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G. Unger

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Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.numerik.math.tu-graz.at`

Convergence analysis of a Galerkin boundary element method for electromagnetic eigenvalue problems

Gerhard Unger

Institut für Numerische Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria

Abstract

In this paper a convergence analysis of a Galerkin boundary element method for eigenvalue problems arising from the time harmonic Maxwell's equations is presented. The interior and the exterior eigenvalue problem with perfect conducting boundary conditions and the transmission eigenvalue problem at a dielectric interface are considered. The underlying operators of the boundary integral formulations of these eigenvalue problems satisfy a so-called T-Gårding's inequality from which the convergence of a conforming Galerkin approximation is in general only guaranteed if the approximation spaces fulfill special requirements. We use recent abstract results for the convergence of the Galerkin approximation of eigenvalue problems for holomorphic T-Gårding operator-valued functions in order to show that two classical boundary element spaces for Maxwell's equations, the Raviart–Thomas and the Brezzi–Douglas–Marini boundary element spaces, satisfy these requirements. Numerical examples are presented, which confirm the theoretical results.

1 Introduction

The numerical solution of electromagnetic eigenvalue problems is an important task in different fields of engineering and technology. In this paper we consider for a given bounded Lipschitz domain $\Omega^{\mathrm{i}} \subset \mathbb{R}^3$ an interior, an exterior and a transmission eigenvalue problem arising from the time harmonic Maxwell's equations with a time variation of the form $e^{-i\omega t}$. The aim is to provide a convergence analysis of a Galerkin boundary element method for these eigenvalue problems.

As interior eigenvalue problem we consider the cavity resonance problem with perfect conducting boundary conditions of the following form: Find $\kappa \in \mathbb{C}$ and $\mathbf{E}^{\mathrm{i}} \in \mathbf{H}(\mathbf{curl}; \Omega^{\mathrm{i}})$, $\mathbf{E}^{\mathrm{i}} \neq 0$, such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{\mathrm{i}} - \kappa^2 \mathbf{E}^{\mathrm{i}} &= 0 \quad \text{in } \Omega^{\mathrm{i}}, \\ \operatorname{div}(\varepsilon \mathbf{E}^{\mathrm{i}}) &= 0 \quad \text{in } \Omega^{\mathrm{i}}, \\ \mathbf{E}^{\mathrm{i}} \times \mathbf{n} &= 0 \quad \text{on } \Gamma := \partial\Omega^{\mathrm{i}}, \end{aligned} \tag{1}$$

where $\kappa = \omega\sqrt{\varepsilon\mu}$ is the wavenumber, ω is the angular frequency, ε is the electric permittivity, μ is the magnetic permeability, and $\mathbf{n}$ is the unit normal vector field on the boundary Γ pointing into the exterior domain $\Omega^{\mathrm{e}} := \mathbb{R}^3 \setminus \overline{\Omega^{\mathrm{i}}}$. We assume throughout this paper that $\varepsilon > 0$ and $\mu > 0$ are constant and that Ω^{i} is simply connected. These assumptions imply that the eigenvalues of the eigenvalue problem (1) are real [24, Thm. 4.18] and non-zero.

The exterior eigenvalue problem, usually referred to as scattering-resonance problem at a perfect conductor, is formulated in Ω^{e} and is given as follows: Find $\kappa \in \mathbb{C}$ and $\mathbf{E}^{\mathrm{e}} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega^{\mathrm{e}})$, $\mathbf{E}^{\mathrm{e}} \neq 0$, such that:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{\mathrm{e}} - \kappa^2 \mathbf{E}^{\mathrm{e}} = 0 \quad \text{in } \Omega^{\mathrm{e}}, \tag{2a}$$

$$\operatorname{div}(\varepsilon \mathbf{E}^{\mathrm{e}}) = 0 \quad \text{in } \Omega^{\mathrm{e}}, \tag{2b}$$

$$\mathbf{E}^{\mathrm{e}} \times \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{2c}$$

$$\mathbf{E}^{\mathrm{e}} \text{ is “outgoing”}. \tag{2d}$$

As radiation condition in (2d) we impose that each Cartesian component of $\mathbf{E}^{\mathrm{e}}$ has outside of any ball $B_{r_0} := \{x : \|\mathbf{x}\| < r_0\}$ which contains Ω^{i} an expansion in terms of the spherical Hankel functions of the first kind $h_n^{(1)}$ of the form

$$(\mathbf{E}^{\mathrm{e}}(\mathbf{x}))[j] = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m}^{(j)} h_n^{(1)}(\kappa r) Y_n^m \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \quad \text{for } r = |\mathbf{x}| > r_0,\ j \in \{1,2,3\}, \tag{3}$$

where Y_n^m are the spherical harmonics. In the case of wavenumbers κ with $0 \leq \arg\kappa < \pi$ the radiation condition (3) for a solution $\mathbf{E} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega^{\mathrm{e}})$ of Maxwell's equations in Ω^{e} is equivalent to that the Cartesian components of $\mathbf{E}$ satisfy the Sommerfeld radiation condition [11, Thm. 2.15], [30, Appendix]. The latter condition is for $0 \leq \arg\kappa < \pi$ again equivalent to that $\mathbf{E}$ satisfies the Silver-Müller radiation condition [11, Thm. 6.8]. The Silver-Müller radiation condition is usually imposed for scattering problems for wavenumbers κ with $0 \leq \arg\kappa < \pi$. But it is well known that for wavenumbers with nonnegative imaginary part the Silver-Müller radiation condition does not not correctly characterize outgoing waves, see e. g., [25, Sect. 1]. Since the eigenvalues of the exterior eigenvalue problem (2) have negative imaginary part, instead of the Silver-Müller radiation condition the radiation condition (3) has to be used.

Boundary integral formulations and boundary element methods have been considered for different kinds of electromagnetic eigenvalue problems. Examples are the interior eigenvalue problem [13, 32], the interior transmission eigenvalue problem [12, 21] or the transmission eigenvalue problem at a dielectric interface [23], to mention just a few. A rigorous numerical analysis of the boundary element approximations of these eigenvalue problems has not been established so far. In this paper we provide a convergence analysis of a Galerkin boundary element approximation of a boundary integral formulation of the interior and exterior eigenvalue problem. In addition we extend the derived results to the transmission eigenvalue problem for the scattering at a dielectric interface. The presented analysis is based on the classical framework of regular approximations of eigenvalue problems for holomorphic Fredholm operator-valued functions [16, 17], which was already applied to

boundary integral formulations of acoustic eigenvalue problems [29] and to coupled FEM-BEM formulations of vibro-acoustic eigenvalue problems [19]. For these cases sufficient conditions for the convergence of conforming Galerkin approximations follow from the fact that the underlying boundary integral operators of the eigenvalue problems satisfy a standard Gårding's inequality. For Maxwell's eigenvalue problems the underlying boundary integral operators satisfy only a so-called T-Gårding's inequality. In such a case additional properties of the approximation spaces are required in order to guarantee convergence. In [15] sufficient conditions for the convergence of a conforming Galerkin approximation for eigenvalue problems for holomorphic T-Gårding operator-valued functions are derived in an abstract setting. In this paper we show that these conditions are satisfied when classical boundary element spaces for Maxwell's equations, namely the Raviart-Thomas and the Brezzi–Douglas–Marini elements, are used for the Galerkin approximation of the proposed boundary integral formulations of the considered eigenvalue problems.

The rest of the paper is organized as follows: in the next section we introduce a boundary integral formulation for the interior and exterior eigenvalue problem and elaborate crucial properties for the spectral analysis. In Sect. 3 we consider a conforming Galerkin approximation for this boundary integral formulation and specify abstract different sufficient conditions for the spectral convergence. In addition we show that these conditions are satisfied when Raviart-Thomas or Brezzi–Douglas–Marini elements are used for the approximation. The results of Sect. 2 and 3 are extended in Sect. 4 to a boundary integral formulation of the transmission eigenvalue problem and its Galerkin approximation. In Sect. 5 numerical examples are presented which confirm the theoretical results.

2 Boundary integral formulation of the interior and exterior eigenvalue problem

In this section we introduce and analyze a boundary integral formulation for the interior and exterior eigenvalue problem (1) and (2). The analysis of the boundary integral formulation of the eigenvalue problem is done in the framework of eigenvalue problem for holomorphic Fredholm operator-valued functions [22, Appendix]. The main reference for the definitions and properties of the required boundary integral operators is the survey paper [9]. Note that the notations in the present paper only partially coincides with the notations in [9].

2.1 Trace spaces

In this subsection we summarize the properties of the trace spaces which we need for the analysis of the boundary integral formulations of the eigenvalue problems. For a detailed presentation of the traces related to Maxwell's equation for Lipschitz domains we refer to the article [8].

For smooth functions $\mathbf{E}^{\mathrm{i/e}} \in \{\mathbf{E}_{|\overline{\Omega^{\mathrm{i/e}}}} : \mathbf{E} \in \mathbf{C}_0^\infty(\mathbb{R}^3)\}$ we define the interior/exter-ior

tangential trace operators $\gamma_\tau^{\mathrm{i/e}}$ and $\pi_\tau^{\mathrm{i/e}}$ by

$$\gamma_\tau^{\mathrm{i/e}}\mathbf{E}^{\mathrm{i/e}} := \mathbf{E}^{\mathrm{i/e}}_{|\Gamma} \times \mathbf{n} \qquad \text{and} \qquad \pi_\tau^{\mathrm{i/e}}\mathbf{E}^{\mathrm{i/e}} := \mathbf{n} \times \mathbf{E}^{\mathrm{i/e}}_{|\Gamma} \times \mathbf{n}.$$

The operators $\gamma_\tau^{\mathrm{i/e}}$ and $\pi_\tau^{\mathrm{i/e}}$ can be extended for $s \in (0,1)$ to continuous operators

$$\begin{aligned}\gamma_\tau^{\mathrm{i/e}} &: \mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})/\mathbf{H}^{s+1/2}_{\mathrm{loc}}(\Omega^{\mathrm{e}}) \to V_\gamma^s := \gamma_\tau^{\mathrm{i}}(\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})),\\ \pi_\tau^{\mathrm{i/e}} &: \mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})/\mathbf{H}^{s+1/2}_{\mathrm{loc}}(\Omega^{\mathrm{e}}) \to V_\pi^s := \pi_\tau^{\mathrm{i}}(\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})),\end{aligned}$$

where V_γ^s and V_π^s are endowed with the norms

$$\begin{aligned}\|\boldsymbol{\psi}\|_{V_\gamma^s} &:= \inf_{\mathbf{E}\in\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})}\{\|\mathbf{E}\|_{\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})} : \gamma_\tau^{\mathrm{i}}(\mathbf{E}) = \boldsymbol{\psi}\},\\ \|\boldsymbol{\chi}\|_{V_\pi^s} &:= \inf_{\mathbf{E}\in\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})}\{\|\mathbf{E}\|_{\mathbf{H}^{s+1/2}(\Omega^{\mathrm{i}})} : \pi_\tau^{\mathrm{i}}(\mathbf{E}) = \boldsymbol{\chi}\}.\end{aligned}$$

The dual space of V_γ^s/V_π^s, $s \in (0,1)$, is denoted by V_γ^{-s}/V_π^{-s}. For $s = 0$ we set $V_\gamma^0 = V_\pi^0 := \mathbf{L}^2_\tau(\Gamma) := \{\boldsymbol{\psi} \in \mathbf{L}^2(\Gamma) : \boldsymbol{\psi}\cdot\mathbf{n} = 0\}$. Finally, we define the space

