Technische Universität Graz

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Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.numerik.math.tu-graz.ac.at`

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O. Steinbach

Institut für Numerische Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria
o.steinbach@tugraz.at

In memoriam Christof Eck (1968–2011).

Abstract

In this paper we present a priori error estimates for the Galerkin solution of variational inequalities which are formulated in fractional Sobolev trace spaces, i.e. in $\widetilde{H}^{1/2}(\Gamma)$. In addition to error estimates in the energy norm we also provide, by applying the Aubin–Nitsche trick for variational inequalities, error estimates in lower order Sobolev spaces including $L_2(\Gamma)$. The resulting discrete variational inequality is solved by using a semi–smooth Newton method, which is equivalent to an active set strategy. A numerical example is given which confirms the theoretical results.

1 Introduction

In this paper we are interested in the numerical analysis of the Galerkin boundary element approximation of first kind variational inequalities to find

$$u \in \mathcal{K} := \left\{ v \in \widetilde{H}^{1/2}(\Gamma) \,:\, v \leq g \text{ on } \Gamma \right\} \tag{1.1}$$

such that

$$\langle Au, v-u\rangle_\Gamma \,\geq\, \langle f, v-u\rangle_\Gamma \quad \text{for all } v \in \mathcal{K}\,. \tag{1.2}$$

We assume that Γ is either a $(n-1)$–dimensional Lipschitz manifold in $\mathbb{R}^n$, $n = 2,3$, or $\Gamma = \partial\Omega$ is the Lipschitz boundary of a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$. By $\widetilde{H}^{1/2}(\Gamma)$ we denote the Sobolev space of functions which can be extended by zero when $\Gamma \subset \widetilde{\Gamma}$ is embedded in a closed Lipschitz surface $\widetilde{\Gamma}$, i.e.

$$\widetilde{H}^{1/2}(\Gamma) := \left\{ v_{|\Gamma} : v \in H^{1/2}(\widetilde{\Gamma}),\ \operatorname{supp} v \subset \Gamma \right\}, \quad H^{-1/2}(\Gamma) := [\widetilde{H}^{1/2}(\Gamma)]'.$$

In the case of a closed surface Γ we have $\widetilde{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$. We further assume that $A : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a bounded, self–adjoint, and $\widetilde{H}^{1/2}(\Gamma)$–elliptic operator satisfying

$$\langle Av, v\rangle_\Gamma \geq c_1^A \, \|v\|^2_{\widetilde{H}^{1/2}(\Gamma)}, \quad \|Av\|_{H^{-1/2}(\Gamma)} \leq c_2^A \, \|v\|_{\widetilde{H}^{1/2}(\Gamma)} \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma). \tag{1.3}$$

Finally we assume $g \in H^{1/2}(\Gamma)$, and $f \in H^{-1/2}(\Gamma)$.

Variational inequalities of the form (1.2) occur, for example, when considering the variational formulation of second order partial differential equations with boundary conditions of Signorini type, e.g., [12, 19, 24], or when considering contact problems in elasticity without friction, e.g., [4, 7, 8, 11]. Other applications involve Dirichlet boundary control problems with control constraints, e.g., [17, 22].

Unique solvability of the first kind variational inequality (1.2) follows by applying standard arguments, see, e.g., [2, 10, 17, 18]. Boundary element error estimates in the energy norm are discussed, e.g., in [12, 19, 24], related finite element error estimates for Galerkin approximations of variational inequalities formulated in $H^1(\Omega)$ are given, for example, in [3, 9]. Although the energy error estimate for the boundary element approximation of the variational inequality (1.2) follows similar as for the finite element approximation of a variational inequality in $H^1(\Omega)$, we provide a proof for completeness. However, the boundary element error estimate given here differs from the related finite element error estimate due to the different approximation properties of functions defined in a bounded domain Ω, or on its boundary $\Gamma = \partial\Omega$. Note that the latter also requires an increased regularity of the function to be approximated on the boundary, as compared to an approximation defined in Ω. The proof as given here is also different as presented, e.g., in [12, 24]. In particular we present a generalisation of Cea's lemma in the case of variational inequalities, and we prove a related approximation property.

The main interest of this paper is to provide an error estimate in $L_2(\Gamma)$. In the case of variational equations these results are due to the well known Aubin–Nitsche trick, see, e.g., [1, 14, 25]. It seems that related results in the case of variational inequalities are not so well known, and to the best of our knowledge, not available for the problem class as considered in this paper. Note that finite element error estimates in $L_2(\Omega)$ for the solution of variational inequalities in $H^1(\Omega)$ are given in [21], see also [27].

This paper is organized as follows: In Sect. 2 we describe related complementary conditions and discuss a rather general regularity result. Moreover, we introduce the boundary element discretization of the variational inequality (1.2). In Sect. 3 we provide an error estimate for the approximate solution in the energy norm $\| \cdot \|_{H^{1/2}(\Gamma)}$. The Nitsche trick for variational inequalities to derive an error estimate in $L_2(\Gamma)$ is considered in Sect. 4. For the solution of the discrete variational inequality we describe a semi–smooth Newton approach in Sect. 5, which is equivalent to an active set strategy, and in Sect. 6 we discuss some applications and provide a numerical example.

2 Complementary conditions and discretization of variational inequalities

The aim of this section is to describe the Galerkin discretization of the variational inequality (1.2) by using boundary element methods and to present an equivalent characterization of the unique solution of the discrete variational inequality by means of some discrete complementary conditions. But first we consider related complementary conditions in the continuous case.

For $u \in \mathcal{K}$ being the unique solution of the variational inequality (1.2) we introduce the active and inactive boundary parts as

$$\Gamma^{\text{act}} := \Big\{ x \in \Gamma : u(x) = g(x) \Big\}, \quad \Gamma^{\text{in}} := \Big\{ x \in \Gamma : u(x) < g(x) \Big\} = \Gamma \backslash \Gamma^{\text{act}},$$

and we define

$$\lambda := Au - f \in H^{-1/2}(\Gamma). \tag{2.1}$$

Lemma 2.1 *Let $u \in \mathcal{K}$ be the unique solution of the variational inequality* (1.2)*, and let $\lambda \in H^{-1/2}(\Gamma)$ be defined as in* (2.1)*. Then there hold the complementary conditions*

$$u \le g \text{ in } H^{1/2}(\Gamma), \quad \lambda \le 0 \text{ in } H^{-1/2}(\Gamma), \quad \lambda[g-u] = 0 \text{ a.e. on } \Gamma. \tag{2.2}$$

Proof. We first consider the variational inequality (1.2), i.e. for $v = g \in \mathcal{K}$ we have

$$\langle \lambda, g-u \rangle_\Gamma = \langle Au - f, g - u \rangle_\Gamma \ge 0\,.$$

For $w \in H^{1/2}(\Gamma)$ with $w \ge 0$ we have $u - w \le u \le g$ on Γ, and therefore $v := u - w \in \mathcal{K}$. Hence we obtain from (1.2)

$$-\langle \lambda, w \rangle_\Gamma = \langle Au - f, v - u \rangle_\Gamma \ge 0 \quad \text{for all } w \in H^{1/2}(\Gamma), \quad w \ge 0 \quad \text{on } \Gamma,$$

i.e. $\lambda \le 0$ in the sense of $H^{-1/2}(\Gamma)$. In particular for $w := g - u \ge 0$ on Γ we have

$$\langle \lambda, g-u \rangle_\Gamma \le 0,$$

and therefore,

$$\langle \lambda, g-u \rangle_\Gamma = 0$$

follows. Due to $\lambda \le 0$ in $H^{-1/2}(\Gamma)$ and $g - u \ge 0$ in $H^{1/2}(\Gamma)$ we finally conclude

$$\lambda[g-u] = 0 \quad \text{almost everywhere on } \Gamma.$$

■

Note that the complementary conditions (2.2) are nothing than the Karush–Kuhn–Tucker conditions which describe the saddle point (u, λ) of the Lagrange functional

$$\mathcal{L}(v,\mu) := \frac{1}{2}\langle Av, v \rangle_\Gamma - \langle f, v \rangle_\Gamma + \langle \mu, g - v \rangle_\Gamma \quad \text{for } v \in \widetilde{H}^{1/2}(\Gamma),\ \mu \in H^{-1/2}(\Gamma),\ \mu \le 0.$$

Now we are in a position to state some regularity result for the solution of the variational inequality (1.2).

