathbf{b}) - (\nabla q, (\rho \nabla \mathbf{u})\mathbf{u}) + 2\left(\nabla q, (\nabla \mathbf{u})^\top \nabla \mu\right) + \langle \mathbf{n} \times \nabla q, \mu \nabla \times \mathbf{u}\rangle_\Gamma . \tag{18}$$

Once again, the mixed problem can be seen as two linear problems to be solved in sequence, without the need for iterative schemes.

2.4. Pressure scaling

As previously discussed, the standard setting for the PPE is a pure Neumann problem, and therefore a scaling condition is needed to ensure unique solvability. In compressible flows, the pressure is normally treated as a strictly positive quantity, so that the scaling $(p, 1) = 0$ is not appropriate. Alternatively to this global scaling, the problem can be determined by enforcing the pressure at one point in the domain, which is an available information in practical situations (e.g., far-field pressure, inlet/outlet pressure, etc). Also, in internal flows with an open outflow region Γ_{out}, the following scaling usually holds[24, 25]:

$$\int_{\Gamma_{\text{out}}} p \ \mathrm{d}\Gamma = 0. \tag{19}$$

This can be enforced by modifying the right-hand side of Eq. (9) to

$$(\nabla q, \nabla p) + \int_{\Gamma_{\text{out}}} q \ \mathrm{d}\Gamma \int_{\Gamma_{\text{out}}} p \ \mathrm{d}\Gamma \tag{20}$$

and simply looking for $p \in H^1(\Omega)$. On the discrete level, this additional term will lead to a highly sparse symmetric matrix corresponding to a rank-one perturbation. If the right-hand side of Eq. (19) is a known nonzero constant, as in domains with multiple outlets (*c.f.* Ref.[26]), the corresponding value can be used to shift the solution afterwards.

3. On the continuity of flow quantities

In deriving our formulation, we have assumed that p, μ and $\nabla \cdot \mathbf{u}$ are continuous, but the only flow quantities whose continuity is actually dictated by physical principles are velocities and stresses. In fact, a necessary condition we must have is that μ be continuous. If there are jumps in the viscosity field, as in flows of immiscible materials, then both p and $\nabla\cdot\mathbf{u}$ can also have jumps[27]. Moreover, all flow quantities may experience discontinuities if the compressible Navier-Stokes equations are replaced by the Euler equations, which are inviscid. It is important to remark, however, that the continuity assumptions discussed here do hold in several practical situations such as incompressible single-phase flows, continuum hemodynamics and subsonic gas dynamics. Furthermore, if the quantities of interest are at least piecewise continuous (e.g., two-phase flows, inviscid shock-wave problems), such requirements can be relaxed using appropriate domain decomposition techniques.

4. Numerical examples

Standard Lagrangian finite element spaces will be used in the interpolation of the unknowns (pressure, density, velocity divergence, viscosity) and the given velocity field. Optimal convergence rates would require the polynomial degree for the velocity to be one order higher than for the unknowns. In practical cases, however, one cannot always afford to use quadratic interpolation, since the velocity is given/measured only in a reduced set of points. For this reason, we will consider both optimal and suboptimal settings in the numerical examples, always using first-order interpolation for the unknows. In order to assess the accuracy of our formulation, we consider examples with known solutions and use a normalised L^2 norm:

$$\|p - p_h\|_0 := \frac{\|p - p_h\|_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)}}.$$

The spatial coordinates will be denoted by (x, y, z).

4.1. Compressible flow in two-dimensions

We start by considering a compressible flow accelerating through a straight channel $\Omega = (0,3)\times(0,1)$ due to a body force $\mathbf{b} = \left\{ e^x\left(y\left(y-1\right)\right)^2, \frac{(4x-1)(1-2y)}{3} \right\}^\top$. We set a constant dynamic viscosity $\mu = 1$, a mass flow rate per unit width equal to 1/6 and an inlet density $\rho(0, y) \equiv 1$, to get

$$\begin{aligned}
\mathbf{u}(x,y) &= \begin{Bmatrix} y(1-y)e^x \\ 0 \end{Bmatrix}, \\
p(x,y) &= \frac{20 - x\left(4y^2 - 4y + 6\right)}{3}, \\
\rho(x,y) &= e^{-x}.
\end{aligned}$$

For the numerical solution, we use triangular elements with linear basis functions for p, ρ and ϕ, and quadratic interpolation for $\mathbf{u}$. The coarsest mesh has 6 elements, as illustrated in Figure 1, then 6 levels of uniform refinement are considered. Figure 2 shows the error plot with respect to the mesh size h, where quadratic convergence is verified for the pressure approximation.

