Technische Universität Graz

Stable least-squares space-time boundary element methods for the wave equation

D. Hoonhout, R. Löscher, O. Steinbach, C. Urzúa–Torres

Berichte aus dem Institut für Angewandte Mathematik

Bericht 2023/8

Technische Universität Graz

Stable least-squares space-time boundary element methods for the wave equation

D. Hoonhout, R. Löscher, O. Steinbach, C. Urzúa–Torres

Berichte aus dem Institut für Angewandte Mathematik

Bericht 2023/8

Technische Universität Graz
Institut für Angewandte Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.applied.math.tugraz.at`

Stable least-squares space-time boundary element methods for the wave equation

Daniel Hoonhout[1], Richard Löscher[2],
Olaf Steinbach[2], Carolina Urzúa–Torres[1]

[1]Delft Institute of Applied Mathematics, TU Delft,
Mekelweg 4, 2628CD Delft, The Netherlands

[2]Institut für Angewandte Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria

Abstract

In this paper, we recast the variational formulation corresponding to the single layer boundary integral operator V for the wave equation as a minimization problem in $L^2(\Sigma)$, where $\Sigma := \partial\Omega \times (0,T)$ is the lateral boundary of the space-time domain $Q := \Omega \times (0,T)$. For discretization, the minimization problem is restated as a mixed saddle point formulation. Unique solvability is established by combining conforming nested boundary element spaces for the mixed formulation such that the related bilinear form is discrete inf-sup stable. We analyze under which conditions the discrete inf-sup stability is satisfied, and, moreover, we show that the mixed formulation provides a simple error indicator, which can be used for adaptivity. We present several numerical experiments showing the applicability of the method to different time-domain boundary integral formulations used in the literature.

1 Introduction

Time-domain boundary integral equations and boundary element methods (BEM) for evolution problems are well established in the literature, see, for example [12] for an overview. The common procedure is to either first discretize the temporal part using convolution quadrature, and then applying a boundary element method for the spatial variables, see, e.g., [5, 7, 26, 36]; or to use BEM with spatial basis functions and temporal basis functions chosen separately, and then considered together as tensor product. We refer to [1, 2, 3, 20, 21, 22, 23, 25], to name a few. Typically, the choice of temporal basis function is done in order to obtain a marching-on-in-time algorithm, which is an explicit time stepping scheme [47].

Lately, the interest of discretizing space and time simultaneously has been increasing, resulting in so called space-time discretization methods. Admittedly, space-time discretizations lead to larger systems of algebraic equations to be solved. Nevertheless, these methods offer the advantage of having full control of the discretization in space and time at once, allowing for space-time adaptivity. In order to see this, it is worth noting that although space-time discretizations may lead to tensor-product basis functions on structured space-time meshes, they treat time as if it were another spatial variable, and thus, also permit unstructured meshes. Moreover, space-time methods allow for preconditioning and parallelization in the space-time domain, which gives more flexibility in the construction of efficient solvers than time stepping methods, see, e.g., [18, 19].

The success of space-time BEM for parabolic problems [14, 15, 32] and the promising developments for the wave equation when using space-time finite element methods (FEM) [13, 31, 33, 43, 44], encourage us to also study space-time BEM for hyperbolic problems. For this, the first step is to consider the literature on time-domain boundary integral equations for the wave equation.

The standard approaches for BEM for the wave equation started with the groundbreaking works of Bamberger and Ha-Duong [6], and Aimi et al. [2]. The main difficulty in the numerical analysis of these formulations is in the so-called norm gap, coming from continuity and coercivity estimates in different space-time Sobolev norms. When using the energetic BEM from [2], a complete stability and error analysis can be done in $L^2(\Sigma)$, see [28], where $\Sigma := \partial\Omega \times (0,T)$ is the lateral boundary of the space-time domain $Q := \Omega \times (0,T)$. Hence, the energetic BEM is amenable to space-time discretizations. Its disadvantage, though, is that it requires the Dirichlet data to be sufficiently regular, i.e., in $H^1(\Sigma)$.

Using a generalized inf-sup stable variational formulation [44] for the wave equation, in [40] we derived inf-sup stability conditions for all boundary integral operators in related trace spaces, overcoming norm-gaps and also the need for extra regularity of the Dirichlet data. However, the standard discretization of the single layer boundary integral operator V by means of space-time piecewise constant basis functions does not provide an inf-sup stable pair [24] in one spatial dimension, which we believe will also be the case for $d = 2, 3$.

As an alternative, we proposed a regularisation via a modified Hilbert transform [43], the resulting composition with V becomes elliptic in the natural energy space $[H^{1/2}_{,0}(\Sigma)]'$, similarly to what is known for boundary integral operators for second-order elliptic partial differential equations [41]. At the time of writing this article, this strategy had two drawbacks: it introduces further computational costs, and so far it is only applicable to space-time meshes that admit a tensor product structure. However, these obstacles could be circumvented applying techniques used in [45].

Another approach is to replace the straightforward variational formulation by a least-squares/minimal residual equation. For FEM this has been extensively studied for time independent problems, see Bochev and Gunzberger [9, 10, 11]. Time dependent parabolic problems have been investigated in the context of first order least squares systems (FOSLS) in [16] and in the context of minimal residual Petrov–Galerkin methods in [4, 46]. In the context of BEM for elliptic partial differential equations this has been studied in [39] and, recently, for FEM for the wave equation in [17, 29]. In this paper we combine these ideas to also have a least-squares boundary integral formulation that works for hyperbolic problems. In addition to a stable method, we get an error indicator that can be used for space-time adaptivity. It is worth pointing out that, although we present the theory for the wave equation in one spatial dimension, the underlying abstract framework is dimension-independent and, consequently, we expect the theory to carry over to higher dimensions.

This paper is organized as follows. Section 2 introduces our notation and model problem. In particular, we remind the reader that in one spatial dimension the single layer boundary integral operator V is an isomorphism from $[H^1_{,0}(\Sigma)]'$ to $L^2(\Sigma)$, and we derive the associated inf-sup constant. Then, we introduce a least-squares variational formulation for the related boundary integral equation. In Section 3, we present the stable discretization of our least squares formulation. For this, we propose a mixed BEM, and show its unique solvability in Theorems 3.1 and 3.2. Moreover, Lemma 3.3 establishes the convergence of the method, and Lemma 3.4 provides the conditions under which we obtain a reliable error indicator. In Section 4, we provide numerical experiments to verify our theory. There, we investigate the performance of the proposed least-squares formulation for three different first-kind boundary integral equations related to our Dirichlet problem. With this, we compare how different mapping and stability properties affect the numerical behaviour of the proposed method. We pay special attention to the requirements to have a reliable error indicator and the numerical study of the resulting adaptive scheme. Finally, we give some conclusions and comment on ongoing work.

2 Least-Squares Variational Formulation

As in [2, 41], we consider the Dirichlet boundary value problem for the homogeneous wave equation in the one-dimensional spatial domain $\Omega = (0, L)$, with zero initial conditions, and for a given time horizon $T > 0$,

$$\left.\begin{aligned} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) &= 0 && \text{for } (x,t) \in Q := (0,L) \times (0,T), \\ u(x,0) = \partial_t u(x,t)_{|t=0} &= 0 && \text{for } x \in (0,L), \\ u(0,t) &= g_0(t) && \text{for } t \in (0,T), \\ u(L,t) &= g_L(t) && \text{for } t \in (0,T). \end{aligned}\right\} \tag{2.1}$$

In the one-dimensional case, the fundamental solution of the wave equation is the Heaviside function

$$U^*(x,t) = \frac{1}{2}\,\mathsf{H}(t - |x|),$$

and we can represent the solution u of (2.1) by using the single layer potential

$$u(x,t) := (\widetilde{\mathsf{V}}w)(x,t) = \frac{1}{2}\int_0^{t-|x|} w_0(s)\,ds + \frac{1}{2}\int_0^{t-|x-L|} w_L(s)\,ds \quad \text{for } (x,t) \in Q.$$

To determine the yet unknown density functions (w_0, w_L), we consider the boundary integral equations for $x \to 0$,

$$(\mathsf{V}_0\, w)(t) := \frac{1}{2}\int_0^t w_0(s)\,ds + \frac{1}{2}\int_0^{t-L} w_L(s)\,ds = g_0(t) \quad \text{for } t \in (0,T), \tag{2.2}$$

and for $x \to L$,

$$(\mathsf{V}_L\, w)(t) := \frac{1}{2}\int_0^{t-L} w_0(s)\,ds + \frac{1}{2}\int_0^{t} w_L(s)\,ds = g_L(t) \quad \text{for } t \in (0,T). \tag{2.3}$$

We write the boundary integral equations (2.2) and (2.3) in compact form, for $w = (w_0, w_L)$, as

$$(\mathsf{V}\, w)(t) = \begin{pmatrix} (\mathsf{V}_0\, w)(t) \\ (\mathsf{V}_L\, w)(t) \end{pmatrix} = \begin{pmatrix} \mathsf{V}_{00} & \mathsf{V}_{0L} \\ \mathsf{V}_{L0} & \mathsf{V}_{LL} \end{pmatrix} \begin{pmatrix} w_0 \\ w_L \end{pmatrix}(t) = \begin{pmatrix} g_0(t) \\ g_L(t) \end{pmatrix} = g(t), \quad t \in (0,T). \tag{2.4}$$

In energetic BEM [2], instead of (2.4), the time derivative of (2.4) is considered,

$$\partial_t(\mathsf{V}\, w)(t) = \partial_t g(t) \quad \text{for } t \in (0,T). \tag{2.5}$$

Recall the ellipticity estimate [2, Theorem 2.1], see also [41, Theorem 2.1],

$$\langle w, \partial_t \mathsf{V}\, w\rangle_{L^2(\Sigma)} \geq c_S(n)\, \|w\|^2_{L^2(\Sigma)} \quad \text{for all } w = (w_0, w_L) \in L^2(\Sigma) := L^2(0,T) \times L^2(0,T),$$

where

$$c_S(n) := \sin^2 \frac{\pi}{2(n+1)},$$

and

$$n := \min\Big\{ m \in \mathbb{N} : T \leq mL \Big\},$$

is the number of time slices $T_j := ((j-1)L, jL)$ for $j = 1, \dots, n$ when $T = nL$. In the case $T < nL$, we define the last time slice as $T_n := ((n-1)L, T)$, while all the others remain unchanged.

Since $\partial_t \mathsf{V} : L^2(\Sigma) \to L^2(\Sigma)$ is also bounded, unique solvability of the boundary integral equation (2.5) follows. Let us introduce $H^1_{0,}(\Sigma) := H^1_{0,}(0,T) \times H^1_{0,}(0,T)$, and note that $z \in H^1_{0,}(0,T)$ covers the zero initial condition $z(0) = 0$. Moreover, for $u = (u_0, u_L) \in H^1_{0,}(\Sigma)$ we have the norm definition

$$\|u\|^2_{H^1_{0,}(\Sigma)} := \|\partial_t u_0\|^2_{L^2(0,T)} + \|\partial_t u_L\|^2_{L^2(0,T)}.$$

Given that the time derivative $\partial_t : H^1_{0,}(\Sigma) \to L^2(\Sigma)$ is an isomorphism, e.g., [43, Sect. 2.1], we also have that $\mathsf{V} : L^2(\Sigma) \to H^1_{0,}(\Sigma)$ is an isomorphism.