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \{\boldsymbol{\psi} \in V_\pi^{-\frac{1}{2}} : \mathrm{div}_\Gamma\,\boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma)\}$$

endowed with the graph norm $\|\boldsymbol{\psi}\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\boldsymbol{\psi}\|^2_{V_\pi^{-\frac{1}{2}}} + \|\,\mathrm{div}_\Gamma\,\boldsymbol{\psi}\|^2_{H^{-\frac{1}{2}}(\Gamma)}$. The operator div_Γ is the surface divergence operator and for $\boldsymbol{\psi} \in V_\pi^{-\frac{1}{2}}$ it is defined via duality by

$$\langle \mathrm{div}_\Gamma\,\boldsymbol{\psi}, \varphi_{|\Gamma}\rangle_{H^{-\frac{3}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)} = \langle \boldsymbol{\psi}, \pi_\tau\nabla\varphi\rangle_{V_\pi^{-\frac{1}{2}}\times V_\pi^{\frac{1}{2}}}, \qquad \varphi \in H^2(\Omega^{\mathrm{i}}).$$

The space $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is a Hilbert space and the tangential trace operators γ_τ^{i} and γ_τ^{e} can be extended such that

$$\gamma_\tau^{\mathrm{i}} : \mathbf{H}(\mathbf{curl};\Omega^{\mathrm{i}}) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma), \quad \gamma_\tau^{\mathrm{e}} : \mathbf{H}_{\mathrm{loc}}(\mathbf{curl};\Omega^{\mathrm{e}}) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma),$$

are continuous, surjective and possess a continuous right inverse [8, Thm. 4.1]. The anti-symmetric pairing

$$\langle \boldsymbol{\psi}, \boldsymbol{\chi}\rangle_{\tau,\Gamma} := \int_\Gamma (\boldsymbol{\psi}\times\mathbf{n})\cdot\boldsymbol{\chi}\,ds, \quad \boldsymbol{\psi},\boldsymbol{\chi} \in \mathbf{L}^2_{\mathrm{t}}(\Gamma),$$

can be extended to $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ such that $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ becomes its own dual [10, Thm. 3.3], i.e., there exists a linear and isometric isomorphism $J_\times : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to (\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$ such that

$$\langle \boldsymbol{\psi}, \boldsymbol{\chi}\rangle_{\tau,\Gamma} = (J_\times\boldsymbol{\psi})(\boldsymbol{\chi}) \qquad \text{for all } \boldsymbol{\psi},\boldsymbol{\chi} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma).$$

The operator $J_\times : \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to (\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$ is the extension of the mapping $\tilde{J}_\times : \mathbf{L}^2_\tau(\Gamma) \to \mathbf{L}^2_\tau(\Gamma)$ defined by $\tilde{J}_\times(\boldsymbol{\psi}) := \boldsymbol{\psi}\times\mathbf{n}$, see [10, Thm. 3.3]. Since $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is

a Hilbert space we can identify the pairing $\langle \cdot, \overline{\cdot} \rangle_{\tau,\Gamma}$ with the inner product $(\cdot,\cdot)_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$ by

$$\langle \boldsymbol{\psi}, \overline{\boldsymbol{\chi}} \rangle_{\tau,\Gamma} = (JJ_\times \boldsymbol{\psi}, \boldsymbol{\chi})_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \qquad \boldsymbol{\psi}, \boldsymbol{\chi} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \tag{4}$$

where $J : (\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))' \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is a linear, isometric isomorphism. In the following we will use the shorthand notation

$$\mathbf{V} := \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma).$$

As additional trace we introduce the trace $\gamma_{\mathrm{N}}^{\mathrm{i/e}} := \gamma_\tau^{\mathrm{i/e}} \circ \mathbf{curl}$, which is a linear and continuous mapping from $\mathbf{H}(\mathbf{curl}^2;\Omega^{\mathrm{i}})/\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2;\Omega^{\mathrm{e}})$ to $\mathbf{V}$, and for which the integration by parts formula

$$\pm \int_{\Omega^{\mathrm{i/e}}} \mathbf{curl}\, \mathbf{E}^{\mathrm{i/e}} \cdot \mathbf{curl}\, \boldsymbol{\Phi} - \mathbf{curl}\, \mathbf{curl}\, \mathbf{E}^{\mathrm{i/e}} \cdot \boldsymbol{\Phi}\, d\mathbf{x} = \langle \gamma_{\mathrm{N}}^{\mathrm{i/e}} \mathbf{E}^{\mathrm{i/e}}, \gamma_\tau^{\mathrm{i/e}} \boldsymbol{\Phi} \rangle_{\tau,\Gamma} \tag{5}$$

holds for all $\mathbf{E}^{\mathrm{i/e}} \in \mathbf{H}(\mathbf{curl}^2;\Omega^{\mathrm{i}})/\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2;\Omega^{\mathrm{e}})$ and all $\boldsymbol{\Phi} \in \mathbf{C}_0^\infty(\mathbb{R}^3)$ [9, p. 96, Eq. (23)].

2.2 Derivation and analysis of the boundary integral formulation

The boundary integral formulation of the eigenvalue problems (1) and (2) is based on the Stratton-Chu representation formula for the solution of Maxwell's equations. For exterior problems this formula is in the literature only considered for wavenumbers with non-negative imaginary part and together with the Silver-Müller radiation condition; see, e.g., [9, Sect. 4], [20, Thm. 5.49], [26, Sect. 5.5]. For wavenumbers with negative imaginary part the Stratton-Chu representation formula is also valid if instead of the Silver-Müller radiation condition the radiation condition (3) is imposed. This can be shown in the same way as it is done for positive wavenumbers in [9, Sect. 4, p. 95–97] by considering the Cartesian components of the solution of Maxwell's equations. Since these have to satisfy the Helmholtz equation, the representation formula for outgoing solutions of the Helmholtz equation, which is also valid for wavenumbers with negative imaginary part [30, Appendix, Cor. 6.5], yields together with the integration by parts formula (5) the Stratton-Chu representation formula in the following form: any solution $\mathbf{E}$ of Maxwell's equations in $\Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}}$ with wavenumber $\kappa \in \mathbb{C} \setminus \{0\}$ which satisfies the radiation condition (3) is given by

$$\mathbf{E}(\mathbf{x}) = -\left(\boldsymbol{\Psi}_{\mathrm{DL}}(\kappa)\left(\gamma_\tau^{\mathrm{e}}\mathbf{E} - \gamma_\tau^{\mathrm{i}}\mathbf{E}\right)\right)(\mathbf{x}) - \left(\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\left(\gamma_{\mathrm{N}}^{\mathrm{e}}\mathbf{E} - \gamma_{\mathrm{N}}^{\mathrm{i}}\mathbf{E}\right)\right)(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}}, \tag{6}$$

where

$$\left(\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi}\right)(\mathbf{x}) := \left(\boldsymbol{\Psi}_{\mathbf{A}}(\kappa)\boldsymbol{\psi}\right)(\mathbf{x}) + \frac{1}{\kappa^2}\nabla\left(\Psi_V(\kappa)\,\mathrm{div}_\Gamma\,\boldsymbol{\psi}\right)(\mathbf{x}), \quad \mathbf{x} \in \Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}},$$

is the Maxwell single layer potential and where

$$\left(\boldsymbol{\Psi}_{\mathrm{DL}}(\kappa)\boldsymbol{\psi}\right)(\mathbf{x}) := \mathbf{curl}\left(\boldsymbol{\Psi}_{\mathbf{A}}(\kappa)\boldsymbol{\psi}\right)(\mathbf{x}) \quad \mathbf{x} \in \Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}},$$

is the Maxwell double layer potential. Here, $\boldsymbol{\Psi}_{\mathbf{A}}(\kappa)$ and $\Psi_V(\kappa)$ are the vectorial and scalar single layer potentials related to the Helmholtz equation, which have the integral representations

$$(\boldsymbol{\Psi}_{\mathbf{A}}(\kappa)\boldsymbol{\psi}))(\mathbf{x}) := \int_\Gamma \boldsymbol{\psi}(\mathbf{y}) E_\kappa(\mathbf{x}-\mathbf{y}) ds_{\mathbf{y}}, \qquad (\Psi_V(\kappa)\phi))(\mathbf{x}) := \int_\Gamma \phi(\mathbf{y}) E_\kappa(\mathbf{x}-\mathbf{y}) ds_{\mathbf{y}}$$

with $E_\kappa(\mathbf{x}) = \exp(i\kappa\|\mathbf{x}\|)/4\pi\|\mathbf{x}\|$.

Let $(\kappa^{\mathrm{i}}, \mathbf{E}^{\mathrm{i}})$ be an eigenpair of (1) and let us extend $\mathbf{E}^{\mathrm{i}}$ in Ω^{e} by zero, then the Stratton-Chu representation formula (6) gives

$$\Big((\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa^{\mathrm{i}}))(\gamma_{\mathrm{N}}^{\mathrm{i}}\,\mathbf{E}^{\mathrm{i}})\Big)(\mathbf{x}) = \begin{cases} \mathbf{E}^{\mathrm{i}}(\mathbf{x}), & \mathbf{x}\in\Omega^{\mathrm{i}}, \\ 0, & \mathbf{x}\in\Omega^{\mathrm{e}}. \end{cases} \tag{7}$$

If $(\kappa^{\mathrm{e}}, \mathbf{E}^{\mathrm{e}})$ is an eigenpair of (2) and if we extend $\mathbf{E}^{\mathrm{e}}$ by zero in Ω^{i}, then we have

$$-\Big((\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa^{\mathrm{e}}))(\gamma_{\mathrm{N}}^{\mathrm{e}}\,\mathbf{E}^{\mathrm{e}})\Big)(\mathbf{x}) = \begin{cases} 0, & \mathbf{x}\in\Omega^{\mathrm{i}}, \\ \mathbf{E}^{\mathrm{e}}(\mathbf{x}), & \mathbf{x}\in\Omega^{\mathrm{e}}. \end{cases} \tag{8}$$

We consider a boundary integral formulation of the eigenvalue problems (1) and (2) in terms of the single layer boundary integral operator $\mathsf{S}(\kappa)$ which is defined by

$$\mathsf{S}(\kappa)\boldsymbol{\psi} := \tfrac{1}{2}\left(\gamma_\tau^{\mathrm{i}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa) + \gamma_\tau^{\mathrm{e}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\right)\boldsymbol{\psi}, \quad \boldsymbol{\psi}\in\mathbf{V}.$$

The operator $\mathsf{S}(\kappa) : \mathbf{V} \to \mathbf{V}$ is linear and continuous [9, Cor. 2] and it holds $\mathsf{S}(\kappa) = \gamma_\tau^{\mathrm{i}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa) = \gamma_\tau^{\mathrm{e}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)$ [9, Thm. 7]. Further, we have the following representation [9, Eq. (31)]

$$\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma} = -\langle \boldsymbol{\xi}, \mathbf{A}(\kappa)\boldsymbol{\psi}\rangle_{\tau,\Gamma} + \frac{1}{\kappa^2}\langle \mathrm{div}_\Gamma\,\boldsymbol{\xi}, V(\kappa)\,\mathrm{div}_\Gamma\,\boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}},$$

where $\mathbf{A}(\kappa) := \gamma_\tau^{\mathrm{i}}\,\boldsymbol{\Psi}_{\mathbf{A}}(\kappa)$ and where $V(\kappa)$ is the single layer boundary integral operator of the Helmholtz equation. The pairing $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing of $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ By applying the tangential trace to (7) and (8) we see that $(\kappa^{\mathrm{i}}, \mathbf{E}^{\mathrm{i}})$ and $(\kappa^{\mathrm{e}}, \mathbf{E}^{\mathrm{e}})$ satisfy the following boundary integral equation

$$\mathsf{S}(\kappa^{\mathrm{i/e}})(\gamma_{\mathrm{N}}^{\mathrm{i/e}}\mathbf{E}^{\mathrm{i/e}}) = 0. \tag{9}$$

As boundary integral formulation of the eigenvalue problems (1) and (2) we consider the following eigenvalue problem: Find $\kappa\in\mathbb{C}\setminus\{0\}$ and $\boldsymbol{\psi}\in\mathbf{V}\setminus\{0\}$ such that:

$$\mathsf{S}(\kappa)\boldsymbol{\psi} = 0. \tag{10}$$

Note that this eigenvalue problem is nonlinear with respect to the eigenvalue parameter κ. The eigenvalue problem (10) is referred to as eigenvalue problem for the operator-valued function $\mathsf{S} : \mathbb{C}\setminus\{0\} \to \mathcal{L}(\mathbf{V})$. Here $\mathcal{L}(\mathbf{V})$ denotes the space of linear and bounded operators mapping from $\mathbf{V}$ into itself. The equivalence of the eigenvalue problem for S with the interior and exterior eigenvalue problem (1) and (2) is specified next.