Theorem 2.2 *Let $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ be bijective and bounded for some $s > 0$. Assume $g \in H^{1/2+s}(\Gamma)$ and $f \in H^{-1/2+s}(\Gamma)$. Then $u \in \widetilde{H}^{1/2+s}(\Gamma)$ and $\lambda \in H^{-1/2+s}(\Gamma)$.*

Proof. We consider the complementary conditions (2.2) on the inactive boundary part Γ^{in}, i.e.

$$\lambda = Au - f = 0 \quad \text{on } \Gamma^{\text{in}}.$$

The unique solution $u \in \mathcal{K}$ of the variational inequality (1.2) can be written as $u = g + w$ for $w \in \widetilde{H}^{1/2}(\Gamma^{\text{in}})$ (and satisfying $w < 0$ on Γ^{in}). Hence we find that $w \in \widetilde{H}^{1/2}(\Gamma^{\text{in}})$ is the unique solution of the operator equation

$$Aw = f - Ag \quad \text{in } H^{-1/2}(\Gamma^{\text{in}}).$$

Note that $A : \widetilde{H}^{1/2}(\Gamma^{\text{in}}) \to H^{-1/2}(\Gamma^{\text{in}})$ is bounded and $\widetilde{H}^{1/2}(\Gamma^{\text{in}})$–elliptic, and hence invertible. Now the assertion follows from the mapping properties of A and the assumptions made on g and f. ∎

Next we consider the Galerkin discretization of the variational inequality (1.2) by using boundary element methods. Let $S_h^1(\Gamma) = \text{span}\{\varphi_k\}_{k=1}^M$ be the space of piecewise linear and continuous nodal basis functions φ_k which are defined with respect to an admissible and quasi–uniform boundary element mesh $\Gamma_h = \cup_{\ell=1}^N \tau_\ell$ of mesh size h, and with nodal points x_k, $k = 1, \ldots, M$. In the three–dimensional case $n = 3$ we assume that the triangular boundary elements τ_ℓ are shape regular. Let

$$g_h(x) = I_h g(x) := \sum_{k=1}^M g_k \varphi_k(x), \quad g_k = g(x_k) \quad \text{for } k = 1, \ldots, M,$$

be the piecewise linear interpolation of the barrier function g which now is assumed to be continuous. Then we define

$$\mathcal{K}_h := \left\{ v_h = \sum_{k=1}^M v_k \varphi_k \in S_h^1(\Gamma) : v_k \leq g_k \quad \text{for all } k = 1, \ldots, M \right\}$$

and we consider the variational inequality to find $u_h \in \mathcal{K}_h$ such that

$$\langle Au_h, v_h - u_h \rangle_\Gamma \geq \langle f, v_h - u_h \rangle_\Gamma \quad \text{for all } v_h \in \mathcal{K}_h. \tag{2.3}$$

As in the continuous case we conclude unique solvability of the discrete variational inequality (2.3), see, e.g., [10]. Note that (2.3) is equivalent to the discrete variational inequality to find $\underline{u} \in \mathbb{R}^M \leftrightarrow u_h \in \mathcal{K}_h$ such that

$$(A_h \underline{u}, \underline{v} - \underline{u}) \geq (\underline{f}, \underline{v} - \underline{u}) \quad \text{for all } \underline{v} \in \mathbb{R}^M \leftrightarrow v_h \in \mathcal{K}_h, \tag{2.4}$$

where

$$A_h[\ell, k] = \langle A\varphi_k, \varphi_\ell \rangle_\Gamma, \quad f_\ell = \langle f, \varphi_\ell \rangle_\Gamma \quad \text{for } k, \ell = 1, \ldots, M.$$

Lemma 2.3 *Let $u_h \in \mathcal{K}_h \leftrightarrow \underline{u} \in \mathbb{R}^M$ be the unique solution of the variational inequalities* (2.3) *and* (2.4)*, respectively. Let*

$$\underline{\lambda} := A_h \underline{u} - \underline{f} \in \mathbb{R}^M.$$

Then there hold the discrete complementary conditions

$$u_k \leq g_k, \quad \lambda_k \leq 0, \quad \lambda_k [g_k - u_k] = 0 \quad \textit{for all } k = 1, \ldots, M. \tag{2.5}$$

Proof. For $\underline{u} \in \mathbb{R}^M \leftrightarrow u_h \in \mathcal{K}_h$ we obviously have $u_k \leq g_k$ for all $k = 1, \ldots, M$. From the variational inequality (2.4) we then find

$$0 \leq (A_h\underline{u} - \underline{f}, \underline{v} - \underline{u}) = (\underline{\lambda}, \underline{v} - \underline{u}) = \sum_{k=1}^{M} \lambda_k(v_k - u_k) \quad \text{for all } \underline{v} \in \mathbb{R}^M \leftrightarrow v_h \in \mathcal{K}_h.$$

For $\ell = 1, \ldots, M$ arbitrary but fixed we chose

$$v_\ell < u_\ell \leq g_\ell, \quad v_k = u_k \quad \text{for } k \neq \ell,$$

i.e. we have $\underline{v} \in \mathbb{R}^M \leftrightarrow v_h \in \mathcal{K}_h$. This gives

$$0 \leq \lambda_\ell(v_\ell - u_\ell), \quad \text{i.e.} \quad \lambda_\ell \leq 0 \quad \text{for all } \ell = 1, \ldots, M.$$

On the other hand, for $v_k = g_k$ for all $k = 1, \ldots, M$ we have

$$0 \leq \sum_{k=1}^{M} \lambda_k[g_k - u_k] \leq 0,$$

and therefore

$$\lambda_k[g_k - u_k] = 0 \quad \text{for all } k = 1, \ldots, M.$$

■

Note that Lemma 2.3 is the discrete counterpart of Lemma 2.1, and that (2.5) are the Karush–Kuhn–Tucker conditions which are related to the discrete variational inequality (2.4).

Remark 2.1 *The discrete Lagrange multiplier* $\underline{\lambda} = A_h\underline{u} - \underline{f} \in \mathbb{R}^M$ *is in general not an approximation of the continuous Lagrange multiplier* $\lambda = Au - f \in H^{-1/2}(\Gamma)$*. Instead, considering* $\lambda_h \in span\{\psi_k\}_{k=1}^M \subset H^{-1/2}(\Gamma)$ *and* $u_h \in S_h^1(\Gamma)$ *we find*

$$\langle \lambda_h, v_h \rangle_\Gamma = \langle Au_h, v_h \rangle_\Gamma - \langle f, v_h \rangle_\Gamma \quad \textit{for all } v_h \in S_h^1(\Gamma),$$

i.e.

$$M_h^\top \widetilde{\underline{\lambda}} = A_h\underline{u} - \underline{f},$$

with the mass matrix defined by

$$M_h[\ell, k] = \langle \varphi_k, \psi_\ell \rangle_\Gamma \quad \textit{for } k, \ell = 1, \ldots, M.$$

Hence, only when using bi–orthogonal basis functions [15] *satisfying* $\langle \varphi_k, \psi_\ell \rangle_\Gamma = \delta_{k\ell}$ *we conclude* $\underline{\lambda} = \widetilde{\underline{\lambda}}$*. However, in our approach as presented in this paper we do not consider any approximation of the continuous Lagrange parameter* λ*, we just introduce the discrete counterpart* $\underline{\lambda}$*.*

3 Error estimates in the energy norm

In this section we present an a priori error estimate in the energy norm $\|u - u_h\|_{H^{1/2}(\Gamma)}$ for the approximate solution $u_h \in \mathcal{K}_h$ of the discrete variational inequality (2.3). While the general idea is similar to the related proof in the case of a finite element approximation [3], the handling of the inequality constraints is rather different. An alternative proof for a particular application, following [9], is discussed in [24]. For energy error estimates for *hp* boundary element methods in the case of Signorini problems, see also [19].

The first result is the extension of Cea's lemma to variational inequalities.

Lemma 3.1 *Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the unique solutions of the variational inequalities* (1.2) *and* (2.3)*, respectively. For all $v_h \in \mathcal{K}_h$ satisfying $v_h = g_h$ on Γ^{act} there holds the error estimate*

$$\|u - u_h\|_{H^{1/2}(\Gamma)} \le \frac{c_2^A}{c_1^A} \|u - v_h\|_{H^{1/2}(\Gamma)} \tag{3.1}$$

where the constants c_1^A and c_2^A are the ellipticity and boundedness constants of A as given in (1.3)*.*

Proof. From the variational inequality (2.3) we first have

$$\langle f - Au_h, v_h - u_h \rangle_\Gamma \le 0 \quad \text{for all } v_h \in \mathcal{K}_h.$$

By using $\lambda := Au - f \in H^{-1/2}(\Gamma)$ and the $\widetilde{H}^{1/2}(\Gamma)$–ellipticity of A, see (1.3), we further obtain, for all $v_h \in \mathcal{K}_h$,

$$\begin{aligned} c_1^A \|u - u_h\|^2_{H^{1/2}(\Gamma)} &\le \langle A(u - u_h), u - u_h \rangle_\Gamma \\ &= \langle A(u - u_h), u - v_h \rangle_\Gamma + \langle Au - f, v_h - u_h \rangle_\Gamma + \langle f - Au_h, v_h - u_h \rangle_\Gamma \\ &\le \langle A(u - u_h), u - v_h \rangle_\Gamma + \langle \lambda, v_h - u_h \rangle_\Gamma \\ &= \langle A(u - u_h), u - v_h \rangle_\Gamma + \langle \lambda, v_h - g_h \rangle_\Gamma + \langle \lambda, g_h - u_h \rangle_\Gamma \\ &\le \langle A(u - u_h), u - v_h \rangle_\Gamma + \langle \lambda, v_h - g_h \rangle_\Gamma, \end{aligned}$$

where we have used $\lambda \le 0$ and $u_h \le g_h$ to ensure

$$\langle \lambda, g_h - u_h \rangle_\Gamma \le 0.$$

Taking into account $v_h = g_h$ on Γ^{act} and $\lambda = 0$ on $\Gamma^{\text{in}} = \Gamma \backslash \Gamma^{\text{act}}$ we have

$$\langle \lambda, v_h - g_h \rangle_\Gamma = \int_\Gamma \lambda(x)[v_h(x) - g_h(x)]\, ds_x = 0\,.$$

Now the assertion follows from the boundedness of A. ■

It remains to construct $v_h = u_h^* \in \mathcal{K}_h$ in a suitable way to be able to derive an approximation property in $H^{1/2}(\Gamma)$. In particular, we define $u_h^* \in \mathcal{K}_h$ by, see also Fig. 1,

$$u_k^* := \begin{cases} g(x_k) & \text{for } x_k \in \tau_\ell : \tau_\ell \cap \Gamma^{\text{act}} \ne \emptyset, \\ u(x_k) & \text{else.} \end{cases} \tag{3.2}$$

Note that u_h^* coincides with g_h in all boundary elements τ_ℓ which include some part of the active boundary $\Gamma^{\rm act}$. In all other nodes, u_h^* is the piecewise linear interpolation of u. We start to give an error estimate for $u - u_h^*$ in $L_2(\Gamma)$. By $H^\sigma_{\rm pw}(\Gamma)$ we denote the space of all $L_2(\Gamma)$ functions which are piecewise in $H^\sigma(\Gamma_j)$ when $\Gamma = \cup \overline{\Gamma}_j$, $\Gamma_j \cap \Gamma_i = \emptyset$ for $j \neq i$.