We define $H^1_{,0}(\Sigma) := H^1_{,0}(0,T) \times H^1_{,0}(0,T)$ in a similar way, but with a zero terminal condition at $t = T$. As in [41, eqn. (2.9)] we also have the ellipticity estimate

$$-\langle \overline{\partial}_t^{-1} \mathsf{V}\, w, w\rangle_\Sigma \geq c_S(n)\, \|w\|^2_{[H^1_{,0}(\Sigma)]'} \quad \text{for all } w \in [H^1_{,0}(\Sigma)]',$$

where

$$(\overline{\partial}_t^{-1} f)(t) = -\int_t^T f(s)\,ds, \quad t \in (0,T),$$

is the inverse of $\partial_t : H^1_{,0}(\Sigma) \to L^2(\Sigma)$, and $[H^1_{,0}(\Sigma)]'$ denotes the dual space of $H^1_{,0}(\Sigma)$ with respect to $L^2(\Sigma)$, which is equipped with norm

$$\|w\|_{[H^1_{,0}(\Sigma)]'} = \sup_{0 \neq v \in H^1_{,0}(\Sigma)} \frac{\langle w, v\rangle_\Sigma}{\|v\|_{H^1_{,0}(\Sigma)}},$$

where $\langle \cdot,\cdot\rangle_\Sigma : [H^1_{,0}(\Sigma)]' \times H^1_{,0}(\Sigma) \to \mathbb{R}$ denotes the duality pairing as extension of the inner product $\langle \cdot,\cdot\rangle_{L^2(\Sigma)}$ in $L^2(\Sigma)$.

Hence we conclude that $\mathsf{V} : [H^1_{,0}(\Sigma)]' \to L^2(\Sigma)$ is an isomorphism and, in particular, bounded and satisfying the inf-sup stability condition. Our next aim is to find the inf-sup constant $\widetilde{c}_S(n) > 0$ of

$$\widetilde{c}_S(n)\|w\|_{[H^1_{,0}(\Sigma)]'} = \sup_{0 \neq q \in L^2(\Sigma)} \frac{\langle \mathsf{V}\, w, q\rangle_{L^2(\Sigma)}}{\|q\|_{L^2(\Sigma)}} = \|\mathsf{V}\, w\|_{L^2(\Sigma)}.$$

With this goal in mind, we first consider two auxiliary lemmas, where we follow the ideas and notation of [43].

Lemma 2.1 *Let $\{\mathcal{W}_k(t)\}_{k=0}^\infty := \{\cos(\alpha_k t)\}_{k=0}^\infty$ with $\alpha_k := \left(\frac{\pi}{2} + k\pi\right)\frac{1}{T}$, and $w \in [H^1_{,0}(0,T)]'$. Then there exists a unique $v \in L^2(0,T)$ such that $\partial_t v = w$ and*

$$\|w\|^2_{[H^1_{,0}(0,T)]'} = \frac{2}{T}\sum_{k=0}^\infty \alpha_k^{-2}\overline{w}_k^2 = \|v\|^2_{L^2(0,T)},$$

where $\overline{w}_k := \langle w, \mathcal{W}_k\rangle_{(0,T)}$.

Proof. First, let us remind the reader that $\{\mathcal{W}_k(t)\}_{k=0}^\infty$ forms an orthogonal basis for $L^2(0,T)$ and $H^1_{,0}(0,T)$, while $\{\mathcal{V}_k(t)\}_{k=0}^\infty := \{\sin(\alpha_k t)\}_{k=0}^\infty$ is an orthogonal basis for $L^2(0,T)$ and $H^1_{0,}(0,T)$. Since $\partial_t : L^2(0,T) \to [H^1_{,0}(0,T)]'$ is an isomorphism, we can write $w = \partial_t v$ for a unique $v \in L^2(0,T)$. By representing v by the orthogonal basis $\{\mathcal{V}_k\}_{k=0}^\infty$, i.e., $v = \sum_{k=0}^\infty v_k \mathcal{V}_k$, with $v_k := \frac{2}{T}\langle v, \mathcal{V}_k\rangle_{L^2(0,T)}$, we obtain

$$\|v\|_{L^2(0,T)} = \left(\frac{T}{2}\sum_{k=0}^\infty |v_k|^2\right)^{1/2},$$

and further

$$w = \partial_t v = \sum_{k=0}^\infty v_k \alpha_k \mathcal{W}_k. \tag{2.6}$$

Using, that we can represent any function $q \in H^1_{,0}(0,T)$ as $q(t) = \sum_{k=0}^{\infty} \overline{q}_k \mathcal{W}_k(t)$, with $\overline{q}_k = \frac{2}{T}\langle q, \mathcal{W}_k\rangle_{L^2(0,T)}$, we first compute that

$$\|\partial_t q\|_{L^2(0,T)} = \left(\frac{T}{2}\sum_{k=0}^{\infty} |\overline{q}_k|^2 \alpha_k^2\right)^{1/2}.$$

Now, by the definition of $\|\cdot\|_{[H^1_{,0}(0,T)]'}$, we have that

$$\begin{aligned}
\|w\|_{[H^1_{,0}(0,T)]'} &= \sup_{0\neq q\in H^1_{,0}(0,T)} \frac{\langle w, \sum_{k=0}^{\infty}\overline{q}_k \mathcal{W}_k\rangle_{(0,T)}}{\|\partial_t q\|_{L^2(0,T)}} = \sup_{0\neq q\in H^1_{,0}(0,T)} \frac{\sum_{k=0}^{\infty}\overline{q}_k\langle w, \mathcal{W}_k\rangle_{(0,T)}}{\left(\frac{T}{2}\sum_{k=0}^{\infty}\alpha_k^2|\overline{q}_k|^2\right)^{\frac{1}{2}}} \\
&= \sup_{0\neq q\in H^1_{,0}(0,T)} \frac{\sum_{k=0}^{\infty}\alpha_k\overline{q}_k\alpha_k^{-1}\overline{w}_k}{\left(\frac{T}{2}\sum_{k=0}^{\infty}\alpha_k^2|\overline{q}_k|^2\right)^{\frac{1}{2}}} \leq \left(\frac{2}{T}\sum_{k=0}^{\infty}\overline{w}_k^2\alpha_k^{-2}\right)^{1/2}.
\end{aligned}$$

By picking

$$\hat{q} = \sum_{k=0}^{\infty}\alpha_k^{-2}\overline{w}_k\mathcal{W}_k \in H^1_{,0}(0,T),$$

we can bound $\|w\|_{[H^1_{,0}(0,T)]'}$ from below by the same estimate, i.e.,

$$\|w\|_{[H^1_{,0}(0,T)]'} = \sup_{0\neq q\in H^1_{,0}(0,T)} \frac{\langle w, q\rangle_{(0,T)}}{\|\partial_t q\|_{L^2(0,T)}} \geq \frac{\langle w, \hat{q}\rangle_{(0,T)}}{\|\partial_t \hat{q}\|_{L^2(0,T)}} = \left(\frac{2}{T}\sum_{k=0}^{\infty}\overline{w}_k^2\alpha_k^{-2}\right)^{1/2}.$$

Thus, we have that

$$\|w\|_{[H^1_{,0}(0,T)]'} = \sqrt{\frac{2}{T}\left(\sum_{k=0}^{\infty}\overline{w}_k^2\alpha_k^{-2}\right)}.$$

By (2.6), we get $\overline{w}_k = \alpha_k v_k$ and compute,

$$\|w\|^2_{[H^1_{,0}(0,T)]'} = \frac{2}{T}\sum_{k=0}^{\infty}\alpha_k^{-2}\overline{w}_k^2 = \frac{T}{2}\sum_{k=0}^{\infty}|v_k|^2 = \|v\|^2_{L^2(0,T)}.$$

■

Remark 2.1 *The results of Lemma 2.1 also hold when considering the lateral boundary* Σ *as the domain, instead of* $(0,T)$.

Using the compact form (2.4) for $w = (w_0, w_L)$, we define the operators V_D and V_{OD} as

$$\mathsf{V}_D(t) := \begin{pmatrix} \mathsf{V}_{00} & 0 \\ 0 & \mathsf{V}_{LL} \end{pmatrix} \qquad \text{and} \qquad \mathsf{V}_{OD}(t) := \begin{pmatrix} 0 & \mathsf{V}_{0L} \\ \mathsf{V}_{L0} & 0 \end{pmatrix}. \tag{2.7}$$

Given Lemma 2.1 and definition (2.7), we can relate $\|\mathsf{V}_D\, w\|_{L^2(\Sigma)}$ to $\|w\|_{[H^1_{,0}(\Sigma)]'}$ as summarised in the following lemma:

Lemma 2.2 *Let $w \in [H^1_{,0}(\Sigma)]'$ and V_D be defined as in* (2.7)*. Then*

$$\|\mathsf{V}_D\, w\|^2_{L^2(\Sigma)} = \frac{1}{4}\, \|w\|^2_{[H^1_{,0}(\Sigma)]'}.$$

Proof. Let $w = \partial_t v$ for some $v \in L^2(\Sigma)$. By definition of V_D we have

$$\mathsf{V}_D\, w = \mathsf{V}_D\, \partial_t v = \frac{1}{2} v(t).$$

Hence, by Lemma 2.1

$$\|\mathsf{V}_D\, w\|^2_{L^2(\Sigma)} = \frac{1}{4}\|v\|^2_{L^2(\Sigma)} = \frac{1}{4}\|w\|^2_{[H^1_{,0}(\Sigma)]'}.$$

■

Now we have all the tools to return to our study of the inf-sup constant $\tilde{c}_S(n)$ and provide the main result of this section.

Theorem 2.3 *Let $w \in [H^1_{,0}(\Sigma)]'$ and let $n \in \mathbb{N}$ denote the number of time-slices. Then the operator $\mathsf{V} : [H^1_{,0}(\Sigma)]' \to L^2(\Sigma)$ is continuously bounded from below by constant $\widetilde{c}_S(n)$, i.e.,*

$$\|\mathsf{V}\, w\|^2_{L^2(\Sigma)} \geq \widetilde{c}_S(n)^2 \|w\|^2_{[H^1_{,0}(\Sigma)]'}, \quad \textit{where} \quad \widetilde{c}_S(n) := \sin\left(\frac{\pi}{2(2n+1)}\right).$$

Proof. We have

$$\begin{aligned} \|\mathsf{V}\, w\|_{L^2(\Sigma)} &= \langle (\mathsf{V}_D + \mathsf{V}_{OD})w, (\mathsf{V}_D + \mathsf{V}_{OD})w \rangle_{L^2(\Sigma)} \\ &= \|\mathsf{V}_D\, w\|^2_{L^2(\Sigma)} + 2\langle \mathsf{V}_D\, w, \mathsf{V}_{OD}\, w\rangle_{L^2(\Sigma)} + \|\mathsf{V}_{OD}\, w\|^2_{L^2(\Sigma)}. \end{aligned} \tag{2.8}$$

Let Σ_j, $j = 1, \dots, n$, denote the lateral trace, restricted to the j^{th} time-slice in time. By using the definitions (2.7), one can verify the following relation for all $w \in [H^1_{,0}(\Sigma)]'$:

$$\|\mathsf{V}_D\, w\|_{L^2(\Sigma_{j-1})} = \|\mathsf{V}_{OD}\, w\|_{L^2(\Sigma_j)}, \quad j = 2, \dots n. \tag{2.9}$$

Using (2.8) and (2.9), we get

$$\|\mathsf{V}\, w\|^2_{L^2(\Sigma)} \geq \sum_{i=1}^{n} \|\mathsf{V}_D\, w\|^2_{L^2(\Sigma_i)} - 2\sum_{i=2}^{n} \|\mathsf{V}_D\, w\|_{L^2(\Sigma_i)} \|\mathsf{V}_D\, w\|_{L^2(\Sigma_{i-1})} + \sum_{i=2}^{n} \|\mathsf{V}_D\, w\|^2_{L^2(\Sigma_{i-1})},$$

which can be represented in matrix form as

$$\|\mathsf{V}\,w\|^2_{L^2(\Sigma)} \geq \left\langle \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} \|\mathsf{V}_D\,w\|_{L^2(\Sigma_1)} \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_2)} \\ \vdots \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_{n-1})} \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_n)} \end{pmatrix}, \begin{pmatrix} \|\mathsf{V}_D\,w\|_{L^2(\Sigma_1)} \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_2)} \\ \vdots \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_{n-1})} \\ \|\mathsf{V}_D\,w\|_{L^2(\Sigma_n)} \end{pmatrix} \right\rangle. \tag{2.10}$$