Proposition 2.1. *The following assertions hold true:*

(i) Suppose that $(\kappa, \mathbf{E})$ *is an eigenpair of either the interior eigenvalue problem* (1) *or the exterior eigenvalue problem* (2)*. Then* $(\kappa, \mathbf{E})$ *is an eigenpair of the eigenvalue problem for* S*.*

(ii) Let $(\kappa, \boldsymbol{\psi})$ *be an eigenpair of the eigenvalue problem for* S*. If* κ *is real, then it is an eigenvalue of the interior eigenvalue problem* (1) *and* $(\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi})_{|\Omega^{\mathrm{i}}}$ *is a corresponding eigenfunction. Otherwise,* κ *is an eigenvalue of* (2) *and* $(\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi})_{|\Omega^{\mathrm{e}}}$ *is a corresponding eigenfunction.*

(iii) The geometric multiplicity of an eigenvalue of (1) *or of* (2) *coincides with the geometric multiplicity as eigenvalue of* (10)*.*

Proof. The assertion (i) has been already shown, see (9). Suppose now that $(\kappa, \boldsymbol{\psi})$ is an eigenpair of (10). We define $\mathbf{E} = \boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi}$ in $\Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}}$. Then $\gamma_\tau^{\mathrm{i}}\,\mathbf{E} = \gamma_\tau^{\mathrm{e}}\,\mathbf{E} = \mathsf{S}(\kappa)\boldsymbol{\psi} = 0$. It remains to show that $\mathbf{E}_{|\Omega^{\mathrm{i}}} \neq 0$ for $\kappa \in \mathbb{R}$ and $\mathbf{E}_{|\Omega^{\mathrm{e}}} \neq 0$ for $\kappa \in \mathbb{C} \setminus \mathbb{R}$.

Suppose that $\kappa \in \mathbb{R}$. Then $\mathbf{E}_{|\Omega^{\mathrm{e}}} = 0$ because of the unique solvability of the related exterior boundary value problem [20, Cor. 5.63]. From this we get $\gamma_{\mathrm{N}}^{\mathrm{e}}\,\mathbf{E} = 0$ and the jump relation $\gamma_{\mathrm{N}}^{\mathrm{e}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi} - \gamma_{\mathrm{N}}^{\mathrm{i}}\,\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa)\boldsymbol{\psi} = -\boldsymbol{\psi}$ [9, Thm. 7] implies $\mathbf{E}_{|\Omega^{\mathrm{i}}} \neq 0$.

Suppose now that κ is non-real. Then $\mathbf{E}_{|\Omega^{\mathrm{i}}} = 0$ because otherwise $(\kappa, \mathbf{E}_{|\Omega^{\mathrm{i}}})$ would be an interior eigenpair which is not possible since the square of all interior eigenvalues is

positive [24, Thm. 4.18]. From the jump relation of the single layer potential we get $\gamma_{\mathrm{N}}^{\mathrm{e}}\,\mathbf{E} = \boldsymbol{\psi}$. Hence we have $\mathbf{E}_{|\Omega^{\mathrm{e}}} \neq 0$.

Assume that $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \mathbf{V}$ are two linearly independent eigenfunctions corresponding to an eigenvalue κ of the eigenvalue problem for S. Define $\mathbf{E}_i = (\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa))\boldsymbol{\psi}_i, i = 1, 2$. Then for any $(\alpha_1, \alpha_2) \in \mathbb{C}\times\mathbb{C}$ with $(\alpha_1, \alpha_2) \neq (0, 0)$ we have $\alpha_1\mathbf{E}_1 + \alpha_2\mathbf{E}_2 = (\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa))(\alpha_1\boldsymbol{\psi}_1 + \alpha_2\boldsymbol{\psi}_2) \neq 0$ in $\Omega^{\mathrm{i}} \cup \Omega^{\mathrm{e}}$ because otherwise the assertion (ii) would imply that $\alpha_1\boldsymbol{\psi}_1 + \alpha_2\boldsymbol{\psi}_2 = 0$. The proof for the other direction can be done analogously. □

For the analysis of the eigenvalue problem for S and its Galerkin approximation it is crucial that the single layer boundary integral operator $\mathsf{S}(\kappa)$ satisfies a generalized Gårding's inequality for all wavenumbers $\kappa \in \mathbb{C}\setminus\{0\}$. This result is well known for wavenumbers with nonnegative imaginary part, see, e. g., [9, Lem. 10]. We will show that this result also holds true for wavenumbers with negative imaginary part. Basically we will use the same arguments as in the case of wavenumbers with nonnegative imaginary part. Substantial for the proof is the direct sum decomposition

$$\mathbf{V} = \mathcal{X} \oplus \mathcal{N}, \tag{11}$$

where $\mathcal{X}$ and $\mathcal{N}$ are closed subspaces of $\mathbf{V}$ with $\mathcal{X} \subset V_\pi^{-\frac{1}{2}}$ and $\mathcal{N} = (\ker \mathrm{div}_\Gamma) \cap \mathbf{V}$ [9, Lem. 2]. We denote by R and Z the associated continuous projectors onto $\mathcal{X}$ and $\mathcal{N}$, respectively. An equivalent norm in $\mathbf{V}$ is given by

$$(\|\,\mathsf{Z}\,\cdot\|^2_{V_\pi^{-\frac{1}{2}}} + \|\,\mathrm{div}_\Gamma\,\mathsf{R}\,\cdot\|^2_{H^{-\frac{1}{2}}(\Gamma)})^{1/2}, \tag{12}$$

see [10, Thm. 3.4]. Further, we define the operator

$$\mathsf{T} := \mathsf{R} - \mathsf{Z} : \mathbf{V} \to \mathbf{V}, \tag{13}$$

which is by construction an isomorphism.

Lemma 2.2. *Let $\kappa \in \mathbb{C} \setminus \{0\}$.*

a) If $\mathrm{Re}(\kappa) \neq 0$, then there exists a compact operator $C(\kappa) : \mathbf{V} \to \mathbf{V}$ and a constant $c(\kappa) > 0$ such that the following generalized Gårding's inequality is satisfied

$$|\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \mathsf{T}\,\overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma} + \langle C(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma}| \geq c(\kappa)\|\boldsymbol{\psi}\|^2_{\mathbf{V}} \quad \textit{for all } \boldsymbol{\psi} \in \mathbf{V}. \tag{14}$$

b) If $\kappa = i\alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$, then there exists a compact operator $C_1(\kappa) : \mathbf{V} \to \mathbf{V}$ and a constant $c_1(\kappa) > 0$ such that the following Gårding's inequality holds

$$|\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma} + \langle C_1(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma}| \geq c_1(\kappa)\|\boldsymbol{\psi}\|^2_{\mathbf{V}} \quad \textit{for all } \boldsymbol{\psi} \in \mathbf{V}. \tag{15}$$

Proof. a) The proof follows [9, Lem. 10], but there only positive wavenumbers have been considered. Let us define the operator $\hat{\mathsf{S}}(\kappa)$ by

$$\langle \hat{\mathsf{S}}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma} := -\langle \boldsymbol{\xi}, \mathbf{A}(0)\boldsymbol{\psi}\rangle_{\tau,\Gamma} + \frac{1}{\kappa^2}\langle \mathrm{div}_\Gamma\, \boldsymbol{\xi}, V(0)\,\mathrm{div}_\Gamma\, \boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

Then $\mathsf{S}(\kappa) - \hat{\mathsf{S}}(\kappa)$ is compact [9, Cor. 4]. Using the decomposition of $\mathbf{V}$ in (11) and that $\mathrm{div}_\Gamma\, \mathsf{Z}\,\boldsymbol{\chi} = 0$ for all $\boldsymbol{\chi} \in \mathbf{V}$, we get

$$\langle \hat{\mathsf{S}}(\kappa)\boldsymbol{\psi}, \mathsf{T}\boldsymbol{\xi}\rangle_{\tau,\Gamma} = K(\boldsymbol{\psi}, \boldsymbol{\xi}) + \langle \mathsf{Z}\,\boldsymbol{\xi}, \mathbf{A}(0)\,\mathsf{Z}\,\boldsymbol{\psi}\rangle_{\tau,\Gamma} + \frac{1}{\kappa^2}\langle \mathrm{div}_\Gamma\, \mathsf{R}\,\boldsymbol{\xi}, V(0)\,\mathrm{div}_\Gamma\, \mathsf{R}\,\boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}},$$

where $K(\boldsymbol{\psi}, \boldsymbol{\xi})$ is defined by

$$K(\boldsymbol{\psi}, \boldsymbol{\xi}) := -\langle \mathsf{R}\,\boldsymbol{\xi}, \mathbf{A}(0)\,\mathsf{R}\,\boldsymbol{\psi}\rangle_{\tau,\Gamma} + \langle \mathsf{Z}\,\boldsymbol{\xi}, \mathbf{A}(0)\,\mathsf{R}\,\boldsymbol{\psi}\rangle_{\tau,\Gamma} - \langle \mathsf{R}\,\boldsymbol{\xi}, \mathbf{A}(0)\,\mathsf{Z}\,\boldsymbol{\psi}\rangle_{\tau,\Gamma}.$$

