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domain decomposition methods

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Abstract

The space-time finite element discretization of parabolic evolution equations results in very large linear systems of algebraic equations. In this context, for the iterative solution we propose the use of finite element tearing and interconnecting (FETI) domain decomposition methods where the local subproblems are solved directly, while the Schur complement system is solved by using a GMRES method globally. We also discuss the handling of floating subdomains and an all-floating (total) FETI approach. First numerical results for the heat equation and the time-dependent Stokes system indicate the potential of space-time FETI methods.

1 Introduction

Finite element tearing and interconnecting (FETI) domain decomposition methods [4] are well established for the parallel solution of elliptic problems. This is mainly due to their simple implementation and the availability of efficient and robust preconditioning strategies. Among other variants to deal with floating subdomains, total FETI [2] or all-floating FETI [8] methods handle all subdomains as floating, incorporating Dirichlet boundary conditions by using Lagrange multipliers as well. This can simplify the implementation, in particular when considering systems of partial differential

equations. While the original derivation of the FETI method was based on a constrained minimization problem, related methods can be formulated for the Helmholtz equation [12] and the Maxwell equation [13] as well, using tearing and interconnecting on the discrete level only.

So far, domain decomposition and FETI methods are mainly restricted to elliptic problems, or to time-dependent problems which are discretized by using tensor-product ansatz spaces. Parallelization in time is in most cases based on the parareal algorithm [7] combining coarse and fine temporal grids.

In recent years, space-time discretization methods have become very popular, see, e.g., the review article [14] and the references given therein. These methods consider time as just another spatial coordinate, using a finite element discretization in the whole space-time domain [10]. As this allows an adaptive resolution in space and time simultaneously, the solution of the resulting linear system of algebraic equations requires the use of efficient solution strategies in parallel. Domain decomposition methods are a natural choice to provide efficient and robust preconditioning, and allow parallelization when considering one subdomain per processor.

While the work presented in [11] considers standard domain decomposition methods [1, 5] for the heat equation, the focus of the present contribution is on FETI methods applied to the Stokes system and the heat equation. In Sect. 2 we describe the space-time finite element discretization of the related model problems. For the solution of the resulting linear systems we present in Sect. 3 a FETI method, including a discussion on floating subdomains. When considering all subdomains as floating, we end up with an all-floating FETI method. First numerical results in Sect. 4 indicate the great potential of space-time FETI domain decomposition methods, including parallel-in-time algorithms.

2 Space-time finite element methods

As a first model problem, we consider the homogeneous Dirichlet problem for the transient heat equation

$$\partial_t u - \Delta_x u = f \quad \text{in } Q, \quad u = 0 \quad \text{on } \Sigma, \quad u = 0 \quad \text{on } \Sigma_0, \quad (1)$$

where for a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, and a finite time horizon T we have the space-time domain $Q := \Omega \times (0, T) \subset \mathbb{R}^{d+1}$ with the

lateral boundary $\Sigma := \partial\Omega \times (0, T)$, and $\Sigma_0 := \Omega \times \{0\}$. For simplicity, we only consider homogeneous boundary and initial conditions, but inhomogeneous data and other types of boundary conditions can be handled as well. The space-time variational formulation of (1) reads to find $u \in X := L^2(0, T; H_0^1(\Omega)) \cap H_0^1(0, T; H^{-1}(\Omega))$ such that

$$\int_0^T \int_{\Omega} [\partial_t u v + \nabla_x u \cdot \nabla_x v] dx dt = \int_0^T \int_{\Omega} f v dx dt \quad (2)$$

is satisfied for all $v \in Y := L^2(0, T; H_0^1(\Omega))$. Note that the ansatz space X covers zero boundary and initial conditions. For a space-time finite element discretization of (2), we introduce conforming finite element spaces $X_h \subset X$ and $Y_h \subset Y$, where we assume $X_h \subset Y_h$. In particular, we use the finite element spaces $X_h = Y_h$ of piecewise linear and continuous basis functions, defined with respect to some admissible decomposition of the space-time domain Q into shape regular simplicial finite elements. For a detailed stability and error analysis of this space-time finite element method we refer to [10, 11]. The space-time finite element discretization of (2) results in a large linear system of algebraic equations which we shall solve using an appropriate tearing and interconnecting domain decomposition method.