Lemma 3.2 *Let $u \in \mathcal{K}$ be the unique solution of the variational inequality* (1.2). *Let $u_h^* \in K_h$ be as constructed in* (3.2). *Assume $u, g \in H^\sigma_{pw}(\Gamma) \cap C(\Gamma)$ for some $\sigma \in (\frac{n-1}{2}, 2]$. Then there holds the error estimate*

$$\|u - u_h^*\|_{L_2(\Gamma)} \,\leq\, c\, h^\sigma \left(|u|^2_{H^\sigma_{pw}(\Gamma)} + |g|^2_{H^\sigma_{pw}(\Gamma)} \right)^{1/2}.$$

Proof. First we recall

$$\|u - u_h^*\|^2_{L_2(\Gamma)} = \int_\Gamma |u(x) - u_h^*(x)|^2 \, ds_x = \sum_{\ell=1}^N \int_{\tau_\ell} |u(x) - u_h^*(x)|^2 ds_x$$

and it remains to consider four different cases:

i. For $\tau_\ell \subset \Gamma_{\rm act}$ we have $u = g$ and $u_h^* = g_h = I_h g$. Hence we have, by using a local interpolation error estimate,

$$\int_{\tau_\ell} |u(x) - u_h^*(x)|^2 \, ds_x = \int_{\tau_\ell} |g(x) - I_h g(x)|^2 \, ds_x \leq c\, h^{2\sigma} \, |g|^2_{H^\sigma(\tau_\ell)}.$$

ii. Next we consider all boundary elements τ_ℓ where $u_h^* = I_h u$ is the piecewise linear interpolation of u. As in the first case we obtain

$$\int_{\tau_\ell} |u(x) - u_h^*(x)|^2 \, ds_x = \int_{\tau_\ell} |u(x) - I_h u(x)|^2 \, ds_x \leq c\, h^{2\sigma} \, |u|^2_{H^\sigma(\tau_\ell)}.$$

It remains to consider two additional cases as depicted in Fig. 1.

iii. We first consider the case of boundary elements τ_ℓ^1 with $\tau_\ell^1 \not\subset \Gamma_{\rm act}$ but $\tau_\ell^1 \cap \Gamma_{\rm act} \neq \emptyset$, i.e. there is a $x_\ell^* \in \tau_\ell^1$ such that $u(x) = g(x)$ for all $x \in \tau_\ell^1$, $|x - x_\ell^*| < \varepsilon$ for some $\varepsilon > 0$. Then,

$$\|u - u_h^*\|_{L_2(\tau_\ell^1)} \leq \|u - g\|_{L_2(\tau_\ell^1)} + \|g - u_h^*\|_{L_2(\tau_\ell^1)} = \|u - g\|_{L_2(\tau_\ell^1)} + \|(I - I_h) g\|_{L_2(\tau_\ell^1)},$$

where the second part again corresponds to the standard local interpolation error. Due to $u(x) = g(x)$ for $x \in U_\varepsilon(x_\ell^*)$ we conclude that also the surface gradients of u and g coincide in x_ℓ^*. Hence, the linear Hermite interpolation polynomials $I_h^1 u = I_h^1 g$ of u and g, which are defined with respect to x_ℓ^*, coincide. With this we conclude

$$\|u - g\|_{L_2(\tau_\ell^1)} \leq \|(I - I_h^1) u\|_{L_2(\tau_\ell^1)} + \|(I - I_h^1) g\|_{L_2(\tau_\ell^1)} \leq c\, h^\sigma \left(|u|_{H^\sigma(\tau_\ell^1)} + |g|_{H^\sigma(\tau_\ell^1)} \right),$$

again by applying standard interpolation error estimates for Hermite interpolation.

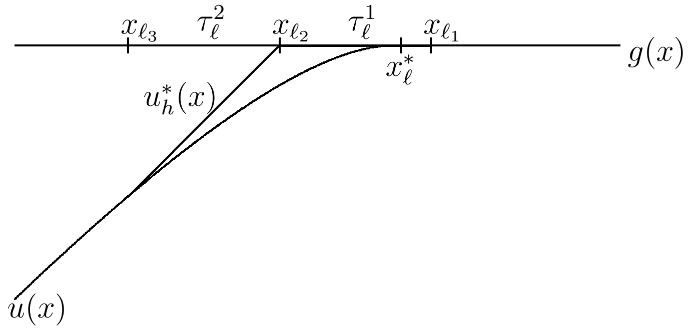

Figure 1: Boundary elements with changing active zones.

iv. Finally we consider the case where the boundary element τ_ℓ^2 does not touch the active part Γ_{act} but joins at least one common node x_{ℓ_2} with a boundary element τ_{ℓ_1}, $\tau_{\ell_1} \cap \Gamma_{\text{act}} \neq \emptyset$. In this case we have

$$\|u - u_h^*\|_{L_2(\tau_\ell^2)} \le \|u - I_h u\|_{L_2(\tau_\ell^2)} + \|I_h u - u_h^*\|_{L_2(\tau_\ell^2)}$$

where the first part again is a standard interpolation error estimate. Let $\eta(x)$ be a sufficiently smooth function satisfying $\eta(x) \in [0,1]$ for all $x \in \tau_\ell^2$, $\eta(x_{\ell_2}) = 1$ for all nodes x_{ℓ_2} with $u_h^*(x_{\ell_2}) = g(x_{\ell_2})$, and $\eta(x_{\ell_3}) = 0$ for all nodes x_{ℓ_3} with $u_h^*(x_{\ell_3}) = u(x_{\ell_3})$, see also Fig. 1. For $w := \eta(u-g)$ we then conclude $I_h w = I_h u - u_h^*$ and therefore

$$\|I_h u - u_h^*\|_{L_2(\tau_\ell^2)} = \|I_h w\|_{L_2(\tau_\ell^2)} \le \|(I - I_h)w\|_{L_2(\tau_\ell^2)} + \|w\|_{L_2(\tau_\ell^2)}$$

follows. As in the third case we further obtain, by using $|\eta| \le 1$,

$$\begin{aligned} \|w\|_{L_2(\tau_\ell^2)} &= \|\eta(u-g)\|_{L_2(\tau_\ell^2)} \le \|u-g\|_{L_2(\tau_\ell^2)} \\ &\le \|u-g\|_{L_2(\tau_\ell^2 \cup \tau_\ell^1)} \le c\, h^\sigma \left(|u|^2_{H^\sigma(\tau_\ell^2 \cup \tau_\ell^1)} + |g|^2_{H^\sigma(\tau_\ell^2 \cup \tau_\ell^1)} \right)^{1/2}. \end{aligned}$$

For the remaining term we first consider the case $\sigma \ge 1$. By using standard interpolation error estimates we then have

$$\begin{aligned} \|(I - I_h)w\|_{L_2(\tau_\ell^2)} &\le c\, h\, \|\nabla_\Gamma w\|_{L_2(\tau_\ell^2)} \\ &\le c\, h \left(\|(\nabla_\Gamma \eta)(u-g)\|_{L_2(\tau_\ell^2)} + \|\eta \nabla_\Gamma (u-g)\|_{L_2(\tau_\ell^2)} \right) \\ &\le c \left(\|u-g\|_{L_2(\tau_\ell^2)} + h\, \|\nabla_\Gamma (u-g)\|_{L_2(\tau_\ell^2)} \right) \\ &\le c\, h^\sigma \left(|u|_{H^\sigma(\tau_\ell^2)} + |g|_{H^\sigma(\tau_\ell^2)} \right). \end{aligned}$$