In [9, Lem. 9] it is shown that $K : \mathbf{V} \to \mathbf{V}$ is compact. Therefore the operator $C(\kappa) := \mathsf{T}^*(\hat{\mathsf{S}}(\kappa) - \mathsf{S}(\kappa)) - K$ is also compact. Here T^* denotes the adjoint with respect to $\langle \cdot, \overline{\cdot}\rangle_{\tau,\Gamma}$. From the ellipticity results for $\mathbf{A}(0)$ and $V(0)$ [9, Lem. 8] and with the norm equivalence (12) we obtain

$$\begin{aligned}
&|\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \mathsf{T}\,\overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma} + \langle C(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma}| \\
&\qquad = \Big|\langle \mathsf{Z}\,\overline{\boldsymbol{\psi}}, \mathbf{A}(0)\,\mathsf{Z}\,\boldsymbol{\psi}\rangle_{\tau,\Gamma} + \frac{1}{\kappa^2}\langle \mathrm{div}_\Gamma\, \mathsf{R}\,\overline{\boldsymbol{\psi}}, V(0)\,\mathrm{div}_\Gamma\, \mathsf{R}\,\boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}}\Big| \\
&\geq \left|\frac{1}{\kappa}\right| \min\{|\,\mathrm{Re}(\kappa)|, |\,\mathrm{Re}\left(\frac{\overline{\kappa}}{|\kappa|^2}\right)|\} \left(c_{\mathbf{A}(0)}\|\,\mathsf{Z}\,\boldsymbol{\psi}\|^2_{V_\pi^{-\frac{1}{2}}} + c_{V(0)}\|\,\mathrm{div}_\Gamma\, \mathsf{R}\,\boldsymbol{\psi}\|^2_{H^{-\frac{1}{2}}(\Gamma)}\right) \\
&\qquad = c(\kappa)\|\boldsymbol{\psi}\|^2_{\mathbf{V}},
\end{aligned}$$

where $c_{\mathbf{A}(0)}, c_{V(0)} > 0$. Here we used that

$$\left|\alpha + \frac{\beta}{\kappa^2}\right| = \left|\frac{1}{\kappa}\right| \left|\kappa\alpha + \frac{\overline{\kappa}\beta}{\overline{\kappa}\kappa}\right| \geq \left|\frac{1}{\kappa}\right| \min\{|\operatorname{Re}(\kappa)|, |\operatorname{Re}\left(\frac{\overline{\kappa}}{|\kappa|^2}\right)|\} \, |\alpha + \beta|$$

holds for all $\alpha, \beta \geq 0$.

b) Let $\kappa = i\alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$. With the same arguments as in part a) one can show that the operator $C_1(\kappa) := \hat{\mathsf{S}}(\kappa) - \mathsf{S}(\kappa) - K_1$ is compact, where

$$K_1(\boldsymbol{\psi}, \boldsymbol{\xi}) := \langle \hat{\mathsf{S}}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma} + \langle \mathsf{Z}\boldsymbol{\xi}, \mathbf{A}(0)\,\mathsf{Z}\boldsymbol{\psi}\rangle_{\tau,\Gamma} - \frac{1}{\kappa^2}\langle \operatorname{div}_\Gamma \mathsf{R}\boldsymbol{\xi}, V(0)\operatorname{div}_\Gamma \mathsf{R}\boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

With the ellipticity results for $\mathbf{A}(0)$ and $V(0)$ [9, Lem. 8] we get

$$\begin{aligned}
&|\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma} + \langle C_1(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}\rangle_{\tau,\Gamma}| \\
&\qquad = |-\langle \mathsf{Z}\overline{\boldsymbol{\psi}}, \mathbf{A}(0)\,\mathsf{Z}\boldsymbol{\psi}\rangle_{\tau,\Gamma} - \frac{1}{\alpha^2}\langle \operatorname{div}_\Gamma \overline{\boldsymbol{\psi}}, V(0)\operatorname{div}_\Gamma \boldsymbol{\psi}\rangle_{-\frac{1}{2},\frac{1}{2}}| \\
&\qquad\qquad \geq \min\{1, \alpha^{-2}\}\left(c_{\mathbf{A}(0)}\|\,\mathsf{Z}\boldsymbol{\psi}\|^2_{V_\pi^{-\frac{1}{2}}} + c_{V(0)}\|\operatorname{div}_\Gamma \mathsf{R}\boldsymbol{\psi}\|^2_{H^{-\frac{1}{2}}(\Gamma)}\right).
\end{aligned}$$

The inequality (15) follows from the norm equivalence (12). □

From Lem. 2.2 it follows that the operator $(T(\kappa))^*\,\mathsf{S}(\kappa)$ is a Fredholm operator of index zero for all $\kappa \in \mathbb{C}\setminus\{0\}$, where $T(\kappa) = I$ for $\operatorname{Re}\kappa = 0$ and otherwise $T(\kappa) = \mathsf{T}$. This implies that also the operator $\mathsf{S}(\kappa)$ is a Fredholm operator of index zero for all $\kappa \in \mathbb{C} \setminus \{0\}$. Next we consider the properties of the mapping $\kappa \mapsto \mathsf{S}(\kappa)$.

Lemma 2.3. *The function* $\mathsf{S} : \mathbb{C} \setminus \{0\} \to \mathcal{L}(\mathbf{V})$, $\kappa \mapsto \mathsf{S}(\kappa)$, *is holomorphic.*

Proof. It is sufficient to show that the mapping

$$\kappa \mapsto \langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma}$$

is holomorphic as mapping from $\mathbb{C} \setminus \{0\}$ into $\mathbb{C}$ for all $\boldsymbol{\psi}, \boldsymbol{\xi}$ of a dense subspace of $\mathbf{V}$, see Theorem III.3.12 in Kato [18] and the remark following it. Let us choose $\gamma_\tau^{\mathrm{i}}(\mathbf{C}_0^\infty(\mathbb{R}^3))$ as dense subspace of $\mathbf{V}$. Then we can use the following integral representation of the pairing $\langle \cdot, \cdot\rangle_{\tau,\Gamma}$ and of the boundary integral operator $\mathsf{S}(\kappa)$ [9, Eq. 32]:

$$\begin{aligned}
\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma} = &-\int_\Gamma \int_\Gamma E(\kappa)(\mathbf{x}-\mathbf{y})\boldsymbol{\psi}(\mathbf{x})\cdot\boldsymbol{\xi}(\mathbf{y})ds_{\mathbf{y}}ds_{\mathbf{x}} \\
&+ \frac{1}{\kappa^2}\int_\Gamma\int_\Gamma E(\kappa)(\mathbf{x}-\mathbf{y})\operatorname{div}_\Gamma\boldsymbol{\psi}(\mathbf{x})\operatorname{div}_\Gamma\boldsymbol{\xi}(\mathbf{y})ds_{\mathbf{y}}ds_{\mathbf{x}}. \quad (16)
\end{aligned}$$

The holomorphy of the mapping $\kappa \mapsto \langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \boldsymbol{\xi}\rangle_{\tau,\Gamma}$ follows now from the holomorphy of $E(\kappa)(\cdot) = \frac{e^{i\kappa\|\cdot\|}}{4\pi\|\cdot\|} = \frac{1}{4\pi}\sum_{n=0}^\infty \frac{(i\kappa)^n}{n!}\|\cdot\|^{n-1}$ by interchanging the order of integration and summation in (16). □

Lem. 2.2 and Lem. 2.3 enable us to analyze the eigenvalue problem for S in the framework of eigenvalue problems for holomorphic Fredholm operator-valued functions.

2.3 Notations and properties of eigenvalue problems for holomorphic Fredholm operator-valued functions

In this subsection we introduce notations of eigenvalue problems for holomorphic Fredholm operator-valued functions and summarize important properties of such kind of eigenvalue problems. Our main reference is [22, Appendix]. Let X be a Hilbert space and let $\Lambda \subset \mathbb{C}$ be open and connected. We assume that $A : \Lambda \to \mathcal{L}(X)$ is a holomorphic operator-valued function and that $A(\lambda) : X \to X$ is a Fredholm operator with index zero for all $\lambda \in \Lambda$. The set

$$\rho(A) := \{\lambda \in \Lambda : \exists A(\lambda)^{-1} \in \mathcal{L}(X)\}$$

is called the resolvent set of A. In the following we will assume that the resolvent set of A is not empty. The complement of the resolvent set $\rho(A)$ in Λ is called the spectrum $\sigma(A)$. A number $\lambda \in \sigma(A)$ is an eigenvalue of A if there exists a non-trivial $x \in X \setminus \{0\}$ such that

$$A(\lambda)x = 0.$$

The element x is called an eigenelement of A corresponding to the eigenvalue λ. The spectrum $\sigma(A)$ has no cluster points in Λ [14, Corollary IV.8.4] and each $\lambda \in \sigma(A)$ is an eigenvalue of A which follows from the Fredholm alternative. The dimension of the null space $\ker A(\lambda)$ of an eigenvalue λ is called the geometric multiplicity of λ. An ordered collection of elements $x_0, x_1, \ldots, x_{m-1}$ in X is called a Jordan chain of λ if x_0 is an eigenelement corresponding to λ and if

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda) x_{n-j} = 0 \quad \text{for all } n = 0, 1, \ldots, m-1 \tag{17}$$

is satisfied, where $A^{(j)}$ denotes the jth derivative. The length of any Jordan chain of an eigenvalue is finite [22, Lem. A.8.3]. Elements of any Jordan chain of an eigenvalue λ are called generalized eigenelements of λ. The closed linear hull of all generalized eigenelements of an eigenvalue λ is called generalized eigenspace of λ and is denoted by $G(A, \lambda)$. The dimension of the generalized eigenspace $G(A, \lambda)$ is finite [22, Prop. A.8.4] and it is referred to as algebraic multiplicity of λ.

2.4 Adjoint eigenvalue problem for S

We consider the adjoint eigenvalue problem for S in order to show that the eigenvalues of S are symmetric about the imaginary axis. Further, we will show that if $\boldsymbol{\psi}$ is an eigenelement of S that then $\overline{\boldsymbol{\psi}}$ is an eigenelement of the adjoint problem. This result will be employed for the error estimates of the Galerkin approximation of the eigenvalue problem for S.

We formulate the adjoint eigenvalue problem for S with respect to the pairing $\langle \cdot, \overline{\cdot} \rangle_{\tau,\Gamma}$. Let us define the so-called adjoint function $\mathsf{S}^*(\kappa) := (\mathsf{S}(\overline{\kappa}))^*$ for $\kappa \in \mathbb{C} \setminus \{0\}$. The adjoint eigenvalue problem for S is then given by

$$\mathsf{S}^*(\kappa)\boldsymbol{\nu} = 0.$$

The Fredholm alternative implies that $\kappa \in \sigma(\mathsf{S})$ if and only if $\overline{\kappa} \in \sigma(\mathsf{S}^*)$.

Lemma 2.4. *The following holds true:*

(i) $(\mathsf{S}(\kappa))^* = -\,\mathsf{S}(-\overline{\kappa})$, *i. e.,* $\mathsf{S}^*(\kappa) = -\,\mathsf{S}(-\kappa)$.

(ii) $\kappa \in \sigma(\mathsf{S})$ *if and only if* $-\overline{\kappa} \in \sigma(\mathsf{S})$.

(iii) $(\kappa, \boldsymbol{\psi})$ *is an eigenpair of* S *if and only if* $(\overline{\kappa}, \overline{\boldsymbol{\psi}})$ *is an eigenpair of* S^*.