The matrix in (2.10) corresponds to the one dimensional finite difference matrix with (zero) initial Dirichlet condition and terminal Neumann condition. The spectral properties of this matrix are henceforth well understood, and its smallest eigenvalue is given by $\lambda_{min} = 4\sin^2\left(\frac{\pi}{2(2n+1)}\right)$. Consequently, we obtain the bound

$$\begin{aligned} \|\mathsf{V}\,w\|^2_{L^2(\Sigma)} &\geq \lambda_{min} \sum_{i=1}^n \|\mathsf{V}_D\,w\|^2_{L^2(\Sigma_i)} \\ &= 4\sin^2\left(\frac{\pi}{2(2n+1)}\right) \|\mathsf{V}_D\,w\|^2_{L^2(\Sigma)} \geq \sin^2\left(\frac{\pi}{2(2n+1)}\right) \|w\|^2_{[H^1_{,0}(\Sigma)]'}, \end{aligned}$$

where we applied Lemma 2.2 in the last step. ■

As a direct consequence of Theorem 2.3, the inf-sup stability condition is given by

$$\widetilde{c}_S(n)\,\|w\|_{[H^1_{,0}(\Sigma)]'} \leq \sup_{0\neq q\in L^2(\Sigma)} \frac{\langle \mathsf{V}\,w, q\rangle_{L^2(\Sigma)}}{\|q\|_{L^2(\Sigma)}} \quad \text{for all } w \in H^1_{,0}(\Sigma), \tag{2.11}$$

with the constant $\widetilde{c}_S(n)$ as given in Theorem 2.3. In order to find its solution $w \in [H^1_{,0}(\Sigma)]'$, we consider the minimization of the functional

$$\mathcal{J}(v) := \frac{1}{2}\,\|\mathsf{V}\,v - g\|^2_{L^2(\Sigma)},$$

over all $v \in [H^1_{,0}(\Sigma)]$, whose minimizer $w \in [H^1_{,0}(\Sigma)]'$ is determined as unique solution of the gradient equation

$$\mathsf{V}^*\,\mathsf{V}\,w = \mathsf{V}^*\,g, \tag{2.12}$$

where $\mathsf{V}^* : L^2(\Sigma) \to H^1_{,0}(\Sigma)$ is the adjoint of $\mathsf{V} : [H^1_{,0}(\Sigma)]' \to L^2(\Sigma)$. When introducing the adjoint $p := g - \mathsf{V}\,w$, we end up with a mixed variational formulation to find $p \in L^2(\Sigma)$ and $w \in [H^1_{,0}(\Sigma)]'$ such that

$$\langle p, q\rangle_{L^2(\Sigma)} + \langle \mathsf{V}\,w, q\rangle_{L^2(\Sigma)} = \langle g, q\rangle_{L^2(\Sigma)}, \quad \langle p, \mathsf{V}\,v\rangle_{L^2(\Sigma)} = 0 \tag{2.13}$$

is satisfied for all $q \in L^2(\Sigma)$, and for all $v \in [H^1_{,0}(\Sigma)]'$. In fact, the gradient equation (2.12) is the Schur complement system of the mixed formulation (2.13). To establish unique solvability of (2.12), and therefore of (2.13), we consider the Schur complement operator

$S := \mathsf{V}^* \mathsf{V} : [H^1_{,0}(\Sigma)]' \to H^1_{,0}(\Sigma)$. For $w \in [H^1_{,0}(\Sigma)]'$ define $p_w = \mathsf{V}\, w \in L^2(\Sigma)$ as unique solution of the variational formulation

$$\langle p_w, q \rangle_{L^2(\Sigma)} = \langle \mathsf{V}\, w, q \rangle_{L^2(\Sigma)} \quad \text{for all } q \in L^2(\Sigma).$$

With this we obtain

$$\langle Sw, w \rangle_\Sigma = \langle \mathsf{V}^* \mathsf{V}\, w, w \rangle_\Sigma = \langle p_w, \mathsf{V}\, w \rangle_{L^2(\Sigma)} = \langle p_w, p_w \rangle_{L^2(\Sigma)} = \|p_w\|^2_{L^2(\Sigma)},$$

and the inf-sup stability condition (2.11) gives

$$\widetilde{c}_S(n)\, \|w\|_{[H^1_{,0}(\Sigma)]'} \leq \sup_{0 \neq q \in L^2(\Sigma)} \frac{\langle \mathsf{V}\, w, q \rangle_{L^2(\Sigma)}}{\|q\|_{L^2(\Sigma)}} = \sup_{0 \neq q \in L^2(\Sigma)} \frac{\langle p_w, q \rangle_{L^2(\Sigma)}}{\|q\|_{L^2(\Sigma)}} \leq \|p_w\|_{L^2(\Sigma)}. \tag{2.14}$$

Hence, $S : [H^1_{,0}(\Sigma)]' \to H^1_{,0}(\Sigma)$ is elliptic satisfying

$$\langle Sw, w \rangle_\Sigma \geq [\widetilde{c}_S(n)]^2\, \|w\|^2_{[H^1_{,0}(\Sigma)]'} \quad \text{for all } w \in [H^1_{,0}(\Sigma)]'.$$

From this, we conclude unique solvability of the gradient equation (2.12) and of the mixed variational formulation (2.13).

3 A Mixed Boundary Element Method

Let

$$\mathcal{S}^0_H(\Sigma) := \mathcal{S}^0_{H,0}(0,T) \times \mathcal{S}^0_{H,L}(0,T) = \operatorname{span}\{\phi_\ell\}_{\ell=1}^{N_{H,0}} \times \operatorname{span}\{\phi_\ell\}_{\ell=N_{H,0}+1}^{N_H}$$

and

$$\mathcal{S}^0_h(\Sigma) := \mathcal{S}^0_{h,0}(0,T) \times \mathcal{S}^0_{h,L}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^{N_{h,0}} \times \operatorname{span}\{\varphi_k\}_{k=N_{h,0}+1}^{N_h}$$

be two conforming nested boundary element spaces, i.e.,

$$\mathcal{S}^0_H(\Sigma) \subset \mathcal{S}^0_h(\Sigma) \subset L^2(\Sigma) \subset [H^1_{,0}(\Sigma)]',$$

spanned by piecewise constant basis functions ϕ_ℓ and φ_k, which are defined with respect to some nested decomposition of Σ into boundary elements τ^H_ℓ and τ^h_k with local mesh sizes H_ℓ and h_k, respectively. For $\tau^h_k \subset \tau^H_\ell$ we assume $H_\ell = m h_k$ for some $m \in \mathbb{N}$. So we may define a coarse grid mesh Σ_H first, and any element τ^H_ℓ of Σ_H is decomposed into m equal sized elements τ^h_k of the fine mesh Σ_h.

The Galerkin formulation of (2.13) is to find $p_h \in \mathcal{S}^0_h(\Sigma)$ and $w_H \in \mathcal{S}^0_H(\Sigma)$ such that

$$\langle p_h, q_h \rangle_{L^2(\Sigma)} + \langle \mathsf{V}\, w_H, q_h \rangle_{L^2(\Sigma)} = \langle g, q_h \rangle_{L^2(\Sigma)}, \quad \langle p_h, \mathsf{V}\, v_H \rangle_{L^2(\Sigma)} = 0 \tag{3.1}$$

is satisfied for all $q_h \in \mathcal{S}^0_h(\Sigma)$ and for all $v_H \in \mathcal{S}^0_H(\Sigma)$. This is equivalent to a linear system of algebraic equations,

$$\begin{pmatrix} D_h & V_h \\ V_h^\top & \end{pmatrix} \begin{pmatrix} \underline{p} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{0} \end{pmatrix}, \tag{3.2}$$

where for $j, k = 1, \ldots, N_h$ and for $\ell = 1, \ldots, N_H$ we have

$$D_h[j,k] = \int_\Sigma \varphi_k(x)\varphi_j(x)\,ds_x, \quad V_h[j,\ell] = \int_\Sigma (\mathsf{V}\,\phi_\ell)(x)\varphi_j(x)\,ds_x, \quad g_j = \int_\Sigma g(x)\varphi_j(x)\,ds_x.$$

Since the diagonal matrix D_h is invertible, we can eliminate $\underline{p} = D_h^{-1}[\underline{g} - V_h\underline{w}]$ to end up with the Schur complement system

$$S_h\underline{w} := V_h^\top D_h^{-1} V_h \underline{w} = V_h^\top D_h^{-1}\underline{g}, \tag{3.3}$$

which is nothing more than a Galerkin approximation of the gradient equation (2.12). By construction, the Schur complement matrix $S_h = V_h^\top D_h^{-1} V_h$ is symmetric and at least positive semi-definite. We will prove that the matrix S_h is actually positive definite and hence that (3.3) and therefore (3.2) admits a unique solution.

Theorem 3.1 *Assume $T = L$, i.e., $n = 1$. Let Σ_H be a mesh of Σ, which may be non-uniform and adaptive. Let Σ_h be the fine mesh where each element τ_ℓ^H of Σ_H is decomposed into m equal sized elements τ_k^h. Then the Schur complement matrix S_h^L is positive definite for all $m > 2$, i.e.,*

$$(S_h^L\underline{w}, \underline{w}) \ge \left(\frac{1}{2} - \frac{1}{m}\right)^2 \|w_H\|^2_{[H^1_{,0}(\Sigma)]'} \quad \text{for all } \underline{w} \in \mathbb{R}^{N_H} \leftrightarrow w_H \in \mathcal{S}_H^0(\Sigma).$$

Here S_h^L denotes the Schur complement matrix for a single time slice.

Proof. For $w_H \in \mathcal{S}_H^0(\Sigma) \subset [H^1_{,0}(\Sigma)]'$ the application of the Schur complement operator S reads $Sw_H = \mathsf{V}^*\,\mathsf{V}\,w_H = \mathsf{V}^*\,p_{w_H}$ where $p_{w_H} \in L^2(\Sigma)$ is the unique solution of the variational formulation

$$\langle p_{w_H}, q\rangle_{L^2(\Sigma)} = \langle \mathsf{V}\,w_H, q\rangle_{L^2(\Sigma)} \quad \text{for all } q \in L^2(\Sigma).$$

Now we consider the related Galerkin approximation $p_{w_H,h} \in \mathcal{S}_h^0(\Sigma)$ as unique solution of the variational formulation

$$\langle p_{w_H,h}, q_h\rangle_{L^2(\Sigma)} = \langle p_{w_H}, q_h\rangle_{L^2(\Sigma)} = \langle \mathsf{V}\,w_H, q_h\rangle_{L^2(\Sigma)} \quad \text{for all } q_h \in \mathcal{S}_h^0(\Sigma), \tag{3.4}$$

i.e., we have to solve the linear system

$$D_h\underline{p} = V_h\underline{w}\,.$$

Instead of $Sw_H = \mathsf{V}^*\,p_{w_H}$ we now define the approximation $\widetilde{S}w_H := \mathsf{V}^*\,p_{w_H,h}$ for which we derive the matrix representation

$$S_h^L = V_h^\top D_h^{-1} V_h\,.$$

Hence, we can write

$$\begin{aligned} (S_h^L\underline{w}, \underline{w}) &= \langle \widetilde{S}w_H, w_H\rangle_\Sigma = \langle \mathsf{V}^*\,p_{w_H,h}, w_H\rangle_\Sigma \\ &= \langle p_{w_H,h}, \mathsf{V}\,w_H\rangle_{L^2(\Sigma)} = \langle p_{w_H,h}, p_{w_H,h}\rangle_{L^2(\Sigma)} = \|p_{w_H,h}\|^2_{L^2(\Sigma)}. \end{aligned} \tag{3.5}$$

From the triangle inequality

$$\|p_{w_H}\|_{L^2(\Sigma)} = \|p_{w_H} - p_{w_H,h} + p_{w_H,h}\|_{L^2(\Sigma)} \leq \|p_{w_H} - p_{w_H,h}\|_{L^2(\Sigma)} + \|p_{w_H,h}\|_{L^2(\Sigma)}$$

we get, by using (2.14),

$$\begin{aligned} \|p_{w_H,h}\|_{L^2(\Sigma)} &\geq \|p_{w_H}\|_{L^2(\Sigma)} - \|p_{w_H} - p_{w_H,h}\|_{L^2(\Sigma)} \\ &\geq \widetilde{c}_S(n)\, \|w_H\|_{[H^1_{,0}(\Sigma)]'} - \|p_{w_H} - p_{w_H,h}\|_{L^2(\Sigma)}, \end{aligned} \tag{3.6}$$

and it remains to estimate the approximation error of $\|p_{w_H} - p_{w_H,h}\|_{L^2(\Sigma)}$.