As a second model problem, we consider the time-dependent Stokes system

$$\partial_t u - \nu \Delta_x u + \nabla_x p = f, \quad \nabla_x \cdot u = 0 \text{ in } Q, \quad u = 0 \text{ on } \Sigma, \quad u = 0 \text{ on } \Sigma_0. \quad (3)$$

Again we only consider homogeneous boundary and initial conditions, for simplicity. The variational formulation of (3) seeks $u \in X^d$ and $p \in L^2(Q)$ such that

$$\int_0^T \int_{\Omega} [\partial_t u \cdot v + \nu \nabla_x u : \nabla_x v - p \nabla_x \cdot u] dx dt = \int_0^T \int_{\Omega} f \cdot v dx dt \quad (4)$$

is satisfied for all $v \in Y^d$ and

$$\int_0^T \int_{\Omega} \nabla_x \cdot u q dx dt + \int_0^T \left(\int_{\Omega} p dx \int_{\Omega} q dx \right) dt = 0 \quad (5)$$

is satisfied for all $q \in L^2(Q)$. Note that the additional term in (5) ensures the scaling condition $p(t) \in L_0^2(\Omega)$ for all $t \in (0, T)$. The space-time variational formulation (4), (5) can be analyzed similarly to what was done in [10] in

the case of the heat equation, extending to the space-time setting the spatial inf-sup stability condition for the divergence. Note that inhomogeneous essential boundary and initial conditions g and u_0 can be handled through homogenization by using suitable extensions of such data into the space-time domain. For the space-time finite element discretization of (4) and (5) we use inf-sup stable pairs to approximate u_h and p_h . In particular, we extend the well established Taylor–Hood elements to the space-time setting using simplicial finite elements. As an alternative we may also use prismatic space-time Taylor–Hood elements, see [9] for first numerical results. A more detailed stability and error analysis will be published elsewhere.

3 Tearing and interconnecting domain decomposition methods

The space-time finite element discretization of the heat equation (1) and of the Stokes system (3) results in very large linear systems of algebraic equations which must be solved in parallel, and if possible, simultaneously in space and time. One possible approach is to use space-time finite element tearing and interconnecting methods, which are well established for elliptic problems. Here we generalize this approach to parabolic time-dependent problems.

The space-time domain $Q = \Omega \times (0, T)$ is decomposed into s non-overlapping space-time subdomains Q_i which can be rather general, see Fig. 1 for a selection of possible simple decompositions. With respect to this space-time domain decomposition we consider the localized problems, where the continuity of the primal unknowns along the interface is enforced via discrete Lagrange multipliers. This results in the global linear system

$$\begin{pmatrix} K_1 & & & B_1^\top \\ & \ddots & & \vdots \\ & & K_s & B_s^\top \\ B_1 & \cdots & B_s & \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_s \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_s \\ \underline{0} \end{pmatrix}, \quad (6)$$

where the K_i are the local space-time finite element stiffness matrices, and the B_i are Boolean matrices. While (6) corresponds directly to the heat equation (1), it formally also includes the Stokes problem (3) with all quantities defined accordingly. Although we have chosen to enforce the interface continuity

of the pressure field, this is in principle not necessary since the variational problem allows $p \in L^2(Q)$.

At this time we assume that all local matrices K_i are invertible, so that when using direct solvers locally, we end up with the Schur complement system

$$\sum_{i=1}^s B_i K_i^{-1} B_i^\top \underline{\lambda} = \sum_{i=1}^s B_i K_i^{-1} \underline{f}_i. \quad (7)$$

The global linear system (7) is solved here by a GMRES method, either without preconditioning or with a simple diagonal preconditioner. More advanced preconditioning strategies also including some coarse grid contributions seem to be mandatory for more complex problems, and this is a topic of further research.

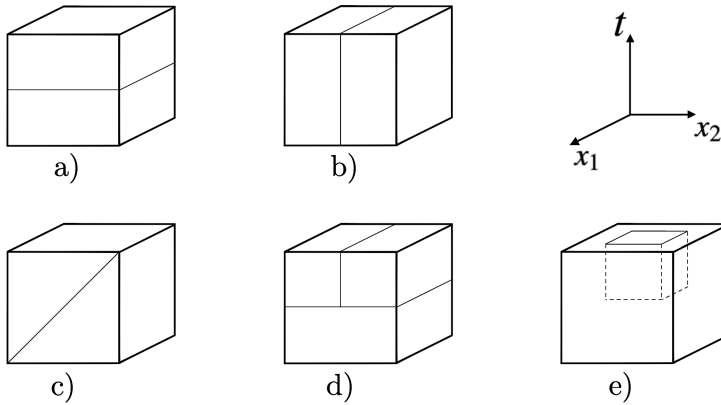


Figure 1: Different decompositions for the space-time domain $Q = \Omega \times (0, T) \subset \mathbb{R}^3$.