In the two–dimensional case $n = 2$ and for $\sigma \in (\frac{1}{2}, 1)$ we first have

$$\|(I - I_h)w\|_{L_2(\tau_\ell^2)} \;\leq\; c\, h^\sigma \, |w|_{H^\sigma(\tau_\ell^2)}$$

and it remains to consider the Sobolev–Slobodeckii norm

$$\begin{aligned}
|w|^2_{H^\sigma(\tau_\ell^2)} &= \int_{\tau_\ell^2}\int_{\tau_\ell^2} \frac{[w(x)-w(y)]^2}{|x-y|^{1+2\sigma}} ds_x ds_y \\
&= \int_{\tau_\ell^2}\int_{\tau_\ell^2} \frac{[\eta(x)(u(x)-g(x)) - \eta(y)(u(y)-g(y))]^2}{|x-y|^{1+2\sigma}} ds_x ds_y \\
&= \int_{\tau_\ell^2}\int_{\tau_\ell^2} \frac{[(\eta(x)-\eta(y))(u(x)-g(x)) + \eta(y)(u(x)-g(x)) - (u(y)-g(y))]^2}{|x-y|^{1+2\sigma}} ds_x ds_y \\
&\leq 2\,|u-g|^2_{H^\sigma(\tau_\ell^2)} + 2 \int_{\tau_\ell^2}\int_{\tau_\ell^2} \frac{(\eta(x)-\eta(y))^2 (u(x)-g(x))^2}{|x-y|^{1+2\sigma}} ds_x ds_y \\
&\leq 2\,|u-g|^2_{H^\sigma(\tau_\ell^2)} + 2 \sup_{\xi\in\tau_\ell^2} |\eta'(\xi)|^2 \int_{\tau_\ell^2}\int_{\tau_\ell^2} |x-y|^{1-2\sigma} (u(x)-g(x))^2 ds_x ds_y \\
&\leq 2\,|u-g|^2_{H^\sigma(\tau_\ell^2)} + 2\, c\, h^{-2} \int_{\tau_\ell^2}\int_{\tau_\ell^2} |x-y|^{1-2\sigma} (u(x)-g(x))^2 ds_x ds_y \\
&\leq 2\,|u-g|^2_{H^\sigma(\tau_\ell^2)} + 2\, c\, h^{-2}\, h^{2-2\sigma} \int_{\tau_\ell^2} (u(x)-g(x))^2 ds_x \\
&\leq 2\,|u-g|^2_{H^\sigma(\tau_\ell^2)} + 2\, c\, h^{-2\sigma} \|u-g\|^2_{L_2(\tau_\ell^2)}.
\end{aligned}$$

Now the final error estimate follows as above.

Joining all four cases we have shown the desired error estimate. ■

As in the standard case of a variational equation, and by using an inverse estimate we are now able to give an error estimate in the energy norm.

Lemma 3.3 *Let* $u \in \mathcal{K}$ *be the unique solution of the variational inequality* (1.2). *Let* $u_h^* \in \mathcal{K}_h$ *be as constructed in* (3.2). *Assume* $u, g \in H^\sigma_{pw}(\Gamma) \cap C(\Gamma)$ *for some* $\sigma \in (\frac{n-1}{2}, 2]$. *Let the mesh be globally quasi–uniform. Then there holds the error estimate*

$$\|u - u_h^*\|_{H^{1/2}(\Gamma)} \;\leq\; c\, h^{\sigma - 1/2} \left(|u|^2_{H^\sigma_{pw}(\Gamma)} + |g|^2_{H^\sigma_{pw}(\Gamma)} \right)^{1/2}. \tag{3.3}$$

Proof. Let $P_h u \in S_h^1(\Gamma)$ be the Galerkin projection of u defined as the unique solution of the variational problem

$$\langle u - P_h u, v_h \rangle_{H^{1/2}(\Gamma)} = 0 \quad \text{for all } v_h \in S_h^1(\Gamma).$$

Using standard techniques, see, e.g. [25], we conclude the error estimates

$$\|u - P_h u\|_{H^{1/2}(\Gamma)} \leq c_1\, h^{\sigma-1/2}\, |u|_{H^\sigma_{\rm pw}(\Gamma)}, \quad \|u - P_h u\|_{L_2(\Gamma)} \leq c_2\, h^\sigma\, |u|_{H^\sigma_{\rm pw}(\Gamma)}.$$

Hence, by using the inverse inequality, and all previous error estimates, we conclude

$$\begin{aligned}\|u-u_h^*\|^2_{H^{1/2}(\Gamma)} &\leq 2\,\|u-P_hu\|^2_{H^{1/2}(\Gamma)} + 2\,\|P_hu-u_h^*\|^2_{H^{1/2}(\Gamma)} \\ &\leq c_1\,h^{2\sigma-1}\,|u|^2_{H^\sigma_{\rm pw}(\Gamma)} + c_I\,h^{-1}\,\|P_hu-u_h^*\|^2_{L_2(\Gamma)} \\ &\leq c_1\,h^{2\sigma-1}\,|u|^2_{H^\sigma_{\rm pw}(\Gamma)} + 2c_I\,h^{-1}\Big(\|P_hu-u\|^2_{L_2(\Gamma)} + \|u-u_h^*\|^2_{L_2(\Gamma)}\Big) \\ &\leq c\,h^{2\sigma-1}\Big(|u|^2_{H^\sigma_{\rm pw}(\Gamma)} + |g|^2_{H^\sigma_{\rm pw}(\Gamma)}\Big).\end{aligned}$$

■

Now, combining the error estimates (3.1) and (3.3) we can state the main result of this section.

Theorem 3.4 *Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the unique solutions of the variational inequalities* (1.2) *and* (2.3)*, respectively. Assume $u, g \in H^\sigma_{pw}(\Gamma) \cap C(\Gamma)$ for some $\sigma \in (\frac{n-1}{2}, 2]$. Let the mesh be globally quasi–uniform. Then there holds the error estimate*

$$\|u-u_h\|_{H^{1/2}(\Gamma)} \leq c\,h^{\sigma-1/2}\Big(|u|^2_{H^\sigma_{pw}(\Gamma)} + |g|^2_{H^\sigma_{pw}(\Gamma)}\Big)^{1/2}. \tag{3.4}$$

Note that the maximal value of σ as used in the error estimate (3.4) is determined by the choice of the basis functions, i.e. $\sigma \leq 2$ when using piecewise linears, and by the regularity of the solution $u \in H^\sigma_{\rm pw}(\Gamma)$. The latter may be obtained either from the properties of the underlying physical problem, or from the mapping properties of the involved operator A, see Theorem 2.2.

4 Error estimates in $L_2(\Gamma)$: The Nitsche trick

In this section we present the main result of this paper, the Aubin–Nitsche trick to derive an error estimate in $L_2(\Gamma)$ for the approximate solution of the variational inequality (1.2). Although the basic ideas of the proof follow the considerations for finite element approximations of variational inequalities in $H^1(\Omega)$, see [21, 27], the proof as given here requires several considerations which are different.

4.1 The adjoint problem

To obtain error estimates in lower Sobolev norms we first assume that $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is bijective and bounded for some $s \in (0, \frac{1}{2}]$. For this s we consider a variational inequality which is adjoint with respect to the variational inequality (1.2), and we define

$$\Gamma_h^{\rm act} := \{x \in \Gamma : u_h(x) = g_h(x)\}$$

where $u_h \in \mathcal{K}_h$ is the unique solution of the variational inequality (2.3). For $u \in \mathcal{K}$ being the unique solution of the variational inequality (1.2) we use $\lambda = Au - f$ as defined in

(2.1). Then we introduce the closed and convex set

$$G := \left\{ v \in \widetilde{H}^{1/2}(\Gamma) : v \le 0 \text{ on } \Gamma_h^{\text{act}},\ \langle \lambda, v \rangle_\Gamma \le 0 \right\}$$

and we consider the adjoint problem to find $z \in G$ as the unique solution of the variational inequality

$$\langle Az, v - z \rangle_\Gamma \ge \langle u - u_h, v - z \rangle_{H^{1/2-s}(\Gamma)} = \langle B_{1/2-s}(u - u_h), v - z \rangle_\Gamma \quad \text{for all } v \in G, \tag{4.1}$$

where $B_{1/2-s} : \widetilde{H}^{1/2-s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is the associated Riesz operator.

As for the solution $u \in \mathcal{K}$ of the primal variational inequality (1.2) we first state some regularity result for the solution $z \in G$ of the variational inequality (4.1). For this we rewrite the variational inequality as a saddle point problem, i.e. by using the Lagrange multiplier $\mu \in \widetilde{H}^{-1/2}(\Gamma_h^{\text{act}})$, $\mu \le 0$ on Γ_h^{act}, and $\alpha \in \mathbb{R}_+$, we introduce the Lagrange functional

$$\mathcal{L}(v; \mu, \alpha) := \frac{1}{2}\langle Av, v \rangle_\Gamma - \langle B_{1/2-s}(u - u_h), v \rangle_\Gamma - \langle \mu, v \rangle_{\Gamma_h^{\text{act}}} + \alpha \langle \lambda, v \rangle_\Gamma, \quad v \in \widetilde{H}^{1/2}(\Gamma). \tag{4.2}$$

The Karush–Kuhn–Tucker conditions which are related to the Lagrange functional (4.2) then read to find $(z; \overline{\mu}, \overline{\alpha}) \in \widetilde{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$Az - B_{1/2-s}(u - u_h) - \overline{\mu} + \overline{\alpha}\lambda = 0 \quad \text{on } \Gamma, \tag{4.3}$$

$$z \le 0, \quad \overline{\mu} \le 0, \quad z\,\overline{\mu} = 0 \quad \text{on } \Gamma_h^{\text{act}}, \quad \overline{\mu} = 0 \quad \text{on } \Gamma \backslash \Gamma_h^{\text{act}}, \tag{4.4}$$

$$\overline{\alpha} \ge 0, \quad \langle \lambda, z \rangle_\Gamma \le 0, \quad \overline{\alpha}\langle \lambda, z \rangle_\Gamma = 0. \tag{4.5}$$