In the case $T = L$, the application of the wave single layer boundary integral operator V is decoupled, i.e.,

$$p_{w_H,0}(t) = \frac{1}{2}\int_0^t w_{H,0}(s)\, ds, \quad p_{w_H,L}(t) = \frac{1}{2}\int_0^t w_{H,L}(s)\, ds, \quad t \in (0,T).$$

For the coefficients of the piecewise constant approximation at $x = 0$,

$$p_{w_H,0,h}(t) = \sum_{k=1}^{N_{h,0}} p_{0,k}\varphi_k(t),$$

we find from (3.4) that

$$p_{0,k} = \frac{1}{h_k}\int_{t_{k-1}}^{t_k} p_{w_H,0}(s)\, ds \quad \text{for } k = 1,\ldots,N_{h,0}.$$

By using standard arguments, see, e.g., [38], and $p'_{w_H,0}(t) = \frac{1}{2}w_{H,0}(t)$, we obtain the error estimate

$$\int_{t_{k-1}}^{t_k} [p_{w_H,0}(t) - p_{w_H,0,h}(t)]^2\, dt \leq \frac{1}{3}\, h_k^2 \int_{t_{k-1}}^{t_k} [p'_{w_H,0}(t)]^2 dt = \frac{1}{12}\, h_k^2 \int_{t_{k-1}}^{t_k} [w_{H,0}(t)]^2 dt\,,$$

and summing up this gives

$$\int_0^T [p_{w_H,0}(t) - p_{w_H,0,h}(t)]^2\, dt \leq \frac{1}{12}\sum_{k=1}^{N_{h,0}} h_k^2 \int_{t_{k-1}}^{t_k} [w_{H,0}(t)]^2 dt\,.$$

When inserting

$$w_{H,0}(t) = \sum_{\ell=1}^{N_{H,0}} w_\ell \phi_\ell(t),$$

assembling all fine grid contributions from the elements $\tau_k^h \subset \tau_\ell^H$, and using $H_\ell = m h_k$, we further conclude

$$\int_0^T [p_{w_H,0}(t) - p_{w_H,0,h}(t)]^2\, dt \leq \frac{1}{12}\,\frac{1}{m^2}\sum_{\ell=1}^{N_{H,0}} H_\ell^3 w_\ell^2\,.$$

By doing the same computations at $x = L$, and summing up both contributions, this gives

$$\|p_{w_H} - p_{w_H,h}\|^2_{L^2(\Sigma)} \le \frac{1}{12}\,\frac{1}{m^2}\,\sum_{\ell=1}^{N_H} H_\ell^3 w_\ell^2\,.$$

We now consider a piecewise quadratic function

$$v_H(t) = \sum_{\ell=1}^{N_H} w_\ell \psi_\ell(t),$$

where the bubble function ψ_ℓ in the boundary element τ_ℓ^H is defined by its form function

$$\psi(s) = s(H-s) \quad \text{for } s \in (0,H).$$

For this we compute

$$\int_{\tau_\ell^H} \psi_\ell(t)\,dt = \frac{1}{6}\,H_\ell^3, \qquad \int_{\tau_\ell^H} [\psi_\ell'(t)]^2\,dt = \frac{1}{3}\,H_\ell^3.$$

Thus, we have

$$\langle w_H, v_H\rangle_{L^2(\Sigma)} = \frac{1}{6}\sum_{\ell=1}^{N_H} w_\ell^2\,H_\ell^3, \quad \text{and} \quad \|v_H'\|^2_{L^2(\Sigma)} = \frac{1}{3}\sum_{\ell=1}^{N_H} w_\ell^2\,H_\ell^3.$$

With this, we finally obtain

$$\begin{aligned}
\|p_{w_H} - p_{w_H,h}\|_{L^2(\Sigma)} \;&\le\; \frac{1}{m}\sqrt{\frac{1}{12}\sum_{\ell=1}^{N_H} H_\ell^3 w_\ell^2} = \frac{1}{m}\,\frac{\frac{1}{12}\sum\limits_{\ell=1}^{N_H} H_\ell^3 w_\ell^2}{\sqrt{\frac{1}{12}\sum\limits_{\ell=1}^{N_H} H_\ell^3 w_\ell^2}} = \frac{1}{m}\,\frac{\frac{1}{2}\,\langle w_H, v_H\rangle_{L^2(\Sigma)}}{\frac{1}{2}\,\|v_H'\|_{L^2(\Sigma)}} \\
&\le\; \frac{1}{m}\sup_{0\ne v\in H^1_{,0}(\Sigma)} \frac{\langle w_H, v\rangle_{L^2(\Sigma)}}{\|v'\|_{L^2(\Sigma)}} = \frac{1}{m}\,\|w_H\|_{[H^1_{,0}(\Sigma)]'}\,.
\end{aligned}$$

Hence, we can write (3.6) as

$$\|p_{w_H,h}\|_{L^2(\Sigma)} \;\ge\; \left(\widetilde{c}_S(n) - \frac{1}{m}\right)\|w_H\|_{[H^1_{,0}(\Sigma)]'}, \tag{3.7}$$

which is positive for

$$\frac{1}{m} < \widetilde{c}_S(n) = \sin\left(\frac{\pi}{2(2n+1)}\right) \overset{n=1}{=} \sin\frac{\pi}{6} = \frac{1}{2}, \quad \text{i.e., for } m > 2.$$

The assertion now follows from (3.5). ■

Theorem 3.1 ensures unique solvability of the linear system (3.3), and therefore of the mixed variational formulation (3.1) in the particular case $T = L$. But this result can be generalized as follows.

Theorem 3.2 *Let $T = nL$ for some $n \in \mathbb{N}$ that induces time slices $((j-1)L, jL)$ for $j = 1, \dots, n$. Let Σ_H be a uniform mesh of Σ. Let Σ_h be the fine mesh where each element τ_ℓ^H of Σ_H is decomposed into m equal sized elements τ_k^h. We assume that jL, $j = 0, \dots, n$, are nodes of the mesh Σ_H at $x = 0$ and at $x = L$, respectively. Then,*

$$(S_h^{nL}\underline{w}, \underline{w}) \geq 4\sin^2\left(\frac{\pi}{2(2n+1)}\right)\left(\frac{1}{2} - \frac{1}{m}\right)^2 \|w_H\|^2_{[H^1_{,0}(\Sigma)]'} \quad \textit{for all } \underline{w} \in \mathbb{R}^{N_H} \leftrightarrow w_H \in \mathcal{S}_H^0(\Sigma).$$

Here, S_h^{nL} denotes the Schur complement matrix for n time slices.

Proof. Let Q_h denote the L^2 projection with respect to the fine mesh Σ_h, defined as

$$\langle Q_h u, v_h \rangle_{L^2(\Sigma)} = \langle u, v_h \rangle_{L^2(\Sigma)}, \quad \text{for all } v_h \in \mathcal{S}_h^0(\Sigma), \tag{3.8}$$

when $u \in L^2(\Sigma)$ is given. In case of a uniform refinement, we retain an equality analogous to (2.9), i.e.,

$$\|Q_h \mathsf{V}_D\, w\|_{L^2(\Sigma_{j-1})} = \|Q_h \mathsf{V}_{OD}\, w\|_{L^2(\Sigma_j)}, \quad j = 2, \dots, n. \tag{3.9}$$

Hence, we get

$$(S_h^{nL}\underline{w}, \underline{w}) = (V_h^T D_h^{-1} V_h, \underline{w}, \underline{w}) = \left(\sup_{0 \neq q_h \in \mathcal{S}_h^0(\Sigma)} \frac{\langle V w_H, q_h \rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \right)^2 = \|Q_h V w_H\|^2_{L^2(\Sigma)}. \tag{3.10}$$

Given (3.9), and following the lines of the proof of Theorem 2.3, we get

$$\|Q_h \mathsf{V}\, w_H\|^2_{L^2(\Sigma)} \geq 4\,[\widetilde{c}_S(n)]^2\, \|Q_h \mathsf{V}_D\, w_H\|^2_{L^2(\Sigma)}. \tag{3.11}$$

Next, we make the observation that $\|Q_h \mathsf{V}_D\, w_H\|^2_{L^2(\Sigma)}$ is equivalent to considering the discretized operator $Q_h \mathsf{V}$ on one time-slice, which suggests that we can apply Theorem 3.1. The result now follows from this observation, (3.10) and (3.11),

$$\begin{aligned}(S_h^{nL}\underline{w}, \underline{w}) = \|Q_h \mathsf{V}\, w_H\|^2_{L^2(\Sigma)} &\geq\ 4\,[\widetilde{c}_S(n)]^2\, \|Q_h \mathsf{V}_D\, w_H\|^2_{L^2(\Sigma)} \\ &\geq\ 4\,[\widetilde{c}_S(n)]^2 \left(\frac{1}{2} - \frac{1}{m}\right)^2 \|w_H\|^2_{[H^1_{,0}(\Sigma)]'}. \end{aligned} \tag{3.12}$$

■

Remark 3.1 *The relation* (3.9) *does not need to hold on non-uniform refinements. However, in absence of rounding errors, adaptively refined meshes should retain the property* (3.9) *if one starts with a uniform initial mesh, and, as a consequence, the mixed boundary element method remains discrete inf-sup stable. We will see later, in the numerical experiments, how numerical errors do break this condition and how to remedy it.*

Remark 3.2 *Note, that in the limit case $h \to 0$, we have $m \to \infty$ and the bound of Theorem 3.2 becomes exactly the bound of Theorem 2.3 in the continuous case.*

It remains to provide an a priori error estimate for the unique solution of the mixed variational formulation (3.1). Although this follows as in the elliptic case for the Laplace equation [39], here we present the main steps:

Lemma 3.3 *Let the assumption of Theorem 3.2 hold. Then, for the unique solutions* $w \in [H^1_{,0}(\Sigma)]'$ *of* (2.4) *and* $w_H \in \mathcal{S}^0_H(\Sigma)$ *of* (3.1), *there holds*

$$\|w - w_H\|_{[H^1_{,0}(\Sigma)]'} \leq \left(1 + \frac{2mc_2^{\mathsf{V}}}{\sin\left(\frac{\pi}{2(2n+1)}\right)(m-2)}\right) \inf_{v_H \in \mathcal{S}^0_H(\Sigma)} \|w - v_H\|_{[H^1_{,0}(\Sigma)]'}.$$

Proof. When combining (3.10) and (3.12) we immediately obtain the discrete inf-sup stability condition

$$2\widetilde{c}_S(n)\left(\frac{1}{2} - \frac{1}{m}\right) \|v_H\|_{[H^1_{,0}(\Sigma)]'} \leq \sup_{0 \neq q_h \in \mathcal{S}^0_h(\Sigma)} \frac{\langle \mathsf{V}\, v_H, q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \quad \text{for all } v_H \in \mathcal{S}^0_H(\Sigma).$$

Then, for the solution w_H of (3.1) and for arbitrary $v_H \in \mathcal{S}^0_H(\Sigma)$, we obtain, by using the triangle inequality,

$$\|w - w_H\|_{[H^1_{,0}(\Sigma)]'} \leq \|w - v_H\|_{[H^1_{,0}(\Sigma)]'} + \|v_H - w_H\|_{[H^1_{,0}(\Sigma)]'}.$$

Now, for the second term, we can use the discrete inf-sup stability condition and (3.1) for $g = \mathsf{V}\, w$ to estimate

$$\begin{aligned}
2\widetilde{c}_S(n)\left(\frac{1}{2} - \frac{1}{m}\right) \|v_H - w_H\|_{[H^1_{,0}(\Sigma)]'} &\leq \sup_{0 \neq q_h \in \mathcal{S}^0_h(\Sigma)} \frac{\langle \mathsf{V}(v_H - w_H), q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \\
&= \sup_{0 \neq q_h \in \mathcal{S}^0_h(\Sigma)} \frac{\langle \mathsf{V}\, v_H - (g - p_h), q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} = \sup_{0 \neq q_h \in \mathcal{S}^0_h(\Sigma)} \frac{\langle \mathsf{V}(v_H - w) + p_h, q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \\
&\leq c_2^{\mathsf{V}} \|v_H - w\|_{[H^1_{,0}(\Sigma)]'} + \|p_h\|_{L^2(\Sigma)}.
\end{aligned}$$