(iv) The ordered collection $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1 \ldots, \boldsymbol{\psi}_m$ *is a Jordan chain corresponding to* κ *for* S *if and only if* $\overline{\boldsymbol{\psi}}_0, -\overline{\boldsymbol{\psi}}_1, \overline{\boldsymbol{\psi}}_2 \ldots, (-1)^m \overline{\boldsymbol{\psi}}_m$ *is a Jordan chain corresponding to* $\overline{\kappa}$ *for* S^*

Proof. In the proof of the assertions we employ the identity:

$$e^{i\kappa} = e^{i\,\mathrm{Re}(\kappa)} e^{-\,\mathrm{Im}(\kappa)} = \overline{e^{-i\,\mathrm{Re}(\kappa)}}\;\overline{e^{-\,\mathrm{Im}(\kappa)}} = \overline{e^{-i(\mathrm{Re}(\kappa) - i\,\mathrm{Im}(\kappa))}} = \overline{e^{-i\overline{\kappa}}}. \tag{18}$$

(i) It is sufficient to show that

$$\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\xi}} \rangle_{\tau,\Gamma} = \langle \boldsymbol{\psi}, -\overline{\mathsf{S}(-\overline{\kappa})\boldsymbol{\xi}} \rangle_{\tau,\Gamma} \tag{19}$$

holds for all $\boldsymbol{\psi}, \boldsymbol{\xi} \in \gamma_\tau^{\mathrm{i}}(\mathbf{C}_0^\infty(\mathbb{R}^3))$. Let $\boldsymbol{\psi}, \boldsymbol{\xi} \in \gamma_\tau^{\mathrm{i}}(\mathbf{C}_0^\infty(\mathbb{R}^3))$, then we can use the integral representation for the the pairings in (19) and get with (18):

$$\begin{aligned}\langle \mathsf{S}(\kappa)\boldsymbol{\psi}, \overline{\boldsymbol{\xi}} \rangle_{\tau,\Gamma} = &- \int_\Gamma \int_\Gamma \frac{e^{i\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|} \boldsymbol{\psi}(\mathbf{x}) \cdot \overline{\boldsymbol{\xi}(\mathbf{y})} ds_{\mathbf{y}} ds_{\mathbf{x}} \\ &+ \frac{1}{\kappa^2} \int_\Gamma \int_\Gamma \frac{e^{i\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|} \,\mathrm{div}_\Gamma\, \boldsymbol{\psi}(\mathbf{x})\, \mathrm{div}_\Gamma\, \overline{\boldsymbol{\xi}(\mathbf{y})} ds_{\mathbf{y}} ds_{\mathbf{x}} = \langle \overline{\mathsf{S}(-\overline{\kappa})\boldsymbol{\xi}}, \boldsymbol{\psi} \rangle_{\tau,\Gamma}.\end{aligned}$$

From the anti-symmetry of the pairing, the assertion follows.

(ii) The Fredholm alternative implies that $\kappa \in \sigma(\mathsf{S})$ if and only if $(\mathsf{S}(\kappa))^*\boldsymbol{\nu} = 0$ for some $\boldsymbol{\nu} \in \mathbf{V} \setminus \{0\}$. The latter is by the result in (i) equivalent to $\mathsf{S}(-\overline{\kappa})\boldsymbol{\nu} = 0$.

(iii) Because of (18) we have $\overline{\mathsf{S}(\kappa)\boldsymbol{\xi}} = \mathsf{S}(-\overline{\kappa})\overline{\boldsymbol{\xi}}$. If $(\kappa, \boldsymbol{\psi})$ is an eigenpair of S, then we get by i)

$$0 = \overline{\mathsf{S}(\kappa)\boldsymbol{\psi}} = \mathsf{S}(-\overline{\kappa})\overline{\boldsymbol{\psi}} = -(\mathsf{S}(\kappa))^*\overline{\boldsymbol{\psi}} = -\,\mathsf{S}^*(\overline{\kappa})\overline{\boldsymbol{\psi}}.$$

(iv) Again, integral representations of the pairing $\langle \cdot, \cdot \rangle_{\tau,\Gamma}$ and of the functions S and S^* show that $\mathsf{S}^{*(j)}(\kappa) = (-1)^{j+1}(\mathsf{S}^{(j)}(-\kappa))$ and $\overline{(\mathsf{S}^{(j)}(-\kappa))\boldsymbol{\xi}} = \mathsf{S}^{(j)}(-\overline{\kappa})\overline{\boldsymbol{\xi}}$. From this, the assertion follows directly from the definition of the generalized eigenelements. □

3 Galerkin approximation of the interior and exterior eigenvalue problem

For the approximation of the eigenvalue problem for S we consider a conforming Galerkin method. Let $\{\mathbf{V}_h\}_{h\in\mathbb{H}}$ be the sequence of finite dimensional subspaces of $\mathbf{V}$ employed in

the Galerkin method. The Galerkin approximation of the eigenvalue problem (10) then reads as follows: Find $\kappa_h \in \mathbb{C}$ and $\boldsymbol{\psi}_h \in \mathbf{V}_h \setminus \{0\}$ such that

$$\langle \mathsf{S}(\kappa_h)\boldsymbol{\psi}_h, \overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma} = 0 \quad \text{for all } \boldsymbol{\chi}_h \in \mathbf{V}_h. \tag{20}$$

For the convergence analysis it is convenient to represent the Galerkin eigenvalue problem (20) in terms of the inner product in $\mathbf{V}$ in the form

$$(\mathsf{J}\,\mathsf{S}(\kappa_h)\boldsymbol{\psi}_h, \boldsymbol{\chi}_h)_{\mathbf{V}} = 0 \quad \text{for all } \boldsymbol{\chi}_h \in \mathbf{V}_h, \tag{21}$$

where $\mathsf{J} := JJ_\times : \mathbf{V} \to \mathbf{V}$ is the isometric isomorphism in (4) which relates the pairing $\langle \cdot, \overline{\cdot}\rangle_{\tau,\Gamma}$ to the inner product $(\cdot,\cdot)_{\mathbf{V}}$ by $\langle \cdot, \overline{\cdot}\rangle_{\tau,\Gamma} = (\mathsf{J}\,\cdot, \cdot)_{\mathbf{V}}$. Let $P_h : \mathbf{V} \to \mathbf{V}_h$ be the orthogonal projection, then $(\kappa_h, \boldsymbol{\psi}_h)$ is an eigenpair of the Galerkin eigenvalue problem (20) if and only if it is an eigenpair of

$$P_h\,\mathsf{J}\,\mathsf{S}(\kappa_h)\boldsymbol{\psi}_h = 0. \tag{22}$$

Since the eigenvalues and eigenvectors of the eigenvalue problems for S and $\mathsf{J}\,\mathsf{S}$ are identical, the convergence analysis will be carried out for the approximation of the eigenvalues and eigenvectors of the eigenvalue problem $\mathsf{J}\,\mathsf{S}(\kappa)\boldsymbol{\psi} = 0$ by the eigenvalues and eigenvectors of the eigenvalue problem (22).

For the sequence $\{\mathbf{V}_h\}_{h\in\mathbb{H}}$ we assume that the following two conditions are satisfied:

(A1) $\|P_h\boldsymbol{\chi} - \boldsymbol{\chi}\|_{\mathbf{V}} \to 0$ as $h \to 0$ for all $\boldsymbol{\chi} \in \mathbf{V}$.

(A2) For T as given in (13) there exists a sequence $\{\mathsf{T}_h\}_{h\in\mathbb{H}}$, $\mathsf{T}_h : \mathbf{V}_h \to \mathbf{V}_h$ linear and continuous, such that

$$\sup_{\boldsymbol{\psi}_h \in \mathbf{V}_h\setminus\{0\}} \frac{\|(\mathsf{T} - \mathsf{T}_h)\boldsymbol{\psi}_h\|_{\mathbf{V}}}{\|\boldsymbol{\psi}_h\|_{\mathbf{V}}} \to 0 \quad \text{as } h \to 0. \tag{23}$$

The assumption (A1) is the standard approximation assumption for the sequence of the approximation spaces of a Galerkin method. For eigenvalue problems for holomorphic Fredholm operator-valued functions where the underlying operators satisfy a standard Gårding's inequality, the assumption (A1) ensures already the convergence of the Galerkin approximations [29, Sect. 4]. The additional assumption (A2) is needed since the operator $\mathsf{S}(\kappa)$ satisfies only a generalized Gårding's inequality. The assumption (A1) together with the assumption (A2) guarantees a regular approximation of the operator $\mathsf{J}\,\mathsf{S}(\kappa)$ by the sequence $\{P_h\,\mathsf{J}\,\mathsf{S}(\kappa)P_h\}_{h\in\mathbb{H}}$ which follows from an abstract result in [15, Lem. 2.6]. For convenience of the reader we show this result for $\mathsf{S}(\kappa)$ in Thm. 3.1. The property of the regular approximation is required in order to apply the abstract spectral convergence results from [16, 17] to the Galerkin eigenvalue problem (20).

Theorem 3.1. *Suppose that $(\mathbf{V}_h)_{h\in\mathbb{H}}$ is sequence of subspaces of $\mathbf{V}$ for which the conditions (A1) and (A2) are satisfied. Let $\kappa \in \mathbb{C} \setminus \{0\}$ and define $\mathsf{S}_h(\kappa) := P_h\,\mathsf{J}\,\mathsf{S}(\kappa)P_h$. Then the sequence $(\mathsf{S}_h(\kappa))_{h\in\mathbb{H}}$ is a regular approximation of $\mathsf{J}\,\mathsf{S}(\kappa)$ in the sense of [16], i. e., the following two assertions are satisfied:*

a) For any sequence $(\boldsymbol{\psi}_h)_{h\in\mathbb{H}}$, $\boldsymbol{\psi}_h \in \mathbf{V}_h$, which converges to some $\boldsymbol{\psi} \in \mathbf{V}$ as $h \to 0$, it follows that $\mathsf{S}_h(\kappa)\boldsymbol{\psi}_h \to \mathsf{J}\,\mathsf{S}(\kappa)\boldsymbol{\psi}$ as $h \to 0$.

b) If the sequence $(\mathsf{S}_h(\kappa)\boldsymbol{\psi}_h)_{h\in\mathbb{H}}$, where $\boldsymbol{\psi}_h \in \mathbf{V}_h$ with $\|\boldsymbol{\psi}_h\|_{\mathbf{V}} \leq 1$ for all $h \in \mathbb{H}$, has a convergent subsequence, then the sequence $(\boldsymbol{\psi}_h)_{h\in\mathbb{H}}$ has already a convergent subsequence.

In order to proof Thm. 3.1, we need the following auxiliary result.

Lemma 3.2. *Suppose that $(\mathbf{V}_h)_{h\in\mathbb{H}}$ is a sequence of subspaces of $\mathbf{V}$ for which the conditions (A1) and (A2) are satisfied. Let $\kappa \in \mathbb{C}$ with $\mathrm{Re}(\kappa) \neq 0$ and let $C(\kappa)$ be the compact operator as in Lem. 2.2. Define $\widetilde{C}(\kappa) := (\mathsf{T}^*)^{-1}C(\kappa)$. Then there exist an $h_1 \in \mathbb{H}$ and a $c_1(\kappa) > 0$ such that $P_h\,\mathsf{J}(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))P_h : \mathbf{V}_h \to \mathbf{V}_h$ is invertible and*

$$\|(P_h\,\mathsf{J}(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))P_h)^{-1}\|_{\mathcal{L}(\mathbf{V}_h)} \leq c_1(\kappa)$$

for all $h \leq h_1$.