4.2. Power-law fluid in two-dimensional channel

We now consider a classic benchmark for non-Newtonian fluids: the two-dimensional channel flow of a power-law fluid[28, 29, 30]. This is a popular model for hemodynamic

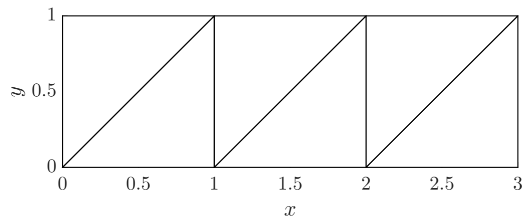

Figure 1: Compressible flow benchmark: coarsest mesh used in the refinement study.

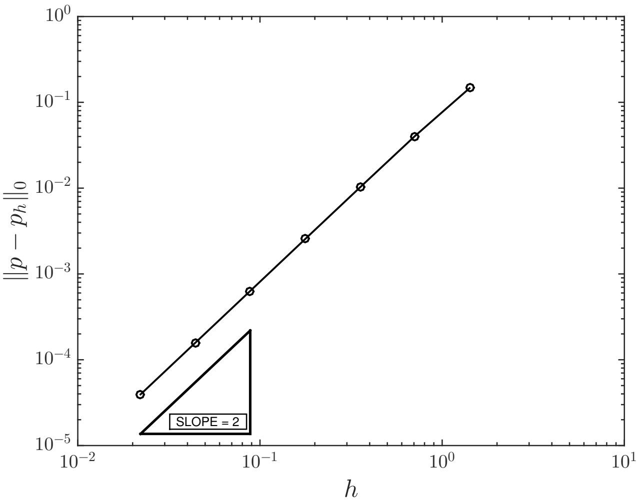

Figure 2: Compressible flow benchmark: uniform refinement study.

and polymeric flows, and its rheological law is given by

$$\eta(\dot{\gamma}) = k\dot{\gamma}^{n-1}, \tag{21}$$

where k is a positive parameter and $n < 1$ for shear-thinning fluids. In a straight channel $\Omega = (0, L) \times \left(-\frac{H}{2}, \frac{H}{2}\right)$ with volumetric flow rate per unit width equal to Q, the velocity and pressure fields for the fully developed flow are given by

$$\mathbf{u}(x, y) = \begin{Bmatrix} \left(\frac{2n+1}{n+1}\right) \frac{Q}{H} \left(1 - |2y/H|^{\frac{n+1}{n}}\right) \\ 0 \end{Bmatrix},$$

$$p(x, y) = \frac{4k}{H} \left[\left(\frac{2n+1}{n}\right) \frac{Q}{H^2} \right]^n (L - x).$$

For this example we use the hemodynamic parameters[31] $\rho = 1050$ kg/m^3, $k = 0.035$ Pa.s$^{0.6}$ and $n = 0.6$, with $Q = 100$ mm^2/s and $L = 3H = 3$ mm. We start from a mesh composed of 6 identical square elements and consider 6 levels of uniform refinement, this time using bilinear basis functions for all quantities of interest. As shown in Figure 3, we now get a linear approximation, as first-order interpolation is used for the given velocity field.

4.3. Carreau fluid in cylindrical pipe

Another classic benchmark problem in hemodynamics, now in three dimensions, is the Carreau pipe flow[32, 33]. The Carreau model describes the shear-thinning behaviour of

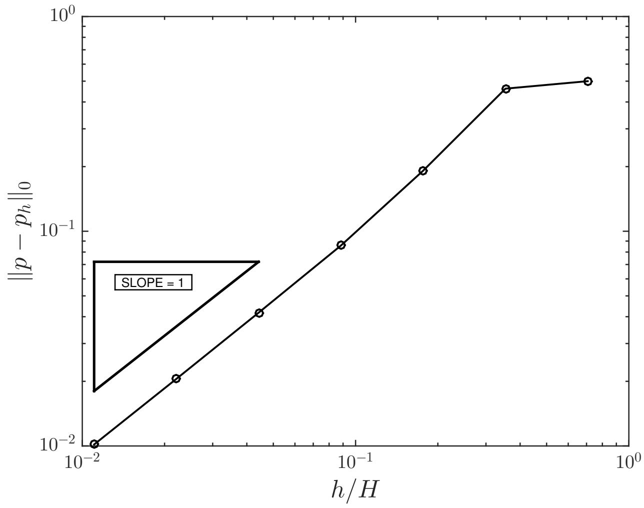

Figure 3: Power-law benchmark: uniform refinement study.