Thus, it remains to estimate $\|p_h\|_{L^2(\Sigma)}$. Therefore, we consider (3.1) with $q_h = p_h$, to get

$$\begin{aligned}
\|p_h\|^2_{L^2(\Sigma)} &= \langle p_h, p_h\rangle_{L^2(\Sigma)} = \langle g - V w_H, p_h\rangle_{L^2(\Sigma)} = \langle \mathsf{V}(w - w_H), p_h\rangle_{L^2(\Sigma)} \\
&= \langle \mathsf{V}(w - v_H), p_h\rangle_{L^2(\Sigma)} + \langle \mathsf{V}(v_H - w_H), p_h\rangle_{L^2(\Sigma)} \\
&= \langle \mathsf{V}(w - v_H), p_h\rangle_{L^2(\Sigma)} \leq c_2^{\mathsf{V}} \|w - v_H\|_{[H^1_{,0}(\Sigma)]'} \|p_h\|_{L^2(\Sigma)}.
\end{aligned}$$

Now, combining the estimates and taking the infimum over all $v_H \in \mathcal{S}^0_H(\Sigma)$ we obtain

$$\begin{aligned}
\|w - w_H\|_{[H^1_{,0}(\Sigma)]'} &\leq \left(1 + \frac{c_2^{\mathsf{V}}}{\widetilde{c}_S(n)\left(\frac{1}{2} - \frac{1}{m}\right)}\right) \inf_{v_H \in \mathcal{S}^0_H(\Sigma)} \|w - v_H\|_{[H^1_{,0}(\Sigma)]'} \\
&= \left(1 + \frac{2mc_2^{\mathsf{V}}}{\widetilde{c}_S(n)(m-2)}\right) \inf_{v_H \in \mathcal{S}^0_H(\Sigma)} \|w - v_H\|_{[H^1_{,0}(\Sigma)]'}.
\end{aligned}$$

∎

Let $0 < h < \underline{H} < H$ be given such that the inclusion $\mathcal{S}_H^0(\Sigma) \subset \mathcal{S}_{\underline{H}}^0(\Sigma) \subset \mathcal{S}_h^0(\Sigma)$ holds, and assume that there exists $\widetilde{c}_S(n) > 0$ such that

$$\widetilde{c}_S(n)\,\|v_{\underline{H}}\|_{[H^1_{,0}(\Sigma)]'} \le \sup_{0 \ne q_h \in \mathcal{S}_h^0(\Sigma)} \frac{\langle \mathsf{V}\, v_{\underline{H}}, q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \quad \text{for all } v_{\underline{H}} \in \mathcal{S}_{\underline{H}}^0(\Sigma) \tag{3.13}$$

is satisfied. Then (3.1) admits a unique solution $(\overline{p}_h, w_{\underline{H}}) \in \mathcal{S}_h^0(\Sigma) \times \mathcal{S}_{\underline{H}}^0(\Sigma)$. Note that, due to the inclusion $\mathcal{S}_H^0(\Sigma) \subset \mathcal{S}_{\underline{H}}^0(\Sigma)$, we have that

$$\widetilde{c}_S(n)\,\|v_H\|_{[H^1_{,0}(\Sigma)]'} \le \sup_{0 \ne q_h \in \mathcal{S}_h^0(\Sigma)} \frac{\langle \mathsf{V}\, v_H, q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}}$$

holds true for all $v_H \in \mathcal{S}_H^0(\Sigma)$, and thus (3.1) also admits a unique solution $(p_h, w_H) \in \mathcal{S}_h^0(\Sigma) \times \mathcal{S}_H^0(\Sigma)$. Under a saturation assumption, we can now show that $p_h \in \mathcal{S}_h^0(\Sigma)$ is an error estimator.

Lemma 3.4 *Let $w \in [H^1_{,0}(\Sigma)]'$ be the unique solution of (2.4). Further, let $(p_h, w_{\underline{H}}) \in \mathcal{S}_h^0(\Sigma) \times \mathcal{S}_{\underline{H}}^0(\Sigma)$ and $(p_h, w_H) \in \mathcal{S}_h^0(\Sigma) \times \mathcal{S}_H^0(\Sigma)$ be the unique solution of (3.1). If the saturation assumption*

$$\|w - w_{\underline{H}}\|_{[H^1_{,0}(\Sigma)]'} \le \delta\,\|w - w_H\|_{[H^1_{,0}(\Sigma)]}, \quad \textit{for } \delta \in (0,1) \tag{3.14}$$

holds, then

$$\frac{1}{c_2^{\mathsf{V}}}\,\|p_h\|_{L^2(\Sigma)} \le \|w - w_H\|_{[H^1_{,0}(\Sigma)]} \le \frac{c_2^{\mathsf{V}}}{[\widetilde{c}_S(n)]^2}\,\frac{1}{1-\delta}\,\|p_h\|_{L^2(\Sigma)}.$$

Proof. First, using (3.1) and $\mathsf{V}\,w = g$ and the boundedness of V, we compute

$$\begin{aligned}
\|p_h\|^2_{L^2(\Sigma)} &= \langle p_h, p_h\rangle_{L^2(\Sigma)} = \langle g - \mathsf{V}\,w_H, p_h\rangle_{L^2(\Sigma)} \\
&= \langle \mathsf{V}(w - w_H), p_h\rangle_{L^2(\Sigma)} \le c_2^{\mathsf{V}}\,\|w - w_H\|_{[H^1_{,0}(\Sigma)]}\,\|p_h\|_{L^2(\Sigma)},
\end{aligned}$$

from which we conclude the first bound. To bound the error by $\|p_h\|_{L^2(\Sigma)}$, let us first estimate

$$\|w - w_H\|_{[H^1_{,0}(\Sigma)]'} \le \|w - w_{\underline{H}}\|_{[H^1_{,0}(\Sigma)]'} + \|w_{\underline{H}} - w_H\|_{[H^1_{,0}(\Sigma)]'}.$$

With the saturation (3.14), we conclude

$$\|w - w_H\|_{[H^1_{,0}(\Sigma)]'} \le \frac{1}{1-\delta}\,\|w_{\underline{H}} - w_H\|_{[H^1_{,0}(\Sigma)]'}.$$

Thus, it is sufficient to bound the discrete error. We note that $w_{\underline{H}} - w_H \in \mathcal{S}_{\underline{H}}^0(\Sigma)$ and we can use the discrete inf-sup stability (3.13), and together with (3.1) we get

$$\begin{aligned}
\widetilde{c}_S(n)\,\|w_{\underline{H}} - w_H\|_{[H^1_{,0}(\Sigma)]'} &\le \sup_{0 \ne q_h \in \mathcal{S}_h^0(\Sigma)} \frac{\langle \mathsf{V}(w_{\underline{H}} - w_H), q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} \\
&= \sup_{0 \ne q_h \in \mathcal{S}_h^0(\Sigma)} \frac{\langle p_h - \overline{p}_h, q_h\rangle_{L^2(\Sigma)}}{\|q_h\|_{L^2(\Sigma)}} = \|p_h - \overline{p}_h\|_{L^2(\Sigma)}.
\end{aligned}$$

We can further bound this term by using again (3.1) as follows

$$\begin{aligned}\|p_h - \overline{p}_h\|^2_{L^2(\Sigma)} &= \langle p_h - \overline{p}_h, p_h - \overline{p}_h\rangle_{L^2(\Sigma)} = \langle \mathsf{V}(w_{\underline{H}} - w_H), p_h - \overline{p}_h\rangle_{L^2(\Sigma)} \\ &= \langle \mathsf{V}(w_{\underline{H}} - w_H, p_h\rangle_{L^2(\Sigma)} \le c_2^{\mathsf{V}} \|w_{\underline{H}} - w_H\|_{[H^1_{,0}(\Sigma)]'} \|p_h\|_{L^2(\Sigma)}.\end{aligned}$$

Altogether, we now obtain that

$$\|w_{\underline{H}} - w_H\|_{[H^1_{,0}(\Sigma)]'} \le \frac{c_2^{\mathsf{V}}}{[\widetilde{c}_S(n)]^2} \|p_h\|_{L^2(\Sigma)},$$

which concludes the proof. ■

Remark 3.3 *The solution* $(\overline{p}_h, w_{\underline{H}}) \in \mathcal{S}^0_h(\Sigma) \times \mathcal{S}^0_{\underline{H}}(\Sigma)$ *is only needed for the proof of the error estimator and does not need to be computed. In general, the idea is to have a stable method and then refine the mesh of the dual variable once more to get an error estimator. In particular, if the method is stable for the choices* $\underline{H} = H/2$ *then the choice* $h = H/4$ *gives an error estimator (or merely* $\underline{H} = H$ *is stable then* $h = H/2$ *gives an estimator). This is in some sense a generalization of the* $h - h/2$ *error estimator for elliptic equations. The behavior is also resembled by our numerical examples, as the choice* $m = 2$ *gives a method that is stable in the primal variable, thus we have* (3.13)*, but only when choosing* $m = 3$ *the dual variable provides an error estimator. Also note, that we chose* $S^0_{\underline{H}}(\Sigma)$ *just for ease of presentation. It is sufficient to have a discrete space* $\mathcal{S}^0_H(\Sigma) \subset X_{\underline{H}} \subset \mathcal{S}^0_h(\Sigma)$ *that fulfills the discrete inf-sup stability* (3.13) *for all* $v_{\underline{H}} \in X_{\underline{H}}$ *and for which the solution* $(\overline{p}_h, w_{\underline{H}}) \in \mathcal{S}^0_h(\Sigma) \times X_{\underline{H}}$ *fulfills the saturation assumption* (3.14)*.*

4 Numerical Experiments

4.1 Set up

We revisit two experiments, introduced in [41], and consider an additional experiment, all of which are posed on the same spatial domain $(0, 3)$, i.e., $L = 3$ and on the time interval $(0, 6)$, i.e., $T = 6$. We consider the following three different Dirichlet data

$$g_1(x,t) := \begin{cases} \frac{1}{2}(t-2)^3(-t)^3 & \text{for } 0 \le t \le 2 \text{ and } x = 0, \\ \frac{1}{2}(t-5)^3(3-t)^3 & \text{for } L \le t \le L+2 \text{ and } x = L, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_2(x,t) := \begin{cases} \frac{1}{2}|\sin(-\pi t)| & \text{for } 0 \le t \text{ and } x = 0, \\ \frac{1}{2}|\sin(\pi(L-t))| & \text{for } L \le t \text{ and } x = L, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_3(x,t) := \begin{cases} t^{1/4} & \text{for } 0 \le t \text{ and } x = 0, \\ (t-L)^{1/4} & \text{for } L \le t \text{ and } x = L, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We will be looking for solutions of the variational formulation for the mixed boundary element method as described in (3.1). For comparison, we will also consider two other variational formulations: the energetic BEM formulation, as described in [2, 41]; and the *modified Hilbert transform (MHT)* formulation from [41]. For clarity, let us restate the mixed boundary element methods for the specific numerical experiments, $i = 1, 2, 3$.

- *standard least squares formulation:* find $p_h \in \mathcal{S}_h^0(\Sigma) \subset L^2(\Sigma)$ and $w_H \in \mathcal{S}_H^0(\Sigma) \subset [H^1_{,0}(\Sigma)]'$ such that

$$\langle p_h, q_h \rangle_{L^2(\Sigma)} + \langle \mathsf{V}\, w_H, q_h \rangle_{L^2(\Sigma)} = \langle g_i, q_h \rangle_{L^2(\Sigma)}, \quad \langle p_h, \mathsf{V}\, v_H \rangle_{L^2(\Sigma)} = 0, \tag{4.2}$$

  is satisfied for all $q_h \in \mathcal{S}_h^0(\Sigma)$ and for all $v_H \in \mathcal{S}_H^0(\Sigma)$.