Proof. We proceed as in [15, proof of Lem. 2.6]. For sufficiently small $h \in \mathbb{H}$ the operator $\mathsf{T}_h : \mathbf{V}_h \to \mathbf{V}_h$ is uniformly bounded from below by a constant $c_0 > 0$ since from

$$\|\mathsf{T}_h\boldsymbol{\psi}_h\|_{\mathbf{V}} = \|\mathsf{T}\boldsymbol{\psi}_h - (\mathsf{T}\boldsymbol{\psi}_h - \mathsf{T}_h\boldsymbol{\psi}_h)\|_{\mathbf{V}} \geq \|\mathsf{T}\boldsymbol{\psi}_h\|_{\mathbf{V}} - \|(\mathsf{T} - \mathsf{T}_h)\boldsymbol{\psi}_h\|_{\mathbf{V}}$$

it follows from assumption (A2) and the bijectivity of T that there exists a constant $h_0 > 0$ such that for all $h \leq h_0$

$$\inf_{\substack{\boldsymbol{\psi}_h\in\mathbf{V}_h\\ \|\boldsymbol{\psi}_h\|_{\mathbf{V}}=1}} \|\mathsf{T}_h\boldsymbol{\psi}_h\|_{\mathbf{V}} \geq \inf_{\substack{\boldsymbol{\psi}\in\mathbf{V}\\ \|\boldsymbol{\psi}\|_{\mathbf{V}}=1}} \|\mathsf{T}\boldsymbol{\psi}\|_{\mathbf{V}} - \sup_{\substack{\boldsymbol{\psi}_h\in\mathbf{V}_h\\ \|\boldsymbol{\psi}_h\|_{\mathbf{V}}=1}} \|(\mathsf{T} - \mathsf{T}_h)\boldsymbol{\psi}_h\|_{\mathbf{V}} > c_0.$$

This implies that T_h is bijective for $h \leq h_0$. Moreover, T_h is uniformly bounded from above since $\|\mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)} \leq \|\mathsf{T}\|_{\mathcal{L}(\mathbf{V})} + \|\mathsf{T} - \mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)} \leq \tilde{c}$ for all $h \in \mathbb{H}$.

Let $\boldsymbol{\psi}_h \in \mathbf{V}_h \setminus \{0\}$, then we obtain for $h \leq h_0$:

$$\begin{aligned}
\sup_{\boldsymbol{\chi}_h\in\mathbf{V}_h\setminus\{0\}} \frac{|\langle(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))\boldsymbol{\psi}_h, \overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma}|}{\|\boldsymbol{\psi}_h\|_{\mathbf{V}}\|\boldsymbol{\chi}_h\|_{\mathbf{V}}} &\geq \sup_{\boldsymbol{\chi}_h\in\mathbf{V}_h\setminus\{0\}} \frac{|\langle(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))\boldsymbol{\psi}_h, \mathsf{T}_h\overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma}|}{\|\mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)}\|\boldsymbol{\psi}_h\|_{\mathbf{V}}\|\boldsymbol{\chi}_h\|_{\mathbf{V}}} \\
&\geq \sup_{\boldsymbol{\chi}_h\in\mathbf{V}_h\setminus\{0\}} \frac{|\langle(\mathsf{S}(\kappa) + \widetilde{C}(\kappa)))\boldsymbol{\psi}_h, \mathsf{T}\overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma}|}{\|\mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)}\|\boldsymbol{\psi}_h\|_{\mathbf{V}}\|\boldsymbol{\chi}_h\|_{\mathbf{V}}} - \frac{|S(\kappa) + \widetilde{C}(\kappa)\|_{\mathcal{L}(\mathbf{V})}\|\mathsf{T} - \mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)}}{\|\mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)}} \\
&\geq \frac{\left|\langle(\mathsf{T}^*\,\mathsf{S}(\kappa) + C(\kappa))\,\boldsymbol{\psi}_h, \overline{\boldsymbol{\psi}}_h\rangle_{\tau,\Gamma}\right|}{\tilde{c}\|\boldsymbol{\psi}_h\|_{\mathbf{V}}\|\boldsymbol{\psi}_h\|_{\mathbf{V}}} - \frac{\|S(\kappa) + \widetilde{C}(\kappa)\|_{\mathcal{L}(\mathbf{V})}\|\mathsf{T} - \mathsf{T}_h\|_{\mathcal{L}(\mathbf{V}_h)}}{c_0}. \qquad (24)
\end{aligned}$$

By Lem. 2.2 the minuend in (24) is uniformly bounded from below by a positive constant. The subtrahend converges by assumption (A2) to zero as $h \to 0$. Hence, we get from (24) that there exist constants $h_1, \tilde{c}_1(\kappa) > 0$ such that

$$\inf_{\boldsymbol{\psi}_h\in\mathbf{V}_h\setminus\{0\}} \sup_{\boldsymbol{\chi}_h\in\mathbf{V}_h\setminus\{0\}} \frac{|\langle(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))\boldsymbol{\psi}_h, \overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma}|}{\|\boldsymbol{\psi}_h\|_{\mathbf{V}}\|\boldsymbol{\chi}_h\|_{\mathbf{V}}} > \frac{1}{\tilde{c}_1(\kappa)} \quad \text{for all } h \leq h_1.$$

It is well known that the inf-sup condition implies that for each given $\boldsymbol{f}_h \in \mathbf{V}_h$ the variational problem

$$\langle (\mathsf{S}(\kappa) + \widetilde{C}(\kappa))\boldsymbol{\psi}_h, \overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma} = \langle \boldsymbol{f}_h, \overline{\boldsymbol{\chi}}_h\rangle_{\tau,\Gamma} \quad \text{for all } \boldsymbol{\chi}_h \in \mathbf{V}_h \tag{25}$$

has a unique solution $\boldsymbol{\psi}_h$ with $\|\boldsymbol{\psi}_h\|_{\mathbf{V}} \le c_1(\kappa)\|\boldsymbol{f}_h\|_{\mathbf{V}}$ for all $h \le h_1$, see, e. g., [33]. Since the variational problem (25) is equivalent to the problem $P_h\,\mathsf{J}(\mathsf{S}(\kappa) + \widetilde{C}(\kappa))P_h\boldsymbol{\psi}_h = \mathsf{J}\boldsymbol{f}_h$, the assertion follows. □

Proof of Thm. 3.1. We have to show that that the sequence $(\mathsf{S}_h(\kappa))_{h\in\mathbb{H}}$, where $\mathsf{S}_h(\kappa) = P_h\,\mathsf{J}\mathsf{S}(\kappa)P_h$, is a regular approximation of $\mathsf{J}\mathsf{S}(\kappa)$.

a) Let $(\boldsymbol{\psi}_h)_{h\in\mathbb{H}}$ be a sequence with $\boldsymbol{\psi}_h \in \mathbf{V}_h$ and such that it converges to some $\boldsymbol{\psi} \in \mathbf{V}$ as $h \to 0$. From $P_h\boldsymbol{\psi}_h = \boldsymbol{\psi}_h$ and $\|(I - P_h)\boldsymbol{\chi}\|_{\mathbf{V}} \to 0$ as $h \to 0$ for all $\boldsymbol{\chi} \in \mathbf{V}$, it follows

$$\|\,\mathsf{S}_h(\kappa)\boldsymbol{\psi}_h - \mathsf{J}\mathsf{S}(\kappa)\boldsymbol{\psi}\|_{\mathbf{V}} \le \|P_h\,\mathsf{J}\mathsf{S}(\kappa)(\boldsymbol{\psi}_h - \boldsymbol{\psi})\|_{\mathbf{V}} + \|(P_h - I)\,\mathsf{J}\mathsf{S}(\kappa)\boldsymbol{\psi}\|_{\mathbf{V}} \to 0 \text{ as } h \to 0.$$

b) If $\mathrm{Re}(\kappa) = 0$, then $\mathsf{S}(\kappa)$ satisfies by Lem. 2.2 b) a standard Gårding's inequality. This implies that the Galerkin approximation is regular, see [31, Sect. 2, Prop. 5].

Let now $\mathrm{Re}(\kappa) \neq 0$. Suppose that $(\mathsf{S}_h(\kappa)\boldsymbol{\psi}_h)_{h\in\mathbb{H}}$, $\boldsymbol{\psi}_h \in \mathbf{V}_h$ and $\|\boldsymbol{\psi}_h\|_{\mathbf{V}} \le 1$ for all $h \in \mathbb{H}$, has a convergent subsequence $(\mathsf{S}_{h'}(\kappa)\boldsymbol{\psi}_{h'})_{h'\in\mathbb{H}'}$. Let $\boldsymbol{\varphi} \in \mathbf{V}$ with $\|\,\mathsf{S}(\kappa)\boldsymbol{\psi}_{h'} - \boldsymbol{\varphi}\|_{\mathbf{V}} \to 0$ as $h' \to 0$. Further, let $\widetilde{C}(\kappa) = (\mathsf{T}^*)^{-1}C(\kappa)$ be as defined in Lem. 3.2. Further, we define the operators

$$\widetilde{\mathsf{S}}(\kappa) := \mathsf{S}(\kappa) + \widetilde{C}(\kappa) \quad \text{and} \quad \widetilde{\mathsf{S}}_h(\kappa) := P_h\,\mathsf{J}\widetilde{\mathsf{S}}(\kappa)P_h.$$

Since $\widetilde{C}(\kappa)$ is compact, there exist a subsequence $(\widetilde{C}(\kappa)\boldsymbol{\psi}_{h''})_{h''\in\mathbb{H}''}$ of $(\widetilde{C}(\kappa)\boldsymbol{\psi}_{h'})_{h'\in\mathbb{H}'}$ and a $\boldsymbol{\chi} \in \mathbf{V}$ such that $\|\widetilde{C}(\kappa)\boldsymbol{\psi}_{h''} - \boldsymbol{\chi}\|_{\mathbf{V}} \to 0$ as $h'' \to 0$. This gives

$$\|\widetilde{\mathsf{S}}_{h''}(\kappa)\boldsymbol{\psi}_{h''} - (\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi})\|_{\mathbf{V}} \le \|\,\mathsf{S}_{h''}(\kappa)\boldsymbol{\psi}_{h''} - \boldsymbol{\varphi}\|_{\mathbf{V}} + \|P_{h''}\,\mathsf{J}\widetilde{C}(\kappa)\boldsymbol{\psi}_{h''} - \mathsf{J}\boldsymbol{\chi}\|_{\mathbf{V}} \to 0$$

as $h'' \to 0$. Note that $\widetilde{\mathsf{S}}(\kappa) = (T^*)^{-1}(T^*\,\mathsf{S}(\kappa) + C(\kappa))$. Hence, the inverse of the operator $\widetilde{\mathsf{S}}(\kappa) : \mathbf{V} \to \mathbf{V}$ exists. By Lem. 3.2 also the inverse of $\widetilde{\mathsf{S}}_h(\kappa) : \mathbf{V}_h \to \mathbf{V}_h$ exists. Define $\boldsymbol{\psi} := (\mathsf{J}\widetilde{\mathsf{S}}(\kappa))^{-1}(\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi})$ and consider

$$\boldsymbol{\psi}_{h''} - P_{h''}\boldsymbol{\psi} = (\widetilde{\mathsf{S}}_{h''}(\kappa))^{-1}\big(\widetilde{\mathsf{S}}_{h''}(\kappa)\boldsymbol{\psi}_{h''} - P_{h''}(\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi}) + P_{h''}\,\mathsf{J}\widetilde{\mathsf{S}}(\kappa)\boldsymbol{\psi} - \widetilde{\mathsf{S}}_{h''}(\kappa)P_{h''}\boldsymbol{\psi}\big).$$

We have $\|\boldsymbol{\psi}_{h''} - P_{h''}\boldsymbol{\psi}\| \to 0$ as $h'' \to 0$ since by Lem. 3.2 $(\widetilde{\mathsf{S}}_{h''}(\kappa))^{-1}$ is uniformly bounded and since

$$\begin{aligned}\|\widetilde{\mathsf{S}}_{h''}(\kappa)\boldsymbol{\psi}_{h''} - P_{h''}(\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi})\|_{\mathbf{V}} \\ \le \|\widetilde{\mathsf{S}}_{h''}(\kappa)\boldsymbol{\psi}_{h''} - (\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi})\|_{\mathbf{V}} + \|(I - P_{h''})(\boldsymbol{\varphi} + \mathsf{J}\boldsymbol{\chi})\|_{\mathbf{V}} \to 0,\end{aligned}$$

and

$$\|P_{h''}\,\mathsf{J}\widetilde{\mathsf{S}}(\kappa)\boldsymbol{\psi} - \widetilde{\mathsf{S}}_{h''}(\kappa)P_{h''}\boldsymbol{\psi}\|_{\mathbf{V}} = \|P_{h''}\,\mathsf{J}\widetilde{\mathsf{S}}(\kappa)(\boldsymbol{\psi} - P_{h''}\boldsymbol{\psi})\|_{\mathbf{V}} \to 0$$

as $h'' \to 0$. From this it follows that $\|\boldsymbol{\psi}_{h''} - \boldsymbol{\psi}\| \to 0$ as $h'' \to 0$. □

In the following theorem we summarize the convergence results for the Galerkin approximation (20) of the eigenvalue problem for S.