Lemma 4.1 *Let $z \in G$ be the unique solution of the adjoint variational inequality* (4.1), *and let $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ be bijective and bounded. Then there holds the regularity estimate*

$$\|z\|_{H^{1/2+s}(\Gamma)} \le c\, \|u - u_h\|_{H^{1/2-s}(\Gamma)}. \tag{4.6}$$

Proof. Let us first define

$$\Gamma_h^{\text{act,z}} := \left\{ x \in \Gamma_h^{\text{act}} : z(x) = 0 \right\}, \quad \text{i.e. } z \in \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_h^{\text{act,z}}), \quad \overline{\mu} = 0 \quad \text{on } \Gamma \backslash \Gamma_h^{\text{act,z}}.$$

Hence we find from (4.3)

$$Az + \overline{\alpha}\lambda = B_{1/2-s}(u - u_h) \quad \text{on } \Gamma \backslash \Gamma_h^{\text{act,z}},$$

and it remains to consider two cases:

i. For $\overline{\alpha} = 0$ we find that $z \in \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_h^{\text{act,z}})$ is the unique solution of the operator equation $Az = B_{1/2-s}(u - u_h)$ and the assertion follows from the assumptions on A, i.e.

$$\|z\|_{H^{1/2+s}(\Gamma)} \le c\, \|Az\|_{H^{-1/2+s}(\Gamma)} = c\, \|B_{1/2-s}(u - u_h)\|_{H^{-1/2+s}(\Gamma)} \le c\, \|u - u_h\|_{H^{1/2-s}(\Gamma)}.$$

ii. For $\alpha > 0$ we conclude $\langle \lambda, z \rangle_\Gamma = 0$ and it remains to consider the subspace

$$\widetilde{H}_0^{1/2}(\Gamma \backslash \Gamma_h^{\mathrm{act},z}) := \left\{ v \in \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_h^{\mathrm{act},z}) : \langle \lambda, v \rangle_\Gamma = 0 \right\}.$$

Again the assertion follows from the mapping properties of A and $B_{1/2-s}$.

■

4.2 Quasi–interpolation

In order to prove error estimates in lower order Sobolev spaces, for $z \in G$ we need to construct a suitable approximation $z_h \in S_h^1(\Gamma)$ which allows an error estimate in negative Sobolev norms, and which retains the inequality constraints on Γ_h^{act}. For this we consider a quasi–interpolation, see also [5, 23]. Since the discrete active set Γ_h^{act} is given as the union of boundary elements τ_ℓ, we define a dual boundary mesh as follows, see Fig. 2.

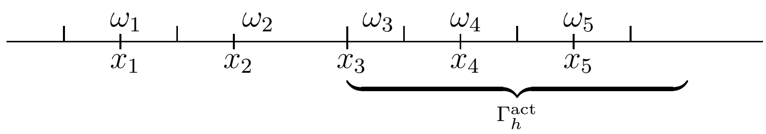

Figure 2: Construction of the dual mesh.

If $x_k \in \Gamma_h^{\mathrm{act}}$ is an interior node of the discrete active set, see for example x_4 and x_5, the dual element ω_k is defined by the midpoints of the adjacent primal elements, in the three–dimensional case we consider the midpoints of the adjacent edges in addition. If x_k is a boundary node of the discrete active set, e.g. x_3, the dual element ω_k is the related part of the primal element in Γ_h^{act}. For all other nodes we define the dual elements accordingly, where the missing parts of the boundary nodes, e.g., x_3, are added to the dual element of a related node of the inactive set, see, e.g., ω_2. For a globally quasi–uniform boundary element mesh Γ_h with mesh size h we conclude that all dual elements are of the same mesh size h.

With respect to the dual boundary element mesh we define the piecewise constant L_2 projection

$$z_h^0(x) = \frac{1}{|\omega_k|} \int_{\omega_k} z(x)\, ds_x \quad \text{for } x \in \omega_k$$

satisfying the local error estimate, for $z \in \widetilde{H}^{1/2+s}(\Gamma)$, $s \in (0, \frac{1}{2}]$,

$$\|z - z_h^0\|_{L_2(\omega_k)} \leq c\, h^{1/2+s}\, |z|_{H^{1/2+s}(\omega_k)}, \tag{4.7}$$

and by applying the standard Aubin–Nitsche trick

$$\|z - z_h^0\|_{\widetilde{H}^{-1}(\omega_k)} \leq c\, h^{3/2+s}\, |z|_{H^{1/2+s}(\omega_k)}. \tag{4.8}$$

For $x \in \omega_k \subset \Gamma_h^{\mathrm{act}}$ we have $z \leq 0$ and therefore $z_h^0 \leq 0$ follows. Now we are in the position to define the quasi–interpolation $z_h \in S_h^1(\Gamma)$,

$$z_h(x) = \sum_{k=1}^{M} z_h^0(x_k)\varphi_k(x) \quad \text{for } x \in \Gamma, \quad z_k := z_h(x_k) \leq 0 \quad \text{for } x_k \in \Gamma_h^{\mathrm{act}}. \tag{4.9}$$

Lemma 4.2 *Let* $z \in \widetilde{H}^{1/2+s}(\Gamma)$, $s \in (0, \frac{1}{2}]$, *and let* $z_h \in S_h^1(\Gamma)$ *be the quasi–interpolation as given in* (4.9). *Then there hold the error estimates*

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \leq c\, h^s\, |z|_{H^{1/2+s}(\Gamma)} \tag{4.10}$$

and

$$\|z - z_h\|_{\widetilde{H}^{-1}(\Gamma)} \leq c\, h^{3/2+s}\, |z|_{H^{1/2+s}(\Gamma)}. \tag{4.11}$$

Proof. For piecewise linear basis functions we have

$$\sum_{j=1}^{M} \varphi_j(x) = 1,$$

and for $x \in \omega_k$ we then conclude

$$z_h(x) = \sum_{j=1}^{M} z_h^0(x_j)\varphi_j(x) = z_h^0(x_k) + \sum_{j=1, j\neq k, \mathrm{supp}\varphi_j \cap \omega_k \neq \emptyset}^{M} [z_h^0(x_j) - z_h^0(x_k)]\varphi_j(x),$$

and therefore

$$\|z - z_h\|_{L_2(\omega_k)} \leq \|z - z_h^0\|_{L_2(\omega_k)} + \sum_{j=1, j\neq k, \mathrm{supp}\varphi_j \cap \omega_k \neq \emptyset}^{M} \left| z_h^0(x_j) - z_h^0(x_k) \right| \|\varphi_j\|_{L_2(\omega_k)}.$$

For what follows, let us recall some basic estimates from the numerical analysis of finite and boundary element methods, see, e.g., [1, 25]. Let τ be the reference element to describe all boundary elements τ_ℓ, in particular for $n = 2$ we have $\tau = (0, 1)$, while for $n = 3$ we have

$$\tau = \left\{ \eta \in \mathbb{R}^2 : 0 < \eta_1 < 1, 0 < \eta_2 < 1 - \eta_1 \right\}.$$

For $x \in \tau_\ell$ we obtain the local parametrisation $x = J_\ell(\eta)$ and for the measure of a boundary element we conclude

$$\Delta_\ell = \int_{\tau_\ell} ds_x = \int_\tau \det J_\ell(\eta)\, d\eta, \quad \text{i.e.} \quad \det J_\ell(\eta) \simeq \Delta_\ell \simeq h^{n-1}.$$

For a boundary element basis function $\varphi_j(x) = \varphi(J_\ell(\eta)) = \widetilde{\varphi}_{\ell,j}(\eta)$ we have

$$\|\varphi_j\|^2_{L_2(\tau_\ell)} \simeq \Delta_\ell \, \|\widetilde{\varphi}_{\ell,j}\|^2_{L_2(\tau)} \simeq h^{n-1},$$

as well as

$$\|\nabla_x \varphi\|^2_{L_2(\tau_\ell)} \simeq \Delta_\ell \, h^{-2} \, \|\nabla_\eta \widetilde{\varphi}_{\ell,j}\|^2_{L_2(\tau)} \simeq h^{n-3}\,.$$

Hence we conclude the estimate

$$\|\varphi_j\|_{L_2(\omega_k)} \leq c\, h^{(n-1)/2},$$

and by using

$$\begin{aligned} c\, h^{(n-1)/2} \, |z_h^0(x_j) - z_h^0(x_k)| & \leq \|z_h^0(x_j) - z_h^0(x_k)\|_{L_2(\omega_k)} \\ & \leq \|z_h^0(x_j) - z\|_{L_2(\omega_k)} + \|z - z_h^0(x_k)\|_{L_2(\omega_k)} \end{aligned}$$

we obtain from the error estimate (4.7)

$$\begin{aligned} \|z - z_h\|_{L_2(\omega_k)} & \leq c\, h^{1/2+s} \, |z|_{H^{1/2+s}(\omega_k)} \\ & \quad + \sum_{j=1, j\neq k, \mathrm{supp}\varphi_j \cap \omega_k \neq \emptyset}^{M} \left[\|z_h^0(x_j) - z\|_{L_2(\omega_k)} + c\, h^{1/2+s} \, |z|_{H^{1/2+s}(\omega_k)} \right]. \end{aligned}$$

For $\mathrm{supp}\varphi_j \cap \omega_k \neq \emptyset$ and $x \in \omega_k$ we have

$$z(x) - z_h^0(x_j) = \frac{1}{|\omega_j|} \int_{\omega_j} [z(x) - z(y)] \, ds_y,$$

and by using standard techniques we obtain

$$\|z - z_h^0(x_j)\|_{L_2(\omega_k)} \leq c\, h^{1/2+s} \, |z|_{H^{1/2+s}(\omega_k \cup \omega_j)}.$$

With this we finally conclude the local error estimate

$$\|z - z_h\|^2_{L_2(\omega_k)} \leq c\, h^{1+2s} \sum_{j=1, \mathrm{supp}\varphi_j \cap \omega_k \neq \emptyset}^{M} |z|^2_{H^{1/2+s}(\omega_j)},$$

and by summation over all elements of the dual mesh we obtain the global error estimate

$$\|z - z_h\|^2_{L_2(\Gamma)} \leq c\, h^{1+2s} \, |z|^2_{H^{1/2+s}(\Gamma)}.$$

Note that in the three–dimensional case we use that all boundary elements are assumed to be shape regular, so that the number of terms in the local error estimate is bounded.