In what follows, we will discuss the more general situation in which a local matrix K_i is not invertible, i.e., when the subdomain Q_i is *floating*. Using a pseudo-inverse K_i^+ of K_i , we can describe the solutions of the local subproblems as

$$\underline{u}_i = K_i^+ (\underline{f}_i - B_i^\top \underline{\lambda}) + R_i \underline{\alpha}_i, \quad (8)$$

where the local matrices R_i describe the kernels $\mathcal{N}(K_i)$ of K_i , and $\underline{\alpha}_i$ are unknown coefficients to be determined. The application of the pseudo-inverse K_i^+ also requires the solvability condition $\underline{f}_i - B_i^\top \underline{\lambda} \in \mathcal{R}(K_i)$, which is equivalent to

$$\tilde{R}_i^\top (\underline{f}_i - B_i^\top \underline{\lambda}) = \underline{0},$$

where the local matrices \tilde{R}_i describe the kernels $\mathcal{N}(K_i^\top)$. In the case of floating subdomains we therefore end up with the Schur complement system

$$\begin{pmatrix} S & -G \\ \tilde{G}^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\alpha} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix}, \quad (9)$$

where

$$S = \sum_{i=1}^s B_i K_i^+ B_i^\top, \quad G = (B_1 R_1, \dots, B_s R_s), \quad \tilde{G} = (B_1 \tilde{R}_1, \dots, B_s \tilde{R}_s),$$

and

$$\underline{d} = \sum_{i=1}^s B_i K_i^+ \underline{f}_i, \quad \underline{e} = \begin{pmatrix} \tilde{R}_1^\top \underline{f}_1 \\ \vdots \\ \tilde{R}_s^\top \underline{f}_s \end{pmatrix}.$$

Similarly as in FETI methods for elliptic problems, we introduce a projection

$$P := I - G(\tilde{G}^\top G)^{-1} \tilde{G}^\top,$$

and it remains to solve the constrained linear system

$$PS\underline{\lambda} = P\underline{d}, \quad \tilde{G}^\top \underline{\lambda} = \underline{e}, \quad (10)$$

which can be done by using a projected GMRES method [6]. Afterwards we can compute

$$\underline{\alpha} = (\tilde{G}^\top G)^{-1} \tilde{G}^\top (S\underline{\lambda} - \underline{d}).$$

Notice that the square matrix $\tilde{G}^\top G$ is small, since it does not depend on the finite element mesh but only on the number s of subdomains. In fact, its dimension is simply s for the heat equation, or $s \times d$ for the Stokes problem. Therefore, the inverse $(\tilde{G}^\top G)^{-1}$ can be computed directly and works as a coarse-grid solver.

It remains to characterize the kernels $\mathcal{N}(K_i)$ and $\mathcal{N}(K_i^\top)$ of the local space-time stiffness matrices K_i and their transposed, respectively. For this we consider the heat equation in the floating space-time subdomain $Q_i = \Omega_i \times (t_{i-1}, t_i)$, where the matrix K_i corresponds to the space-time discretization of the heat equation in Q_i with zero Neumann boundary conditions and *without* initial or terminal conditions at t_{i-1} or t_i , respectively. In the continuous case, the solution in Q_i is given by

$$u_i(x, t) = \sum_{k=0}^{\infty} u_{i,k} e^{-\lambda_{i,k} t} v_{i,k}(x) \quad \text{for } (x, t) \in Q_i, \quad (11)$$

where $v_{i,k}$ are the eigenfunctions of the Neumann eigenvalue problem for the spatial Laplacian in Ω_i , with eigenvalues $\lambda_{i,k} \geq 0$. For the space-time finite element discretization we use piecewise linear and continuous basis functions as partition of unity in Q_i , i.e., $v_{i,0} \in X_{h|Q_i}$ for $\lambda_{i,0} = 0$. Due to the exponential decay in the solution (11) for $k \geq 1$, no more eigenfunctions are represented in the local finite element space $X_{h|Q_i}$, and hence we conclude $\mathcal{N}(K_i) = \{\underline{1}\}$ in the case of the heat equation (1). Similarly, for the Stokes problem (3) we have d constant eigenfunctions for the velocity, and additionally null pressure [15]. In both cases, the constant eigenfunctions remain true for general space-time subdomains Q_i .

While the kernel $\mathcal{N}(K_i)$ is trivially constructed, the basis for $\mathcal{N}(K_i^\top)$ is in general mesh-dependent. Such bases are however easily obtained as subproducts of numerical techniques for computing pseudo-inverses K_i^+ , see [3].

To simplify the implementation and to include all subdomains in the coarse-grid matrix $\tilde{G}^\top G$, we may consider all subdomains as floating, incorporating Dirichlet boundary conditions by using Lagrange multipliers as well. This results in the all-floating [8] or total [2] FETI approach.