Theorem 3.3. *Suppose that $(\mathbf{V}_h)_{h\in\mathbb{H}}$ is sequence of subspaces of $\mathbf{V}$ for which the conditions (A1) and (A2) are satisfied. Then the following holds true:*

(i) (Completeness of the discrete spectrum) Let κ be an eigenvalue of the eigenvalue problem (10). Then there exists a sequence $(\kappa_h)_{h\in\mathbb{H}}$ of eigenvalues of the Galerkin eigenvalue problem (20) such that

$$\kappa_h \to \kappa \quad \text{as } h \to 0.$$

(ii) (Non-pollution of the discrete spectrum) Let $K \subset \mathbb{C}\setminus\{0\}$ be compact and connected set with a simple rectifiable boundary. Suppose that there is no eigenvalue of the eigenvalue problem (10) in K. Then there exists an $h_0 > 0$ such that for all $0 < h < h_0$ the Galerkin eigenvalue problem (20) has no eigenvalues in K.

(iii) Let $D \subset \mathbb{C}\setminus\{0\}$ be compact and connected set with a simple rectifiable boundary. Suppose that $\kappa \in \mathring{D}$ is the only eigenvalue of S in D. Then there exist an $h_0 > 0$ and a $c > 0$ such that for all $0 < h \le h_0$ we have:

a) (Stability of the algebraic multiplicities)

$$\dim G(\mathsf{S},\kappa) = \sum_{\kappa_h\in\sigma(P_h\,\mathsf{J}\,\mathsf{S}\,P_h)\cap D} \dim G(P_h\,\mathsf{J}\,\mathsf{S}\,P_h,\kappa_h).$$

b) For all eigenvalues κ_h of the Galerkin eigenvalue problem (20) in D it holds:

$$|\kappa-\kappa_h| \le c\delta(G(\mathsf{S},\kappa),\mathbf{V}_h)^{2/\ell}.$$

The term $\delta(G(\mathsf{S},\kappa),\mathbf{V}_h)$ denotes the gap between the generalized eigenspace $G(\mathsf{S},\kappa)$ corresponding to κ and the space $\mathbf{V}_h$:

$$\delta(G(\mathsf{S},\kappa),\mathbf{V}_h) := \sup_{\substack{\boldsymbol{\psi}\in G(\mathsf{S},\kappa)\\ \|\boldsymbol{\psi}\|_{\mathbf{V}}=1}} \inf_{\boldsymbol{\chi}_h\in\mathbf{V}_h} \|\boldsymbol{\psi}-\boldsymbol{\chi}_h\|_{\mathbf{V}}.$$

The number ℓ is the maximal length of a Jordan chain corresponding to κ.

c) Let

$$\overline{\kappa}_h := \frac{1}{\dim G(\mathsf{S},\kappa)} \sum_{\kappa_h\in\sigma(P_h\,\mathsf{J}\,\mathsf{S}\,P_h)\cap D} \kappa_h \dim G(P_h\,\mathsf{J}\,\mathsf{S}\,P_h,\kappa_h)$$

be the weighted mean of all eigenvalues of the Galerkin eigenvalue problem (20) in D. Then it holds

$$|\kappa-\overline{\kappa}_h| \le c\delta(G(\mathsf{S},\kappa),\mathbf{V}_h)^2. \tag{26}$$

d) If $(\kappa_h, \boldsymbol{\psi}_h)$ is an eigenpair of (20) with $\kappa_h \in D$ and $\|\boldsymbol{\psi}_h\|_{\mathbf{V}} = 1$, then

$$\inf_{\boldsymbol{\psi} \in G(\mathsf{S},\kappa)} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{\mathbf{V}} \leq c \left(|\kappa_h - \kappa| + \delta(G(\mathsf{S},\kappa), \mathbf{V}_h) \right).$$

Proof. Since the sequence $\{P_h \, \mathsf{J}\mathsf{S}(\kappa) P_h\}_{h \in \mathbb{H}}$ provides a regular approximation of the operator $\mathsf{J}\mathsf{S}(\kappa)$ for all $\kappa \in \mathbb{C} \setminus \{0\}$, the assertions (i) to (iii)c) follow from the abstract results in [16, 17], where regular approximations of eigenvalue problems for holomorphic Fredholm operator-valued functions are considered for an abstract approximation scheme. We also want to refer to [15], where the special case of the conforming Galerkin approximation of eigenvalue problems for holomorphic T-Gårding operator-valued functions is analyzed.

In the following we specify the abstract results of [16, 17] as well as of [15] from which the assertions follow. The assertions (i) and (ii) are a consequence of [16, Thm. 2] and [15, Thm. 3.1 i) and iii)]. For assertion (iii)(a) we refer to [16, Thm. 3] and [15, Thm. 3.1 iv)]. The error estimate for the eigenvalues in (iii)b) follows from [17, Thm. 3] and [15, Thm. 3.1 v)] where we use that $G(\mathsf{S}, \kappa) = \overline{G(\mathsf{S}^*, \overline{\kappa})}$, which we have shown in Lem. 2.4. For the assertion (iii)c) we refer to [17, Thm. 3] and [15, Thm. 3.1 vi)].

The error estimate for the eigenfunction in (iii)d) is a consequence of [15, Thm. 3.1 v)]. □

In the following lemma we specify sufficient conditions such that a sequence $(\mathbf{V}_h)_{h \in \mathbb{H}}$ which satisfies the assumption (A1) also satisfies the assumption (A2).

Lemma 3.4. *Let $(\mathbf{V}_h)_{h \in \mathbb{H}}$ be a sequence satisfying the assumption (A1). Suppose that there exist subspaces $\mathcal{X}_h$ and $\mathcal{N}_h$ of $\mathbf{V}_h$ such that*

$$\mathbf{V}_h = \mathcal{X}_h \oplus \mathcal{N}_h \qquad \text{for all } h \in \mathbb{H}. \tag{27}$$

Further, assume that the following gap-property

$$\delta_h := \max\{\delta(\mathcal{X}_h, \mathcal{X}), \delta(\mathcal{N}_h, \mathcal{N})\} \to 0 \quad \text{as } h \to 0 \tag{28}$$

is satisfied. Then, the condition (A2) is fulfilled for $(\mathbf{V}_h)_{h \in \mathbb{H}}$ with $\mathsf{T}_h = P_h \mathsf{T}$.

Proof. Let $\boldsymbol{\psi}_h \in \mathbf{V}_h$ and $\mathsf{T}_h = P_h \mathsf{T}$. Because of the decompositions of $\mathbf{V} = \mathcal{X} \oplus \mathcal{N}$ and of $\mathbf{V}_h = \mathcal{X}_h \oplus \mathcal{N}_h$, there exist $\boldsymbol{\chi} \in \mathcal{X}, \boldsymbol{\nu} \in \mathcal{N}, \boldsymbol{\chi}_h \in \mathcal{X}_h$ and $\boldsymbol{\nu}_h \in \mathcal{N}_h$ such that

$$\boldsymbol{\psi}_h = \boldsymbol{\chi} + \boldsymbol{\nu} = \boldsymbol{\chi}_h + \boldsymbol{\nu}_h.$$

Since $\mathsf{T}\boldsymbol{\psi}_h = \boldsymbol{\chi} - \boldsymbol{\nu}$, we get

$$\begin{aligned} \|(\mathsf{T} - \mathsf{T}_h)\boldsymbol{\psi}_h\|_{\mathbf{V}} = \|(I - P_h)\mathsf{T}\boldsymbol{\psi}_h\|_{\mathbf{V}} &= \|(I - P_h)(\mathsf{T}\boldsymbol{\psi}_h - (\boldsymbol{\chi}_h - \boldsymbol{\nu}_h))\|_{\mathbf{V}} \\ &\leq \|\boldsymbol{\chi} - \boldsymbol{\nu} - (\boldsymbol{\chi}_h - \boldsymbol{\nu}_h))\|_{\mathbf{V}} \leq \|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\mathbf{V}} + \|\boldsymbol{\nu} - \boldsymbol{\nu}_h\|_{\mathbf{V}}. \end{aligned} \tag{29}$$

We will show that

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\mathbf{V}} + \|\boldsymbol{\nu} - \boldsymbol{\nu}_h\|_{\mathbf{V}} \lesssim \delta_h \|\boldsymbol{\psi}_h\|_V \tag{30}$$

holds for sufficiently small $h \in \mathbb{H}$. Then from (29) it follows that (A2) is satisfied.