Now the error estimate (4.10) in $H^{1/2}(\Gamma)$ follows by using the standard $H^{1/2}(\Gamma)$ projection on $S_h^1(\Gamma)$, and the inverse inequality, see, e.g. [25], and the proof of Lemma 3.3.

The error estimate (4.11) in $\widetilde{H}^{-1}(\Gamma)$ follows as the error estimate in $L_2(\Gamma)$, instead of (4.7) we now use (4.8), and

$$\begin{aligned}\|\varphi_j\|_{\widetilde{H}^{-1}(\tau_\ell)} &= \sup_{0\neq v\in H^1(\tau_\ell)} \frac{\langle \varphi_j, v\rangle_{\tau_\ell}}{\|v\|_{H^1(\tau_\ell)}} \\ &\simeq \sup_{0\neq \widetilde{v}_\ell\in H^1(\tau)} \frac{\Delta_\ell \langle \widetilde{\varphi}_{\ell,j}, \widetilde{v}_\ell\rangle_\tau}{h^{(n-3)/2}\|\widetilde{v}_\ell\|_{H^1(\tau)}} \simeq h^{(n+1)/2}\|\widetilde{\varphi}_{\ell,j}\|_{\widetilde{H}^{-1}(\tau)} \simeq h^{(n+1)/2}.\end{aligned}$$

With this we finally conclude the local error estimate

$$\|z - z_h\|^2_{\widetilde{H}^{-1}(\omega_k)} \leq c\, h^{3+2s} \sum_{j=1,\mathrm{supp}\varphi_j\cap\omega_k\neq\emptyset}^{M} |z|^2_{H^{1/2+s}(\omega_j)},$$

and by localizing the dual norm in $\widetilde{H}^{-1}(\Gamma)$ we obtain (4.11). ∎

4.3 Error estimates in lower Sobolev norms

Now we are in a position to present error estimates in lower Sobolev norms for the solution u_h of the discrete variational inequality (2.3). The main result as given in Theorem 4.5 is based on the following two estimates.

Lemma 4.3 *Let $u \in \mathcal{K}$ and $u_h \in \mathcal{K}_h$ be the unique solutions of the variational inequalities* (1.2) *and* (2.3)*, respectively. Let $z \in G$ be the unique solution of the adjoint variational inequality* (4.1)*, and let $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$, $s \in (0, \frac{1}{2}]$. Let $z_h \in S^1_h(\Gamma)$ be an appropriate approximation of $z \in G$. Then there holds the error estimate*

$$\begin{aligned}\|u - u_h\|^2_{H^{1/2-s}(\Gamma)} \leq\;& c_2^A\, \|u - u_h\|_{H^{1/2}(\Gamma)}\|z - z_h\|_{H^{1/2}(\Gamma)} \\ &+ c\, \|u - u_h\|_{H^{1/2-s}(\Gamma)}\|g - g_h\|_{H^{1/2-s}(\Gamma)} + \langle A(u - u_h), z_h\rangle_\Gamma.\end{aligned} \tag{4.12}$$

Proof. For $z \in G$ and $u \in \mathcal{K}$ we have for $x \in \Gamma_h^{\mathrm{act}}$

$$\underbrace{z(x)}_{\leq 0} + \underbrace{u(x) - g(x)}_{\leq 0} + \underbrace{g_h(x) - u_h(x)}_{=0} \leq 0,$$

as well as, by using $\lambda \leq 0$ and $u_h \leq g_h$ on Γ,

$$\langle \lambda, z + u - g + g_h - u_h\rangle_\Gamma = \underbrace{\langle \lambda, z\rangle_\Gamma}_{\leq 0 \text{ for } z\in G} + \underbrace{\langle \lambda, u - g\rangle_\Gamma}_{= 0 \text{ due to (2.2)}} + \underbrace{\langle \lambda, g_h - u_h\rangle_\Gamma}_{\leq 0} \leq 0.$$

Hence we can chose $v = z + u - u_h + g_h - g \in G$ as a test function in the variational inequality (4.1) to obtain

$$
\begin{aligned}
\|u - u_h\|^2_{H^{1/2-s}(\Gamma)} &= \langle u - u_h, u - u_h\rangle_{H^{1/2-s}(\Gamma)} \\
&= \langle u - u_h, (z + u - u_h + g_h - g) - z\rangle_{H^{1/2-s}(\Gamma)} + \langle u - u_h, g - g_h\rangle_{H^{1/2-s}(\Gamma)} \\
&\le \langle Az, (z + u - u_h + g_h - g) - z\rangle_\Gamma + \langle u - u_h, g - g_h\rangle_{H^{1/2-s}(\Gamma)} \\
&= \langle A(u - u_h), z\rangle_\Gamma + \langle Az, g_h - g\rangle_\Gamma + \langle u - u_h, g - g_h\rangle_{H^{1/2-s}(\Gamma)} \\
&= \langle A(u - u_h), z - z_h\rangle_\Gamma + \langle A(u - u_h), z_h\rangle_\Gamma + \langle Az, g_h - g\rangle_\Gamma + \langle u - u_h, g - g_h\rangle_{H^{1/2-s}(\Gamma)} \\
&\le c_2^A \|u - u_h\|_{H^{1/2}(\Gamma)}\|z - z_h\|_{H^{1/2}(\Gamma)} + \langle A(u - u_h), z_h\rangle_\Gamma \\
&\qquad + \|Az\|_{H^{-1/2+s}(\Gamma)}\|g - g_h\|_{H^{1/2-s}(\Gamma)} + \|u - u_h\|_{H^{1/2-s}(\Gamma)}\|g - g_h\|_{H^{1/2-s}(\Gamma)}.
\end{aligned}
$$

Now the assertion follows from the boundedness of $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$, and from the regularity result (4.6). ∎

Lemma 4.4 *Let $u \in \mathcal{K}$, $u_h \in \mathcal{K}_h$, and $z \in G$ be the unique solutions of the variational inequalities* (1.2), (2.3), *and* (4.1), *respectively. Let $z_h \in S_h^1(\Gamma)$ be as constructed in* (4.9). *Then there holds the estimate*

$$\langle A(u - u_h), z_h\rangle_\Gamma \le \langle Au - f, z_h - z\rangle_\Gamma \, . \tag{4.13}$$

Proof. By using $\lambda_k = 0$ for $u_k < g_k$ and $\lambda_k \le 0$, $z_k \le 0$ for $u_k = g_k$ we first have

$$\langle f - Au_h, z_h\rangle_\Gamma = (\underline{f} - A_h\underline{u}, \underline{z}) = -(\underline{\lambda}, \underline{z}) = -\sum_{k=1}^{M} \lambda_k z_k = -\sum_{u_k = g_k} \lambda_k z_k \le 0.$$

Hence we obtain

$$
\begin{aligned}
\langle A(u - u_h), z_h\rangle_\Gamma &= \langle Au - f, z_h\rangle_\Gamma + \langle f - Au_h, z_h\rangle_\Gamma \\
&\le \langle Au - f, z_h\rangle_\Gamma \\
&= \langle Au - f, z_h - z\rangle_\Gamma + \underbrace{\langle Au - f, z\rangle_\Gamma}_{\le 0 \text{ for } z \in G} \le \langle Au - f, z_h - z\rangle_\Gamma.
\end{aligned}
$$

∎

Now we state the main theorem of this paper.

Theorem 4.5 *Let $u \in \mathcal{K}$, $u_h \in \mathcal{K}_h$, and $z \in G$ be the unique solutions of the variational inequalities* (1.2), (2.3), *and* (4.1), *respectively. Let $A : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ be bounded for some $s \in (0, \frac{1}{2}]$, and let $z_h \in S_h^1(\Gamma)$ be given as in* (4.9). *Assume $\lambda \in H^{\sigma-1}(\Gamma)$ for some $\sigma \in (\frac{n-1}{2}, 2]$. Then there holds the error estimate*

$$\|u - u_h\|_{H^{1/2-s}(\Gamma)} \le c_1\, h^s\, \|u - u_h\|_{H^{1/2}(\Gamma)} + c_2\, \|g - g_h\|_{H^{1/2-s}(\Gamma)} + c_3\, h^{\sigma-(1/2-s)}\, \|\lambda\|_{H^{\sigma-1}(\Gamma)}. \tag{4.14}$$

Proof. Combing the error estimates (4.12) and (4.13) we first conclude

$$\begin{aligned}\|u-u_h\|^2_{H^{1/2-s}(\Gamma)} \;\leq\; & c_2^A\,\|u-u_h\|_{H^{1/2}(\Gamma)}\|z-z_h\|_{H^{1/2}(\Gamma)} \\ & +c\,\|u-u_h\|_{H^{1/2-s}(\Gamma)}\|g-g_h\|_{H^{1/2-s}(\Gamma)}+\|\lambda\|_{H^{\sigma-1}(\Gamma)}\|z_h-z\|_{\widetilde{H}^{1-\sigma}(\Gamma)}.\end{aligned}$$

By using the error estimates (4.10) and (4.11), an interpolation argument, as well as the regularity estimate (4.6) we obtain the assertion. ∎

Together with the energy error estimate (3.4) we now obtain the final result of this subsection.