- *energetic BEM:* find $p_h \in \mathcal{S}_h^0(\Sigma) \subset L^2(\Sigma)$ and $z_H \in \mathcal{S}_H^0(\Sigma) \subset L^2(\Sigma)$ such that

$$\langle p_h, q_h \rangle_{L^2(\Sigma)} + \langle \partial_t \mathsf{V}\, z_H, q_h \rangle_{L^2(\Sigma)} = \langle \partial_t g_i, q_h \rangle_{L^2(\Sigma)}, \quad \langle p_h, \partial_t \mathsf{V}\, v_H \rangle_{L^2(\Sigma)} = 0, \tag{4.3}$$

  is satisfied for all $q_h \in \mathcal{S}_h^0(\Sigma)$ and for all $v_H \in \mathcal{S}_H^0(\Sigma)$.

- *modified Hilbert transform formulation:* Let $\mathcal{H}_T$ be the modified Hilbert transform defined in [43, eqn. (2.8)], we want to solve: find $p_h \in \mathcal{S}_h^0(\Sigma) \subset L^2(\Sigma)$ and $w_H \in \mathcal{S}_H^0(\Sigma) \subset [H^1_{,0}(\Sigma)]'$ such that

$$\langle p_h, q_h \rangle_{L^2(\Sigma)} + \langle \mathcal{H}_T \mathsf{V}\, w_H, q_h \rangle_{L^2(\Sigma)} = \langle \mathcal{H}_T g_i, q_h \rangle_{L^2(\Sigma)}, \quad \langle p_h, \mathcal{H}_T \mathsf{V}\, v_H \rangle_{L^2(\Sigma)} = 0, \tag{4.4}$$

  is satisfied for all $q_h \in \mathcal{S}_h^0(\Sigma)$ and for all $v_H \in \mathcal{S}_H^0(\Sigma)$.

Throughout this section, the numerical experiments are implemented in Python. For the solution of all linear systems built-in direct symmetric solvers are used[1].

4.2 Computation of the dual norm

In order to compute the error $\|w - w_H\|_{[H^1_{,0}(\Sigma)]'}$, we require the exact solution w, and a proper representation of the dual norm $\|w - w_H\|_{[H^1_{,0}(\Sigma)]'}$. In general, solutions to the indirect approach as considered in this paper do not yield densities that can be interpreted physically. However, in our specific setting, we are able to derive the exact density w by noting that for all functions g_i we have $g_i(0, t-L) = g_i(L, t)$ for $t \ge 0$. We aim to find the exact solution w_i, satisfying $\mathsf{V}\, w_i = g_i$. Let us define

$$\widetilde{w}_i(x,t) := \begin{cases} 2\partial_t\, g_i(0,t) & \text{for } x = 0, \\ 0 & \text{for } x = L. \end{cases}$$

[1] SciPy.Linalg.solve

Then, one can verify that

$$(\mathsf{V}\,\widetilde{w}_i)\,(x,t) = \begin{cases} g_i(0,t) & \text{for } x = 0, \\ g_i(0,t-L) = g_i(L,t) & \text{for } x = L, \end{cases}$$

i.e., $w_i = \widetilde{w}_i$.

In order to compute the dual norm $\|\cdot\|_{[H^1_{,0}(\Sigma)]'}$, note, that by the Riesz representation theorem, for $w \in [H^1_{,0}(0,T)]'$ there exists exactly one $\phi_w \in H^1_{,0}(0,T)$ such that

$$\langle \phi_w, v \rangle_{H^1_{,0}(0,T)} = \int_0^T \partial_t \phi_w(t) \partial_t v(t)\, dt = \langle w, v \rangle_{(0,T)} \quad \text{for all } v \in H^1_{,0}(0,T), \tag{4.5}$$

and that

$$\|\phi_w\|^2_{H^1_{,0}(0,T)} = \|\partial_t \phi_w\|^2_{L^2(0,T)} = \langle w, \phi_w \rangle_{(0,T)} = \|w\|^2_{[H^1_{,0}(0,T)]'}. \tag{4.6}$$

Note that (4.5) is the variational formulation of the boundary value problem

$$-\partial_{tt}\phi_w(t) = w(t) \text{ for } t \in (0,T), \quad \partial_t \phi_w(0) = 0, \quad \phi_w(T) = 0,$$

for which the solution is given, using Greens function, as

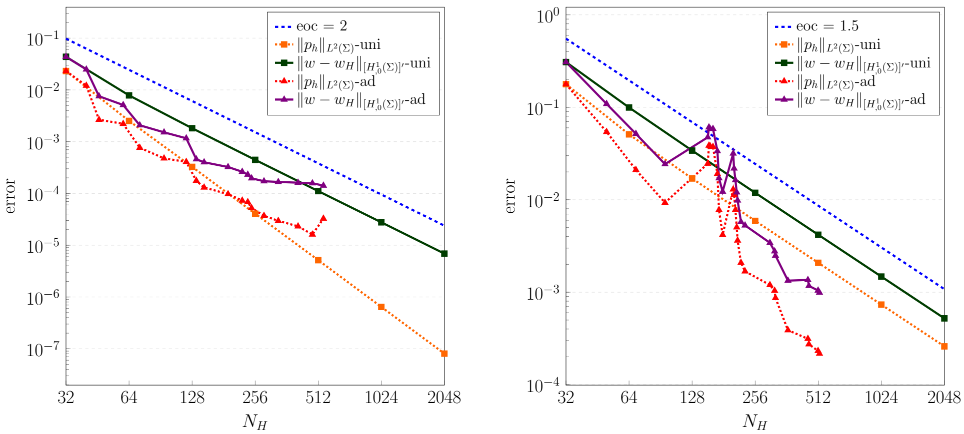
$$\phi_w(t) = \int_0^T G(t,s) w(s)\, ds, \quad \text{where } G(t,s) = \begin{cases} T-t, & s \in (0,t), \\ T-s, & s \in (t,T). \end{cases}$$

4.3 Numerical results

We start by checking numerically if the theoretical results in Theorems 3.1 and 3.2, and Lemma 3.3 are sharp in excluding $m = 2$ (namely, when each element of Σ_H is decomposed into two equally sized elements to obtain the fine mesh Σ_h). Fig. 1 shows the results for the standard formulation (4.2), given $m = 2$, while those of energetic BEM and MHT are displayed in Fig. 2 and 3, respectively. In all cases, the method converges. However, we see that they behave differently when considering adaptive refinements.

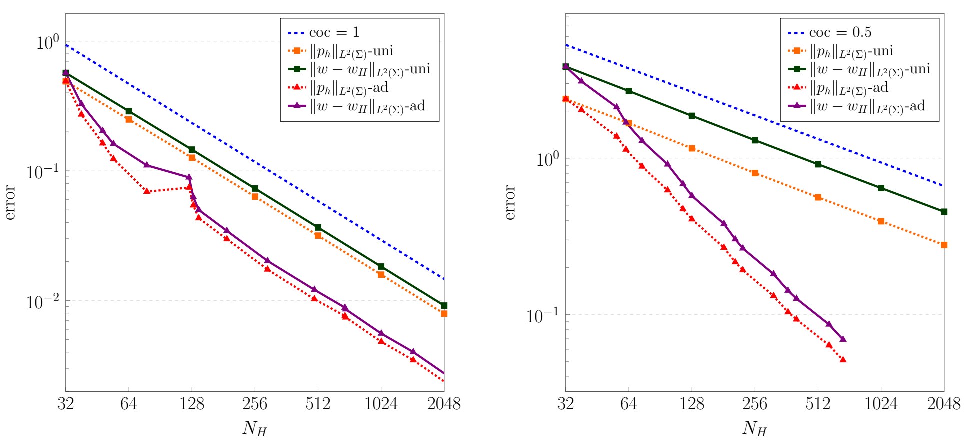
It is clear from Fig. 1 that $\|p_h\|_{L^2(\Sigma)}$ does not provide a reliable error estimator for the standard formulation (4.2) when $m = 2$. This fits the theory presented in Lemma 3.4, which states that, in order to show $\|p_h\|_{L^2(\Sigma)}$ is an error estimator, the saturation assumption (3.14) must hold. This only happens when $m > 2$ for the standard formulation (4.2), while it is already true for $m = 2$ for energetic BEM and MHT, since these formulations are discrete inf-sup stable for the case $h = H$.

In order to verify that $\|p_h\|_{L^2(\Sigma)}$ becomes an error estimator for the standard formulation (4.2) when $m > 2$, Fig. 4 depicts the results for the standard formulation (4.2) when we consider a fine mesh Σ_h such that each element of Σ_H is decomposed into three ($m = 3$) equally sized elements. Interestingly, in the uniform case the exact error $\|w - w_H\|_{[H^1_{,0}(\Sigma)]'}$ does not change significantly, *but* the convergence rate of $\|p_h\|_{L^2(\Sigma)}$ seems to correspond to the convergence rate of the exact error in this case. As shown in Fig. 5, further increasing the value of m does not seem to affect the convergence rate of the the error indicator $\|p_h\|_{L^2(\Sigma)}$ and the error $\|w-w_H\|_{[H^1_{,0}(\Sigma)]'}$ for uniform refinements of the standard formulation (4.2). Moreover, the difference in the error $\|w-w_H\|_{[H^1_{,0}(\Sigma)]'}$ seems to be negligible for different choices of m.



(a) g_1 (b) g_2

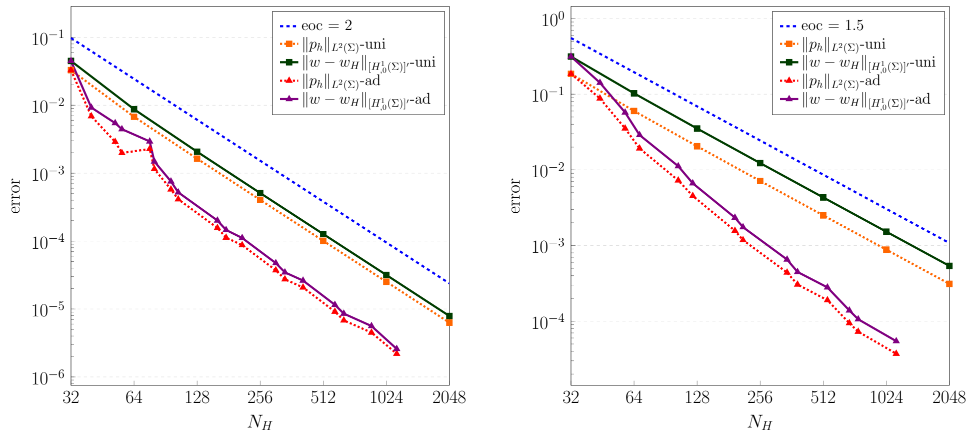
Figure 1: Comparison of errors and error indicators for uniform and adaptive refinement using the standard formulation (4.2), $m = 2$, and different Dirichlet data.



(a) g_1 (b) g_2

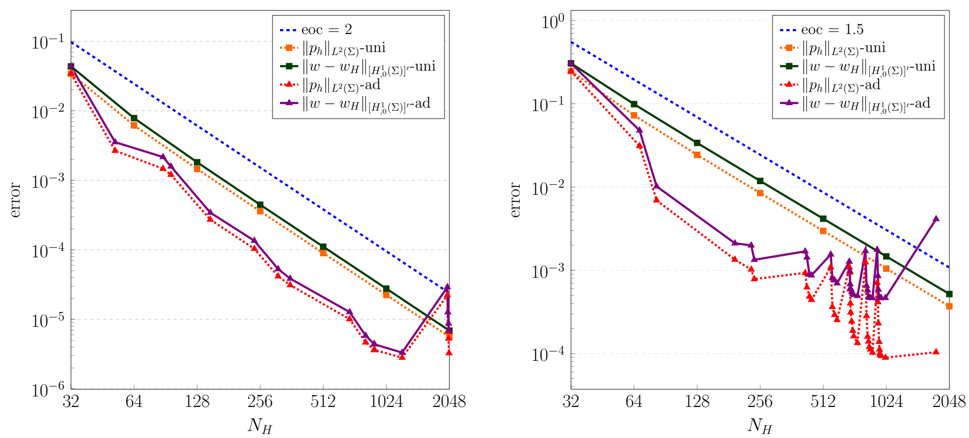
Figure 2: Comparison of errors and error indicators for uniform and adaptive refinement using the energetic formulation (4.3), $m = 2$, and different Dirichlet data.