4 Numerical results

As a first numerical example we consider the Stokes system (3) in the spatial domain $\Omega = (0, 1)^2$ for $T = 1$, i.e., $Q = (0, 1)^3$. In order to check the expected order of convergence we consider for $\nu = 1$ the manufactured solution

$$\begin{aligned} u_1(x, t) &= 2(1 - e^{-t})(x_2 - 3x_2^2 + 2x_2^3)[x_1(1 - x_1)]^2, \\ u_2(x, t) &= 2(1 - e^{-t})(3x_1^2 - x_1 - 2x_1^3)[x_2(1 - x_2)]^2, \\ p(x, t) &= (1 + x_1 - e^{-x_1 x_2 t})t^2, \end{aligned}$$

with the right-hand side f computed accordingly. In this first example we consider decompositions of the space-time domain Q into only a few subdomains, see Fig. 1. Our particular interest is in the effect of the interface orientation on the number of required GMRES iterations to reach a given relative accuracy of $\varepsilon = 10^{-6}$, see also the discussion in [11] in the case of a standard domain decomposition approach for the heat equation. We solve the global Schur complement system without any preconditioning (I), and with a simple diagonal preconditioner (D). In all cases we observe a significant reduction in the number of iterations, with the best results appearing

when considering a decomposition in time (a) or space (b) only, and for the diagonal decomposition (c). The results are not as good when considering the decomposition (d) and the inclusion (e). In general, some coarse-grid preconditioner should be used to further reduce the number of iterations.

Table 1: Space-time FETI domain decomposition method for the time-dependent Stokes system in $Q = (0, 1)^3$. Number of GMRES iterations for the Schur complement system without (I) and with diagonal preconditioning (D), for different numbers N_e of elements.

N_e	$\ \nabla_x(u - u_h)\ _{L^2}$	$\ p - p_h\ _{L^2}$	Domain decomposition											
			a)		b)		c)		d)		e)			
			I	D	I	D	I	D	I	D	I	D	I	D
192	6.86e-3	2.63e-2	15	11	26	13	31	15	36	19				
1536	2.19e-3	1.64	6.53e-3	2.01	25	13	54	17	57	20	79	29	72	28
12288	5.82e-4	1.92	1.57e-3	2.05	36	17	94	22	105	27	165	44	181	50
98304	1.47e-4	1.98	3.81e-4	2.04	55	22	180	34	206	39	374	66	325	83

In the second example we consider the heat equation (1) in the spatially one-dimensional domain $\Omega = (0, 1)$ and with the final time $T = 1$, i.e., $Q = (0, 1)^2$. As solution we have chosen $u(x, t) = \sin \frac{1}{2}\pi t \sin \pi x$. Here we consider a decomposition of the space-time domain Q into up to 64 time slabs, applying both the space-time FETI approach and the all-floating formulation. The results are given in Table 2, where we observe a reasonable number of iterations in all cases. Note that the number of degrees of freedom is significantly larger when using the all-floating approach instead of the standard FETI method. However, the all-floating method may require fewer iterations when the number of subdomains is large enough, see Table 2 and also the discussion in [2, 8]. It is obvious that this approach is strongly related to the parareal algorithm [7] where the coarse grid corresponds to the time slabs of the domain decomposition, see also the results in [11].

5 Conclusions

In this contribution, we have presented and described first results for space-time finite element tearing and interconnecting domain decomposition methods, including also the all-floating approach. Model problems include the heat equation and the Stokes system, but more complex partial differential equations can be considered as well. The space-time finite element discretization and the tearing and interconnecting approach follow the lines of the

Table 2: Classical and all-floating (AF) space-time FETI methods for the heat equation. Number of GMRES iterations for a sequence of time slabs and meshes.

N_e	$s = 2$		$s = 4$		$s = 8$		$s = 16$		$s = 32$		$s = 64$	
	FETI	AF	FETI	AF	FETI	AF	FETI	AF	FETI	AF	FETI	AF
128	5	12	7	12	9	12						
512	7	12	8	14	12	18	17	17				
2048	8	13	10	15	14	21	23	29	34	27		
8192	9	15	11	18	16	24	26	36	40	53	69	49
32768	9	18	12	23	17	29	28	44	47	68	79	104

FETI method for elliptic problems, considering time as just an additional spatial coordinate. The main distinction here lies on the asymmetry of the space-time stiffness matrix, which requires a modified projection operator and also a numerical procedure to construct local kernels. First numerical results show the potential of the proposed method, in particular when using state-of-the-art parallel computing facilities for time-dependent problems. It is clear that a more detailed numerical analysis, in particular with respect to suitable preconditioning strategies for general space-time domain decompositions, is required. Related results will be published elsewhere.

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