Corollary 4.6 *Let $A:\widetilde{H}^{1/2+s}(\Gamma)\to H^{-1/2+s}(\Gamma)$ be bounded for some $s\in(0,\frac{1}{2}]$, and assume $u,g\in H^\sigma_{pw}(\Gamma)$, $\lambda\in H^{\sigma-1}(\Gamma)$ for some $\sigma\in(\frac{n-1}{2},2]$. Then there holds the error estimate*

$$\|u-u_h\|_{H^{1/2-s}(\Gamma)}\leq c\,h^{\sigma-(1/2-s)}\left[|u|^2_{H^\sigma_{pw}(\Gamma)}+|g|^2_{H^\sigma_{pw}(\Gamma)}+\|\lambda\|^2_{H^{\sigma-1}(\Gamma)}\right]^{1/2}. \tag{4.15}$$

In particular for $s=\frac{1}{2}$, i.e. $A:\widetilde{H}^1(\Gamma)\to L_2(\Gamma)$, and when assuming $u,g\in H^2_{\rm pw}(\Gamma)$ and $\lambda\in H^1(\Gamma)$, we obtain the $L_2(\Gamma)$ error estimate

$$\|u-u_h\|_{L_2(\Gamma)}\leq c\,h^2\left[|u|^2_{H^2_{\rm pw}(\Gamma)}+|g|^2_{H^2_{\rm pw}(\Gamma)}+\|\lambda\|^2_{H^1(\Gamma)}\right]^{1/2},$$

i.e., as for the standard approximation property of piecewise linear polynomials we can expect a quadratic order of convergence when measuring the error in the L_2 norm.

5 Semi–smooth Newton method

For the solution of the discrete variational inequality (2.3) we consider the discrete complementary conditions (2.5), i.e.

$$u_k\leq g_k,\quad \lambda_k\leq 0,\quad \lambda_k[u_k-g_k]=0\quad\text{for all }k=1,\ldots,M,\quad \underline{\lambda}=A_h\underline{u}-\underline{f},$$

and which are equivalent to

$$\lambda_k=\min\{0,\lambda_k+c(g_k-u_k)\}\quad\text{for }k=1,\ldots,M,\quad c>0.$$

Hence we have to solve the system of (non)linear equations

$$F_1(\underline{u},\underline{\lambda})=A_h\underline{u}-\underline{\lambda}-\underline{f}=\underline{0},\quad F_2(\underline{u},\underline{\lambda})=\underline{\lambda}-\min\{0,\underline{\lambda}+c(\underline{g}-\underline{u})\}=\underline{0},$$

where the nonlinear equations $F_2(\underline{u},\underline{\lambda})$ have to be considered compenentwise. Since

$$G(x)=\min\{0,x\}$$

is not differentiable in $x = 0$, we introduce the slant derivative

$$G'(x) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0, \end{cases}$$

and the application of a semi–smooth Newton method reads

$$\begin{aligned} \begin{pmatrix} \underline{u}^{m+1} \\ \underline{\lambda}^{m+1} \end{pmatrix} &= \begin{pmatrix} \underline{u}^{m} \\ \underline{\lambda}^{m} \end{pmatrix} \\ &\quad - \begin{pmatrix} A_h & -I \\ cG'(\underline{\lambda}^m + c(\underline{g} - \underline{u}^m)) & I - G'(\underline{\lambda}^m + c(\underline{g} - \underline{u}^m)) \end{pmatrix}^{-1} \begin{pmatrix} F_1(\underline{u}^m, \underline{\lambda}^m) \\ F_2(\underline{u}^m, \underline{\lambda}^m) \end{pmatrix} \end{aligned}$$

where the application of G' has to be understood componentwise. The Newton method requires the solution of the linear system

$$\begin{pmatrix} A_h & -I \\ cG'(\underline{\lambda}^m + c(\underline{g} - \underline{u}^m)) & I - G'(\underline{\lambda}^m + c(\underline{g} - \underline{u}^m)) \end{pmatrix} \begin{pmatrix} \underline{u}^m - \underline{u}^{m+1} \\ \underline{\lambda}^m - \underline{\lambda}^{m+1} \end{pmatrix} = \begin{pmatrix} F_1(\underline{u}^m, \underline{\lambda}^m) \\ F_2(\underline{u}^m, \underline{\lambda}^m) \end{pmatrix},$$

and the first line gives

$$A_h(\underline{u}^m - \underline{u}^{m+1}) - \underline{\lambda}^m + \underline{\lambda}^{m+1} = A_h\underline{u}^m - \underline{\lambda}^m - \underline{f}, \quad \text{i.e.,} \quad A\underline{u}^{m+1} - \underline{\lambda}^{m+1} = \underline{f}.$$

The second equation gives for all $k = 1, \ldots, M$

$$\begin{aligned} [1 - G'(\lambda_k^m + c(g_k - u_k^m))](\lambda_k^m - \lambda_k^{m+1}) + cG'(\lambda_k^m + c(g_k - u_k^m))(u_k^m - u_k^{m+1}) \\ = \lambda_k^m - \min\{0, \lambda_k^m + c(g_k - u_k^m)\}. \end{aligned}$$

For

$$\lambda_k^m + c(g_k - u_k^m) \geq 0, \quad G'(\lambda_k^m + c(g_k - u_k^m)) = 0$$

we then conclude

$$\lambda_k^m - \lambda_k^{m+1} = \lambda_k^m, \quad \text{i.e.} \quad \lambda_k^{m+1} = 0,$$

while for

$$\lambda_k^m + c(g_k - u_k^m) < 0, \quad G'(\lambda_k^m + c(g_k - u_k^m)) = 1$$

we conclude

$$u_k^{m+1} = g_k.$$

This is just the active set strategy, see, e.g., [13, 16], and [15].

6 Applications

6.1 Screen problem

As a first example we consider the variational inequality to find $u \in \mathcal{K}$ such that

$$\langle Du, v - u\rangle_\Gamma \geq \langle f, v - u\rangle_\Gamma \tag{6.1}$$

is satisfied for all $v \in \mathcal{K}$ where $\Gamma = (0, \frac{1}{2})$, $f \equiv 1$, and

$$\mathcal{K} := \left\{ v \in \widetilde{H}^{1/2}(\Gamma) : v(x) \leq 0.4 \text{ for } x \in \Gamma \right\}, \quad \text{i.e. } g \equiv 0.4.$$

In (6.1), $D : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the hypersingular boundary integral operator defined as

$$(Dv)(x) = \frac{1}{2\pi} \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} \log|x - y|\, v(y) ds_y \quad \text{for } x \in \Gamma.$$

Note that the variational inequality (6.1) is related to two–dimensional screen and crack problems, e.g., [26].

Since the variational inequality (6.1) perfectly fits into the framework of the present paper, all theoretical results are valid. In particular, $D : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a bounded and $\widetilde{H}^{1/2}(\Gamma)$–elliptic operator, see [6, 20]. Moreover, $D : \widetilde{H}^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is bounded for all $s \in [0, \frac{1}{2})$ [26, Corollary 1.7]. Hence we can apply Theorem 2.2 for $s = \frac{1}{2} - \varepsilon$ to conclude $u \in H^{1-\varepsilon}(\Gamma)$ for all sufficient small $\varepsilon > 0$. Moreover, by Lemma 4.1 we have $z \in H^{1-\varepsilon}(\Gamma)$ for the solution of the adjoint problem. Hence we can apply Corollary 4.6 for $s = \frac{1}{2} - \varepsilon$ and $\sigma = 1 - \varepsilon$ for all sufficient small $\varepsilon > 0$ to conclude the error estimate

$$\|u - u_h\|_{L_2(\Gamma)} \leq \|u - u_h\|_{H^\varepsilon(\Gamma)} \leq c\, h^{1-2\varepsilon} \left[|u|^2_{H^{1-\varepsilon}_{\text{pw}}(\Gamma)} + |g|^2_{H^{1-\varepsilon}_{\text{pw}}(\Gamma)} + \|\lambda\|^2_{H^{-\varepsilon}(\Gamma)} \right]^{1/2}.$$

Therefore we can expect almost linear convergence. For a numerical experiment we consider a uniform decomposition of the interval $\Gamma = (0, \frac{1}{2})$ into $N = 2^{L+1}$ boundary elements. In addition to the variational inequality (6.1) we also consider the solution of the unconstrained boundary integral equation $Du = f$. Since the exact solutions of both problems are unknown, we chose u_{h_9} as a reference solution to compute approximate errors $\|u_{h_L} - u_{h_9}\|_{L_2(\Gamma)}$, see Table 1. The numerical results show linear convergence for both problems, as expected. Also the number of semi–smooth Newton iterations indicates super–linear convergence. In Fig. 3 we plot the solutions of the unconstrained and of the constrained screen problem.