blood through the law

$$\eta\left(\dot{\gamma}\right) = \mu_\infty + \left(\mu_0 - \mu_\infty\right)\left[1 + \left(\lambda\dot{\gamma}\right)^2\right]^{\frac{n-1}{2}}, \tag{22}$$

where $n < 1$ and the remaining constants are positive material parameters. In the cylindrical domain

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 < R^2, \; 0 < z < L\right\},$$

the developed flow has a semi-analytical solution. Given a constant pressure drop per unit length equal to α, we have

$$\mathbf{u}(x, y, z) = \left\{0, 0, w\left(x, y\right)\right\}^\top, \quad p(x, y, z) = |\alpha|\left(L - z\right),$$

where

$$w\left(x, y\right) = \int_R^{x^2+y^2} f(r)\;\mathrm{d}r,$$

f being the solution of the nonlinear equation

$$\left\{\mu_\infty + \left(\mu_0 - \mu_\infty\right)\left[1 + \left(\frac{\lambda f}{2}\right)^2\right]^{\frac{n-1}{2}}\right\} f = -\frac{|\alpha| r}{2}, \quad r \in [0, R]. \tag{23}$$

For the numerical study, the properties of blood are used again[31]: $\rho = 1050$ kg/m^3, $\mu_\infty = 3.45$ mPa.s, $\mu_0 = 56$ mPa.s, $n = 0.3568$, $\lambda = 3.313$ s. The dimensions are fixed as $L = 2R = 2$ mm. We set a pressure drop of 4.0 Pa accross the length, which corresponds to a Reynolds number of approximately 15. This time, hexahedral elements with trilinear interpolation are employed for all quantities. The coarsest mesh is illustrated in Figure 4, and three levels of refinement are applied. The results of the convergence study are shown in Table 1. Here, again, we cannot expect quadratic convergence, as the velocity is interpolated with the same polynomial degree as the pressure.

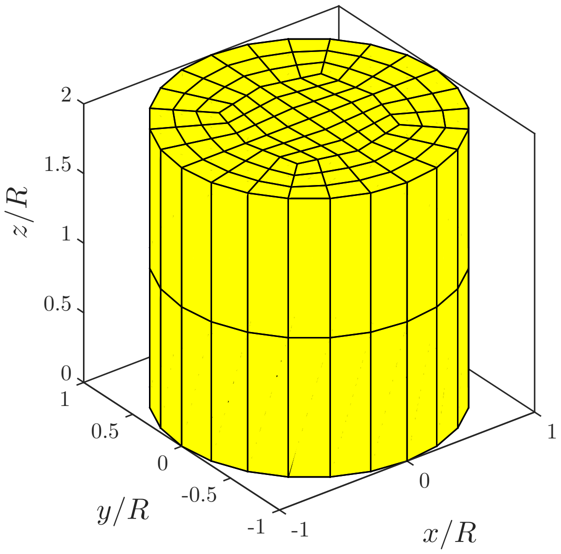

Figure 4: Carreau pipe flow: coarsest mesh used in the refinement study.

Table 1: Carreau pipe flow: pressure error and estimated order of convergence (eoc).

Mesh	$\|p - p_h\|_0$	eoc
1	0.3434	–
2	0.1459	1.23
3	0.0575	1.34
4	0.0220	1.38

5. Concluding remarks

In the present work, we have devised a general mixed framework for the pressure Poisson equation allowing C^0 finite element interpolation of all (given and unknown) quantities. While classical finite element formulations of the PPE require viscous terms to be dropped, ours is able to account even for the nonlinear viscous behaviour of generalised Newtonian fluids, as well as for compressibility. This is accomplished through the use of appropriate vector calculus identities and simple projections of specific flow quantities to circumvent C^1 regularity requirements. We provide numerical examples considering different types of elements, in order to showcase the accuracy of our method in different flow scenarios. While the focus of the present work has been placed on fundamental aspects, the formulation can be readily used in practical applications such as arterial pressure estimation from measured blood flow velocities; the addition of the time-dependent terms brings no extra difficulties. Furthermore, the variational formulation developed here can be used in the design of accurate flow solvers, either by replacing the continuity equation or appropriately modifying it for numerical stabilisation. This is ongoing work to be covered in forthcoming publications. Also ongoing is the development of an ultra-weak variational framework allowing for pressure discontinuities.

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