Up until this point, we have not yet considered results related to g_3. For notational convenience, let us define w as the exact solution to the BIE for either the standard, energetic or MHT formulation. Then, given Dirichlet data g_3 as defined in (4.1), the density w will *not* be in $L^2(\Sigma)$. Hence, solutions of this kind do not fit our current



(a) g_1 (b) g_2

Figure 3: Comparison of errors and error indicators for uniform and adaptive refinement using the MHT formulation (4.4), $m = 2$, and different Dirichlet data.

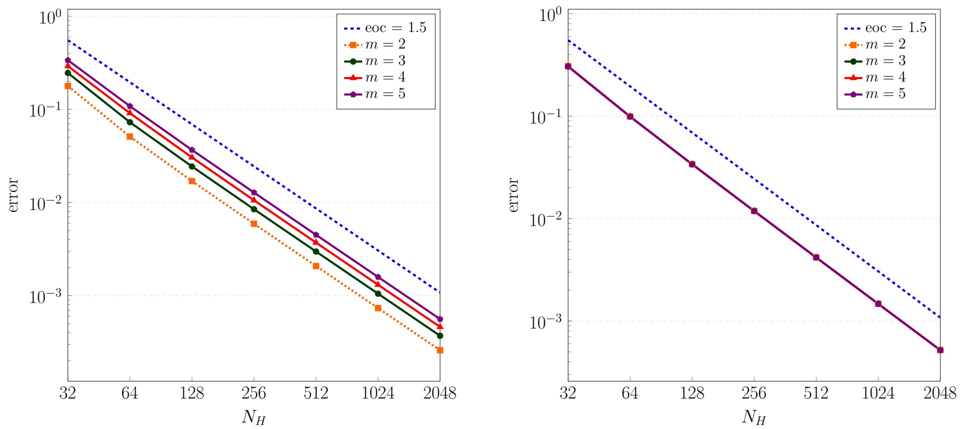


(a) g_1 (b) g_2

Figure 4: Comparison of errors and error indicators for uniform and adaptive refinement using the standard formulation (4.2), $m = 3$, and different Dirichlet data.

framework for the energetic formulation (4.3). With that being said, we still adhere to the same energetic formulation and discretization as already considered for g_1 and g_2. When it comes to energetic formulations with g_3, only the norm in which the error is measured is changed into $[H^1_{,0}(\Sigma)]'$ as opposed to the usual $L^2(\Sigma)$ norm. The results related to g_3, for different formulations, are presented in Fig. 6. There we see that the three different formulations converge with rate 0.75 on uniform meshes. For energetic BEM, $\|p_h\|_{L^2(\Sigma)}$ no longer serves as an error estimator for the error in the $[H^1_{,0}(\Sigma)]'$ norm. This explains why convergence of the adaptive routine for energetic BEM halts after some refinements.

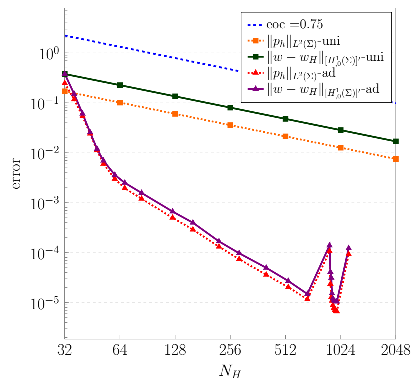
The next point in our 'numerical agenda' is to study the need for *uniform meshes* in



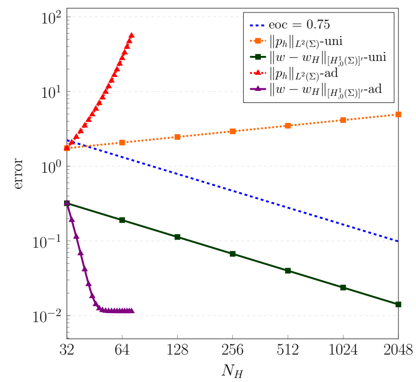
(a) $\|p_h\|_{L^2(\Sigma)}$ (b) $\|w - w_H\|_{[H^1_{,0}(\Sigma)]'}$

Figure 5: The error indicator and the error of the solution when using the standard formulation (4.2), uniform meshes, Dirichlet datum g_2, and different choices of m.

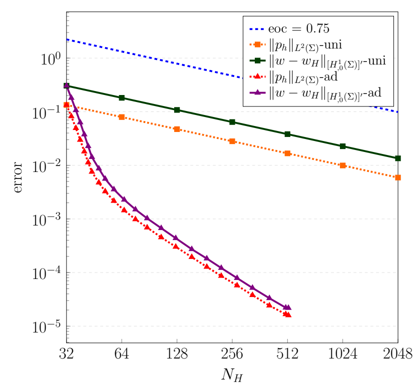
Theorem 3.2. We remind the reader that discrete inf-sup stability of V relies on the assumption that (3.9) is satisfied. As stated in Remark 3.1, the adaptive procedure should uphold the constraint on the mesh given by (3.9) when the initial mesh is uniform. In practice, however, it seems that at some point during the refinement routine the constraint on the mesh is no longer satisfied, resulting in a loss of discrete inf-sup stability. This explains the inability of the adaptively refined formulation to converge after a certain number of refinements, as visualised in Fig. 4. To circumvent this issue, we consider a *constrained* adaptive algorithm with the mesh condition (3.9) hard-coded into the implementation. Ensuring that the mesh on the boundary restricted to a time-slice corresponds to the mesh on the opposite boundary of the subsequent time-slice, provides a sufficient condition to satisfy (3.9). An example of a mesh satisfying this condition is given in Fig. 7. The constrained adaptive refinement routine is realised by enforcing this condition at each iteration. A comparison of the non-constrained and constrained adaptive refinement routines is given in Fig. 8. During the early stages of the refinement procedure, non-constrained adaptive refinement may result in a higher convergence rate compared to the constrained algorithm. This can be explained by the fact the constrained refinement scheme may unnecessarily refine parts where the Galerkin solution is zero. After several refinements, the constrained algorithm overcomes the issue encountered by its unconstrained counterpart.



(a) Standard formulation (4.2)



(b) Energetic BEM (4.3)



(c) MHT formulation (4.4)

Figure 6: Comparison of errors and error indicators for uniform and adaptive refinement schemes given Dirichlet datum g_3.

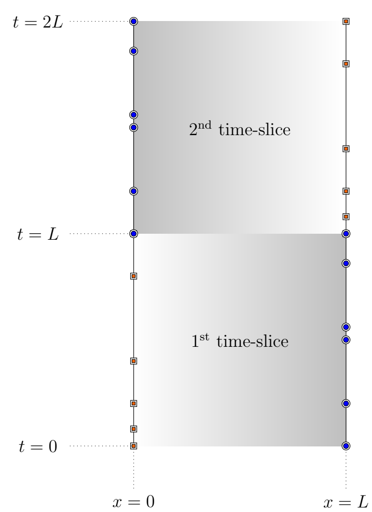
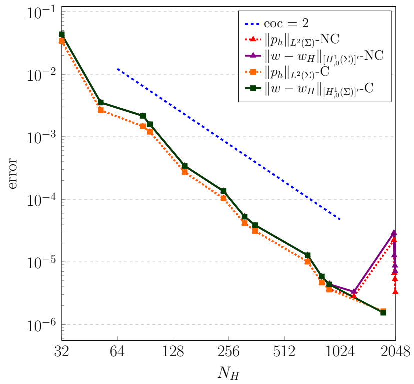
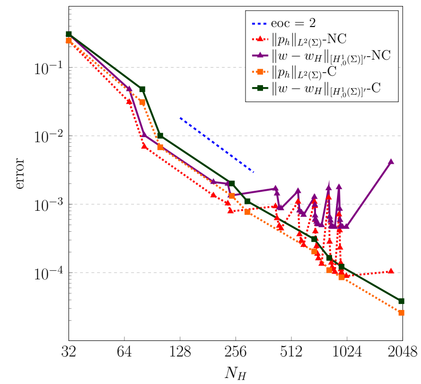


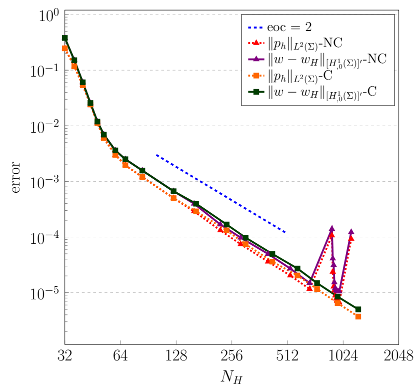
Figure 7: Example mesh satisfying (3.9). On each time-slice, the degrees of freedom (DoFs) of each boundary agree with the DoFs on the opposite boundary shifted in time by L.



(a) g_1



(b) g_2



(c) g_3

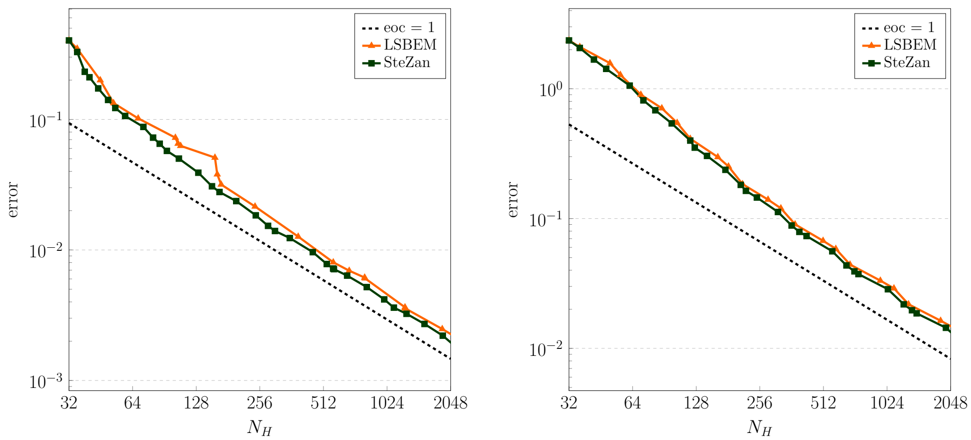
Figure 8: Error convergence comparison between non-constrained (NC) and constrained(C) adaptive refinement algorithms for (4.2), given $m = 3$ and different Dirichlet data.

Finally, we compare the performance of the proposed adaptive algorithm from formulation (4.3), which we will dub *LSBEM*, with an adaptive BEM routine introduced in [37] and applied to the wave equation in [42], here referred to as *SteZan*. Performance is measured by considering the error with respect to the amount of degrees of freedom. For the numerical experiments a Galerkin approximation of the *direct* energetic BIE is considered: For $i \in \{1, 2\}$, find $z_H \in \mathcal{S}^0_H(\Sigma) \subset L^2(\Sigma)$ such that

$$\langle \partial_t \mathsf{V}\, w_H, q_H \rangle_\Sigma = \frac{1}{2} \langle \partial_t g_i, q_H \rangle_\Sigma + \langle \partial_t \mathsf{K}\, g_i, q_H \rangle_\Sigma, \quad \forall q_H \in \mathcal{S}^0_H(\Sigma) \subset L^2(\Sigma), \tag{4.7}$$

where K denotes the *double layer operator*, which is given for $g = 0$ outside of Σ by [42]:

$$\mathsf{K}\, g(x,t) = \begin{cases} -\frac{1}{2} g(L, t - L) & x = 0, \\ -\frac{1}{2} g(0, t - L) & x = L. \end{cases}$$



(a) g_1 (b) g_2

Figure 9: $L^2(\Sigma)$-error convergence comparison for proposed adaptive algorithm (LSBEM) and method from the literature (SteZan).