6.2 Signorini problem

As a second example we consider the Signorini problem for the Laplacian, see, e.g., [24],

$$-\Delta u(x) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad u(x) = g(x) \quad \text{on } \Gamma_D, \quad \frac{\partial}{\partial n_x} u(x) = f(x) \quad \text{on } \Gamma_N,$$

		unconstrained		constrained		
L	N	$\|u_{h_L} - u_{h_9}\|_{L_2(\Gamma)}$		$\|u_{h_L} - u_{h_9}\|_{L_2(\Gamma)}$		Iter
1	4	4.81 –2		4.44 –2		1
2	8	2.41 –2	1.00	2.09 –2	1.09	2
3	16	1.22 –2	0.98	1.04 –2	1.01	2
4	32	6.16 –3	0.99	5.26 –3	0.98	3
5	64	3.08 –3	1.00	2.64 –3	0.99	4
6	128	1.51 –3	1.03	1.30 –3	1.02	6

Table 1: $L_2(\Gamma)$ errors for unconstrained and constrained screen problem.

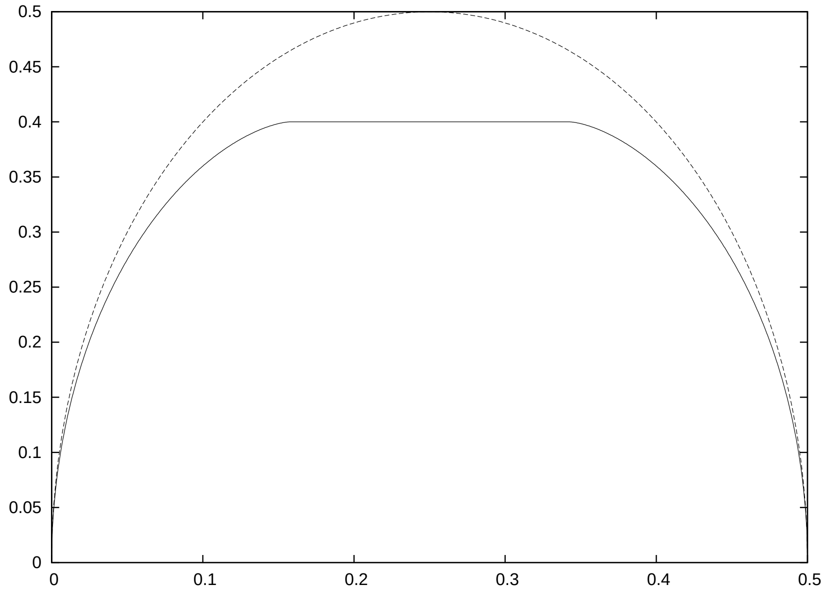

Figure 3: Solutions for the unconstrained and constrained screen problem.

and

$$u(x) \leq g(x), \quad \frac{\partial}{\partial n_x} u(x) \leq f(x), \quad (u(x) - g(x)) \left(\frac{\partial}{\partial n_x} u(x) - f(x) \right) = 0 \quad \text{on } \Gamma_S,$$

where $\Gamma = \partial\Omega$ is the boundary of the Lipschitz domain Ω which is decomposed into mutually disjoint parts Γ_D, Γ_N, and Γ_S. Related to the Laplace equation in Ω we introduce the Dirichlet to Neumann map

$$\frac{\partial}{\partial n} u =: Su \quad \text{on } \Gamma,$$

where the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ can by described, by using boundary integral operators as, see, e.g., [25],

$$S = V^{-1}(\frac{1}{2}I + K) = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K).$$

Note that V is the single layer integral operator, K is the double layer integral operator and K' its adjoint, and D is the hypersingular boundary integral operator. By introducing

$$\mathcal{K} := \left\{ v \in H^{1/2}(\Gamma) : v_{|\Gamma_D} = g,\ v_{|\Gamma_S} \leq g \right\}$$

the solution of the Signorini boundary value problem is equivalent to find the solution $u \in \mathcal{K}$ of the variational inequality

$$\langle Su, v - u \rangle_\Gamma \geq \langle f, v - u \rangle_{\Gamma_S \cup \Gamma_N} \quad \text{for all } v \in \mathcal{K}.$$

Note that the Steklov–Poincaré operator S is $\widetilde{H}^{1/2}(\Gamma_N \cup \Gamma_S)$–elliptic which ensures unique solvability of the variational inequality. Assuming sufficient regularity of the given data one can not expect more than $u \in H^{5/2-\varepsilon}(\Omega)$ for the solution of the Signorini boundary value problem, i.e. $u_{|\Gamma} \in H^{2-\varepsilon}(\Gamma)$. In the case of a Lipschitz boundary Γ we find from the mapping properties of all boundary integral operators [6] that $S : H^1(\Gamma) \to L_2(\Gamma)$. Hence we can apply Corollary 4.6 for $s = \frac{1}{2}$ and $\sigma = 2 - \varepsilon$ to conclude the error estimate, for all sufficient small $\varepsilon > 0$,

$$\|u - u_h\|_{L_2(\Gamma)} \leq c\, h^{2-\varepsilon} \left[|u|^2_{H^{2-\varepsilon}_{\text{pw}}(\Gamma_S)} + |g|^2_{H^{2-\varepsilon}_{\text{pw}}(\Gamma_D \cup \Gamma_S)} + \|f\|^2_{H^{1-\varepsilon}(\Gamma_N \cup \Gamma_S)} \right]^{1/2},$$

i.e. we can expect almost quadratic convergence when using piecewise linear boundary elements. In [24], a proof of the energy error estimate in $H^{1/2}(\Gamma)$ is given for the particular case of a smooth boundary of a bounded domain in two dimensions, and numerical examples are given. For numerical results in the case of contact problems in linear elasticity, see for example [8], and for optimal Dirichlet control problems [22].

References

[1] S. Brenner, R. L. Scott: The Mathematical Theory of Finite Element Methods. Springer, New York, 1994.

[2] H. Brezis: Problémes unilateraux. J. Math. Pures Appl. 51 (1972) 1–168.

[3] F. Brezzi, W. W. Hager, P. A. Raviart: Error estimates for the finite element solution of variational inequalities. Numer. Math. 28 (1977) 431–443.

[4] C. Carstensen, J. Gwinner: FEM and BEM coupling for a nonlinear transmission problem with Signorini contact. SIAM J. Numer. Anal. 34 (1997) 1845–1864.

[5] P. Clement: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. R–2 (1975) 77–84.

[6] M. Costabel: Boundary integral operators on Lipschitz domains: Elementary results. SIAM J. Math. Anal. 19 (1988) 613–626.

[7] C. Eck, J. Jarušek, M. Krbec: Unilateral contact problems. Variational methods and existence theorems. Pure and Applied Mathematics, 270. Chapman & Hall/CRC, Boca Raton, 2005.

[8] C. Eck, O. Steinbach, W. L. Wendland: A symmetric boundary element method for contact problems with friction. Math. Comput. Simulation 50 (1999) 43–61.

[9] R. S. Falk: Error estimates for the approximation of a class of variational inequalities. Math. Comput. 28 (1974) 963–971.

[10] R. Glowinski: Numerical methods for nonlinear variational problems. Springer, Berlin, 1980.

[11] J. Gwinner, E. P. Stephan: A boundary element procedure for contact problems in plane linear elastostatics. RAIRO Model. Math. Anal. Numer. 27 (1993) 457–480.

[12] H. Han: A direct boundary element method for Signorini problems. Math. Comp. 55 (1990) 115–128.

[13] M. Hintermüller, K. Ito, K. Kunisch: The primal–dual active set strategy as a semi–smooth Newton method. SIAM J. Optim. 13 (2002) 865–888.

[14] G. C. Hsiao, W. L. Wendland: The Aubin–Nitsche lemma for integral equations. J. Integral Equations 3 (1981) 299–315.

[15] S. Hüeber, B. Wohlmuth: An optimal a priori error estimate for nonlinear multibody contact problems. SIAM J. Numer. Anal. 43 (2005) 156–173.

[16] K. Ito, K. Kunisch: Semi–smooth Newton methods for the Signorini problem. Appl. Math. 53 (2008) 455–468.

[17] J. L. Lions: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin, Heidelberg, New York, 1971.

[18] J. L. Lions, G. Stampacchia: Variational inequalities. Comm. Pure Appl. Math. 20 (1967) 493–519.

[19] M. Maischak, E. P. Stephan: Adaptive hp–versions of BEM for Signorini problems. Appl. Numer. Math. 54 (2005) 425–449.

[20] W. McLean, O. Steinbach: Boundary element preconditioners for a hypersingular integral equation on an interval. Adv. Comput. Math. 11 (1999) 271–286.

[21] F. Natterer: Optimale L_2–Konvergenz finiter Elemente bei Variationsungleichungen. Bonn. Math. Schrift. 89 (1976) 1–12.

[22] G. Of, T. X. Phan, O. Steinbach: Boundary element methods for Dirichlet boundary control problems. Math. Methods Appl. Sci. 33 (2010) 2187–2205.

[23] L. R. Scott, S. Zhang: Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp. 54 (1990) 483–493.

[24] W. Spann: On the boundary element method for the Signorini problem of the Laplacian. Numer. Math. 65 (1993) 337–356.

[25] O. Steinbach: Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[26] E. P. Stephan, W. L. Wendland: A hypersingular boundary integral method for two–dimensional screen and crack problems. Arch. Rational Mech. Anal. 112 (1990) 363–390.

[27] F.–T. Suttmeier: Numerical solution of variational inequalities by adaptive finite elements. Vieweg+Teubner, Wiesbaden, 2008.

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