A comparison of both methods is presented in Fig. 9 using Dirichlet data g_1 and g_2. Both methods perform similarly when it comes to error convergence. However, in order to obtain a valid error estimator, the LSBEM approach requires solving for a mixed boundary element method with $m = 2$, increasing the computational complexity at each refinement. On the other hand, the SteZan method has limited applicability: it is restricted to direct formulations and requires an implementation or approximation of the adjoint double layer and hypersingular operators.

4.4 Stability Constant

Finally, we compare the theoretical stability constant, as proposed in Theorem 3.2, with the actual discrete inf-sup constant, computed using the method introduced in, e.g., [27, Rem. 3.159]. For notational convenience let us denote the theoretical stability constant by

$$\gamma_n := 2\sin\left(\frac{\pi}{2(2n+1)}\right)\left(\frac{1}{2}-\frac{1}{m}\right).$$

The computation of the discrete inf-sup constant requires the usage of a mass matrix with respect to the $[H^1_{,0}(\Sigma)]'$-inner product, the implementation of this matrix is based on the theory presented in Section 4.2. The results for the stable ($m = 3$) standard formulation (4.2) and the energetic BEM formulation (4.3) without nesting ($m = 1$) are given in Fig. 10. There we see that the proposed stability constant has the same asymptotic behaviour as the actual discrete inf-sup constant. In the case of energetic BEM, which is stable for $m = 1$, we observe that the stability constant coincides with $\tilde{c}_S(n)$, as defined in Theorem 2.3. On each time-slice the coarse mesh consists of 32 uniform elements.

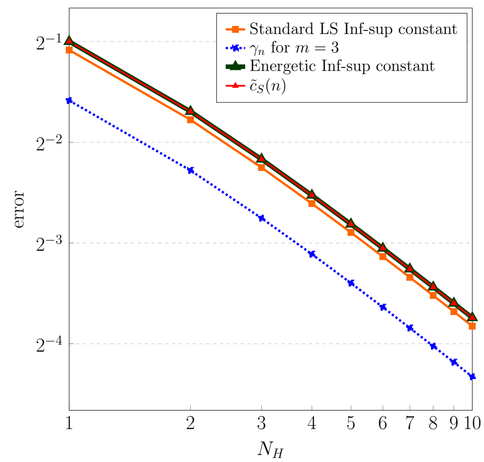


Figure 10: Comparison between γ_n and $\tilde{c}_S(n)$, and the discrete inf-sup constant for formulations (4.2) with $m = 3$ and (4.3) with $m = 1$, on different amount of time-slices.

5 Conclusions

In this paper we have formulated and analyzed a least squares approach for first kind boundary integral equations for the Dirichlet problem for the wave equation. We have established stability of a related boundary element method, from which we can derive a priori error estimates. Moreover, the approximation of the adjoint variable can be used as an error indicator to drive an adaptive algorithm. Numerical results, also for less regular Dirichlet data, confirm the theoretical findings.

It is more or less obvious that this approach can be applied as well to problems with different boundary conditions, and to other boundary integral equations also including the double layer operator and its adjoint, and the hypersingular boundary integral operator for the wave equation. A possible extension to systems such as in elastodynamics will also follow the lines as given for the scalar wave equation. More challenging is the construction of efficient solution methods for the resulting linear systems of algebraic equations, and the construction of appropriate preconditioners. The implementation of the proposed approach to solve problems in higher space dimensions is ongoing work, but the numerical analysis can not be done in such an explicit way as it is possible in one dimension.

References

[1] A. Aimi, M. Diligenti, C. Guardasoni: On the energetic Galerkin boundary element method applied to interior wave propagation problems, J. Comput. Appl. Math. 235 (2011) 1746–1754.

[2] A. Aimi, M. Diligenti, C. Guardasoni, I. Mazzieri, S. Panizzi: An energy approach to space-time Galerkin BEM for wave propagation problems. Internat. J. Numer. Methods Engrg. 80 (2009) 1196–1240.

[3] A. Aimi, M. Diligenti, C. Guardasoni, S. Panizzi: A space-time energetic formulation for wave propagation analysis by BEMs. Riv. Mat. Univ. Parma (7), 8 (2008) 171–207.

[4] R. Andreev: Stability of sparse space-time finite element discretizations of linear parabolic evolution equations. IMA J. Numer. Anal. 33 (2013) 242–260.

[5] H. Antes, T. Rüberg, M. Schanz: Convolution quadrature boundary element method for quasi-static visco- and poroelastic continua. Computers & Structures 83 (2004) 673–684.

[6] A. Bamberger, T. Ha Duong: Formulation variationnelle pour le calcul de la diffraction d'une onde acoustique par une surface rigide. Math. Meth. Appl. Sci. 8 (1986) 598–608.

[7] L. Banjai, S. Sauter: Rapid solution of the wave equation in unbounded domains. SIAM J. Numer. Anal. 47 (2008) 227–249.

[8] L. Banz, H. Gimperlein, Z. Nezhi, E. P. Stephan: Time domain BEM for sound radiation of tires. Comput. Mech. 58 (2016) 45–57.

[9] P. Bochev, M. Gunzburger: Least-squares finite element methods. International Congress of Mathematicians. Vol. III, European Mathematical Society (EMS), Zürich, pp. 1137–1162, 2006.

[10] P. Bochev, M. Gunzburger: Least-squares methods for hyperbolic problems. Handbook of numerical methods for hyperbolic problems, pp. 289–317, Handb. Numer. Anal., 17, Elsevier/North-Holland, Amsterdam, 2016.

[11] P. Bochev, M. Gunzburger: Least-squares finite element methods. Applied Mathematical Sciences, vol. 166, Springer, New York, 2009.

[12] M. Costabel, F.-J. Sayas: Time-dependent problems with the boundary integral equation method. In: Encyclopedia of Computational Mechanics (E. Stein, R. Borst, T. J. R. Hughes eds.), 2nd ed., Wiley, 2017.

[13] W. Dörfler, S. Findeisen, C. Wieners: Space-time discontinuous Galerkin discretizations for linear first-order hyperbolic evolution systems. Comput. Meth. Appl. Math. 16 (2016) 409–428.

[14] S. Dohr. Distributed and Preconditioned Space–Time Boundary Element Methods for the Heat Equation. Phd thesis, TU Graz, 2019.

[15] S. Dohr, O. Steinbach, K. Niino: Space-time boundary element methods for the heat equation. In: Space-Time Methods. Applications to Partial Differential Equations (U. Langer, O. Steinbach eds.) pp. 1–60, de Gruyter, Berlin, Boston, 2019.

[16] T. Führer, M. Karkulik: Space-time least-squares finite elements for parabolic equations. Comput. Math. Appl. 92 (2021) 27–36.

[17] T. Führer, R. Gonzales, M. Karkulik: Well-posedness of first-order acoustuc wave equations and space-time finite element approximation. arXiv:2311.10536, 2023.

[18] M. J. Gander: 50 years of time parallel integration. In: Multiple Shooting and Time Domain Decompositions, pp. 69–114, Springer, Heidelberg, Berlin, 2015.

[19] M. J. Gander, M. Neumüller: Analysis of a new space-time parallel multigrid algorithm for parabolic problems. SIAM J. Sci. Comput. 38 (2016) A2173–A2208.

[20] H. Gimperlein, Z. Nezhi, E. P. Stephan: A priori error estimates for a time-dependent boundary element method for the acoustic wave equation in a half-space. Math. Methods Appl. Sci. 40 (2017) 448–462.

[21] H. Gimperlein, C. Özdemir, D. Stark, E. P. Stephan: hp-version time domain boundary elements for the wave equation on quasi-uniform meshes. Comput. Methods Appl. Mech. Engrg. 356 (2019) 145–174.

[22] H. Gimperlein, C. Özdemir, D. Stark, E. P. Stephan: A residual a posteriori error estimate for the time-domain boundary element method. Numer. Math. 146 (2020) 239–280.

[23] H. Gimperlein, C. Özdemir, E. P. Stephan: Time domain boundary element methods for the Neumann problem: error estimates and acoustic problems. J. Comput. Math. 36 (2018) 70–89.

[24] C. Guardasoni. Wave propogation analysis with boundary element method. PhD thesis, University of Parma, 2010.

[25] T. Ha-Duong, B. Ludwig, I. Terrasse: A Galerkin BEM for transient acoustic scattering by an absorbing obstacle. Internat. J. Numer. Methods Engrg. 57 (2003) 1845–1882.

[26] M. E. Hassell, T. Qiu, T. Sánchez-Vizuet, F.-J. Sayas: A new and improved analysis of the time domain boundary integral operators for the acoustic wave equation. J. Integral Equations Appl. 29 (2017) 107–136.

[27] V. John: Finite Element Methods for Incompressible Flow Problems. Springer Series in Computational Mathematics. vol. 51, Springer, 2016.

[28] P. Joly, J. Rodríguez: Mathematical aspects of variational boundary integral equations for time dependent wave propagation. J. Integral Equations Appl. 29 (2017) 137–187.

[29] C. Köthe, R. Löscher, O. Steinbach: Adaptive least-squares space-time finite element methods. arXiv:2309.14300, 2023.

[30] U. Langer, O. Steinbach: Space-time methods. Applications to partial differential equations. Radon Series on Computational and Applied Mathematics, vol. 25, de Gruyter, Berlin, 2019.

[31] R. Löscher , O. Steinbach M. Zank: Numerical results for an unconditionally stable space-time finite element method for the wave equation. In: Domain Decomposition Methods in Science and Engineering XXVI, Lecture Notes in Computational Science and Engineering, vol. 145, pp. 625–632, Springer, Cham, 2022.

[32] M. Merta, G. Of, R. Watschinger,J.Zapletal: A parallel fast multipole method for a space-time boundary element method for the heat equation. SIAM J. Sci. Comput. 44 (2022) C320–C345.

[33] A. Moiola, I. Perugia: A space-time Trefftz discontinuous Galerkin method for the acoustic wave equation in first-order formulation. Numer. Math. 138 (2018) 389–435.

[34] D. Pölz, M. Schanz: On the space-time discretization of variational retarded potential boundary integral equations. Comput. Math. Appl. 99 (2021) 195–210.

[35] F.-J. Sayas: Energy estimates for Galerkin semidiscretizations of time domain boundary integral equations. Numer. Math. 124 (2013) 121–149.

[36] F.-J. Sayas: Retarded potentials and time domain boundary integral equations. A road map. Springer Series in Computational Mathematics, vol. 50, Springer, Cham, 2016.

[37] H. Schulz, O. Steinbach: A new a posteriori error estimator in adaptive direct boundary element methods: the Dirichlet problem. Calcolo 37 (2000) 79–96.

[38] O. Steinbach: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[39] O. Steinbach: An adaptive least squares boundary element method for elliptic boundary value problems. Berichte aus dem Institut für Angewandte Mathematik, Bericht 2023/1, TU Graz, 2023.

[40] O. Steinbach, C. Urzúa–Torres: A new approach to space-time boundary integral equations for the wave equation. SIAM J. Math. Anal. 54 (2022) 1370–1392.

[41] O. Steinbach, C. Urzúa–Torres, M. Zank: Towards coercive boundary element methods for the wave equation. J. Integral Equations Appl. 34 (2022) 501–515.

[42] O. Steinbach, M. Zank: Adaptive space-time boundary element methods for the wave equation. PAMM. 16 (2016) 777–778.

[43] O. Steinbach, M. Zank: Coercive space–time finite element methods for initial boundary value problems. Electron. Trans. Numer. Anal. 52 (2020) 154–194.

[44] O. Steinbach, M. Zank: A generalized inf-sup stable variational formulation for the wave equation. J. Math. Anal. Appl. 505 (2022) 125457.

[45] O. Steinbach, M. Zank: A note on the efficient evaluation of a modified Hilbert transformation. J. Numer. Math. 29 (2021) 47–61.

[46] R. Stevenson, J. Westerdiep: Stability of Galerkin discretizations of a mixed space-time variational formulation of parabolic equations. IMA J. Numer. Anal. 41 (2021) 28–47.

[47] I. Terrasse: Résolution mathématique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés. Doctoral dissertation, École Polytechnique, 1993.