The gap-property implies that there exists a continuous operator $A_h : \mathcal{X}_h \to \mathcal{X}$ such that $\|\boldsymbol{\chi}_h - A_h\boldsymbol{\chi}_h\|_{\mathbf{V}} \leq 2\delta_h\|\boldsymbol{\chi}_h\|_{\mathbf{V}}$. Further, for the projector $\mathsf{R} : \mathbf{V} \to \mathcal{X}$ which is associated with the splitting $\mathbf{V} = \mathcal{X} \oplus \mathcal{N}$, we have

$$\|\boldsymbol{\chi}_h - \mathsf{R}\,\boldsymbol{\chi}_h\|_{\mathbf{V}} = \|\boldsymbol{\chi}_h - A_h\boldsymbol{\chi}_h + A_h\boldsymbol{\chi}_h - \mathsf{R}\,\boldsymbol{\chi}_h\|_{\mathbf{V}} = \|(I - \mathsf{R})(\boldsymbol{\chi}_h - A_h\boldsymbol{\chi}_h\|_{\mathbf{V}} \lesssim \delta_h\|\boldsymbol{\chi}_h\|_{\mathbf{V}}. \quad (31)$$

Analogously, one can show that for the projector $\mathsf{Z} : \mathbf{V} \to \mathcal{N}$ the inequality

$$\|\boldsymbol{\nu}_h - \mathsf{Z}\,\boldsymbol{\nu}_h\| \lesssim \delta_h\|\boldsymbol{\nu}_h\|_{\mathbf{V}} \quad (32)$$

holds. Using $\boldsymbol{\chi} = \mathsf{R}\boldsymbol{\psi}_h = \mathsf{R}\,\boldsymbol{\chi}_h + \mathsf{R}\,\boldsymbol{\nu}_h$ and $\mathsf{R}\,\boldsymbol{\nu}_h = (I - \mathsf{Z})\boldsymbol{\nu}_h$, we get from (31) and (32) the estimate

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\mathbf{V}} \leq \|\,\mathsf{R}\,\boldsymbol{\chi}_h - \boldsymbol{\chi}_h\|_{\mathbf{V}} + \|\,\mathsf{R}\,\boldsymbol{\nu}_h\|_{\mathbf{V}} \lesssim \delta_h\,(\|\boldsymbol{\chi}_h\|_{\mathbf{V}} + \|\boldsymbol{\nu}_h\|_{\mathbf{V}})\,. \quad (33)$$

It is well known that the gap-property (28) implies that the splitting in (27) of $\mathbf{V}_h$ is uniformly stable for sufficiently small $h \in \mathbb{H}$, i. e., there exists a constant $c > 0$ such that for sufficiently small $h \in \mathbb{H}$ we have

$$\|\boldsymbol{\chi}_h\|_{\mathbf{V}} + \|\boldsymbol{\nu}_h\|_{\mathbf{V}} \leq c\|\boldsymbol{\chi}_h + \boldsymbol{\nu}_h\|_{\mathbf{V}} = c\|\boldsymbol{\psi}_h\|_{\mathbf{V}},$$

see, e.g. [6, Thm. 3.2]. From this we get from (33) that

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\mathbf{V}} \lesssim \delta_h\|\boldsymbol{\psi}_h\|_{\mathbf{V}}.$$

The inequality (30) follows now from $\boldsymbol{\nu} - \boldsymbol{\nu}_h = -(\boldsymbol{\chi} - \boldsymbol{\chi}_h)$. □

3.1 Application of the convergence results to boundary element spaces

In this subsection we apply the abstract results of the last section to some standard boundary element spaces for Maxwell's equations.

We first consider the case that the boundary Γ is piecewise flat. Let $(\mathcal{T}_h)_{h\in\mathbb{H}}$ be a sequence of regular triangulations of Γ with mesh size h. Let us denote by $\mathrm{RT}_k(\mathcal{T}_h)$ the space generated by Raviart–Thomas elements of order k on $\mathcal{T}_h$ and by $\mathrm{BMD}_k(\mathcal{T}_h)$ the space generated by Brezzi–Douglas–Marini elements of order k on $\mathcal{T}_h$. We adopt the convention of [7, 9] that $k = 0$ means lowest order Raviart–Thomas or Brezzi–Douglas–Marini finite elements. Both spaces satisfy the condition (A1), see, e. g., [7, 9]. In [7, Sect. 4.2, Prop. 4.4], it is shown that $(\mathrm{RT}_k(\mathcal{T}_h))_{h\in\mathbb{H}}$ and $(\mathrm{BMD}_k(\mathcal{T}_h))_{h\in\mathbb{H}}$ satisfy in addition the gap property (28) of Lem. 3.4, which implies that they also satisfy assumption (A2). Hence, the convergence results of Thm. 3.3 can be applied to $\mathrm{RT}_k(\mathcal{T}_h)$ and $\mathrm{BMD}_k(\mathcal{T}_h)$. The error estimates for the approximation of an eigenvalue κ of S and the corresponding eigenspace $G(\mathsf{S}, \kappa)$ depend on the approximation property of the approximation spaces. For $\mathbf{V}_{h,k} \in \{\mathrm{RT}_k(\mathcal{T}_h), \mathrm{BMD}_k(\mathcal{T}_h)\}$, it holds for $-\frac{1}{2} \leq s \leq k + 1$ [9, Thm. 14]:

$$\inf_{\boldsymbol{\psi}_h\in\mathbf{V}_{h,k}} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{\mathbf{V}} \leq Ch^{s+\frac{1}{2}}\|\boldsymbol{\psi}\|_{\mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)} \qquad \forall\boldsymbol{\psi} \in \mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma).$$

From Thm. 3.3(iii)b), we conclude if $\kappa \in \sigma(\mathsf{S})$ is the only eigenvalue in a given compact and connected set $D \subset \mathbb{C} \setminus \{0\}$ with rectifiable boundary and if $G(\mathsf{S},\kappa) \subset \mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)$ for some $s \in [-\frac{1}{2}, k+1]$, then there exists a $\boldsymbol{\varphi} \in G(\mathsf{S},\kappa)$ with $\|\boldsymbol{\varphi}\|_{\mathbf{V}} = 1$ such that for all eigenvalues $\kappa_h \in D$ of the Galerkin eigenvalue problem in $\mathbf{V}_{h,k}$ the estimate

$$|\kappa - \kappa_h| \leq c h^{(2s+1)/\ell} \|\boldsymbol{\varphi}\|_{\mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)}$$

holds for sufficiently small $h \in \mathbb{H}$. The number ℓ is the maximal length of a Jordan chain corresponding to κ. For the weighted mean $\overline{\kappa}_h$ of all eigenvalues in D of the Galerkin eigenvalue problem in $\mathbf{V}_{h,k}$ we get by Thm. 3.3(iii)c):

$$|\kappa - \overline{\kappa}_h| \leq c h^{2s+1} \|\boldsymbol{\varphi}\|_{\mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)}. \tag{34}$$

If $\boldsymbol{\psi}_h$ is a corresponding eigenelement of κ_h, then by Thm. 3.3(iii)d) there exists a $\boldsymbol{\psi} \in G(\mathsf{S},\kappa)$ with $\|\boldsymbol{\psi}\|_{\mathbf{V}} = 1$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\| \leq C h^{(s+\frac{1}{2})/\ell} \|\boldsymbol{\psi}\|_{\mathbf{H}^s(\mathrm{div}_\Gamma,\Gamma)}$$

for all sufficiently small $h \in \mathbb{H}$.

Next, we want to briefly address the case of a p-parametric surface approximation Γ_h^p of a curvilinear polyhedral surface Γ. It is possible to utilize the abstract convergence analysis of [16, 17] also in this case when using Raviart–Thomas and Brezzi–Douglas–Marini elements. Let $(\mathbf{V}_{h,k}^p)_{h\in\mathbb{H}}$ be a sequence of such ansatz spaces with respect to the p-parametric surface approximations Γ_h^p and let us denote by $\theta_h^p : \Gamma \to \Gamma_h^p$ the surface liftings and by $\hat{\mathbf{V}}_{h,k}^p = \{v \circ \theta^p : v \in \mathbf{V}_{h,k}^p\}$ the lifted ansatz spaces [27, Chapt. 8]. The approximation of the eigenvalue problem for S can be considered as approximation in $\hat{\mathbf{V}}_{h,k}^p \subset \mathbf{V}$. The proof of the required gap-property of $\hat{\mathbf{V}}_{h,k}^p$ is straightforward with the results in [2], see the comment in [7, Sect 4.2]. The proof of the regular approximation of $\mathsf{S}(\kappa)$ in $\hat{\mathbf{V}}_{h,k}^p$ is more involved but can be done similarly as in Thm. 3.1. From the abstract results in [16, 17], the convergence of the eigenvalue and eigenspace approximations in $(\hat{\mathbf{V}}_{h,k}^p)_{h\in\mathbb{H}}$ follows with similar error estimates as in Thm. 3.3 where $\delta(G(\mathsf{S},\kappa),\mathbf{V}_h)$ has to be replaced by $\delta(G(\mathsf{S},\kappa),\hat{\mathbf{V}}_{h,k}^p)$.

4 Transmission eigenvalue problem for the scattering at a dielectric interface

In this section we consider the transmission eigenvalue problem between two different dielectric media. The proposed boundary integral formulation for the transmission eigenvalue problem is based on the Calderón identities. In [10], this kind of boundary integral formulation has been analyzed for the related source problem. We will show that the boundary integral formulation of the transmission eigenvalue problem exhibit the same basic properties as the boundary integral formulation of the interior and exterior eigenvalue problem

(1) and (2). As a consequence, the same kind of convergence results for the Galerkin approximation of the boundary integral formulation of the transmission eigenvalue problem can be concluded as for the Galerkin approximation of the boundary integral formulation of the interior and exterior eigenvalue problem. The transmission eigenvalue problem reads as follows: Find $\omega \in \mathbb{C}$ and $(0,0) \neq (\mathbf{E}_1, \mathbf{E}_2) \in \mathbf{H}(\mathbf{curl}; \Omega^{\mathrm{i}}) \times \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega^{\mathrm{e}})$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_1 - \omega^2 \varepsilon_1 \mu_1 \mathbf{E}_1 = 0 \quad \text{and} \quad \operatorname{div}(\varepsilon_1 \mathbf{E}_1) = 0 \quad &\text{in } \Omega^{\mathrm{i}}, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_2 - \omega^2 \varepsilon_2 \mu_2 \mathbf{E}_2 = 0 \quad \text{and} \quad \operatorname{div}(\varepsilon_2 \mathbf{E}_2) = 0 \quad &\text{in } \Omega^{\mathrm{e}}, \\ \mathbf{E}_1 \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n} \quad &\text{on } \Gamma, \\ \mu_1^{-1}\,\mathbf{curl}\,\mathbf{E}_1 \times \mathbf{n} = \mu_2^{-1}\,\mathbf{curl}\,\mathbf{E}_2 \times \mathbf{n} \quad &\text{on } \Gamma, \\ \mathbf{E}_2 \text{ is outgoing.} & \end{aligned} \tag{35}$$

We assume that the material parameters ε_1, μ_1 for the interior domain Ω^{i} and ε_2, μ_2 for the exterior domain Ω^{e} are constant and positive. As shown in [10, Sect. 6], then the eigenvalues of (35) are all non-real.

The interior and exterior Calderón identities are used for deriving a boundary integral formulation of the transmission eigenvalue problem (35). A function $\mathbf{u}_1 \in \mathbf{H}(\mathbf{curl}; \Omega^{\mathrm{i}})$ is a solution of the Maxwell's equations in Ω^{i} with wavenumber $\omega\sqrt{\varepsilon_1\mu_1}$ if and only if it satisfies the interior Calderón identity [9, Thm. 8]

$$\begin{pmatrix} \frac{1}{2}I + \mathsf{M}(\omega\sqrt{\varepsilon_1\mu_1}) & \mu_1\,\mathsf{S}(\omega\sqrt{\varepsilon_1\mu_1}) \\ \omega^2\varepsilon_1\,\mathsf{S}(\omega\sqrt{\varepsilon_1\mu_1}) & \frac{1}{2}I + \mathsf{M}(\omega\sqrt{\varepsilon_1\mu_1}) \end{pmatrix} \begin{pmatrix} \gamma_\tau^{\mathrm{i}}\,\mathbf{u}_1 \\ \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{u}_1 \end{pmatrix} = \begin{pmatrix} \gamma_\tau^{\mathrm{i}}\,\mathbf{u}_1 \\ \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{u}_1 \end{pmatrix}, \tag{36}$$

where $\mathsf{M}(\kappa) := \frac{1}{2}(\gamma_N^{\mathrm{i}} + \gamma_N^{\mathrm{e}})\,\mathbf{\Psi}_{\mathrm{SL}}$. We define for $\ell = 1, 2$ the block operator

$$A_\ell(\omega) := \begin{pmatrix} \mathsf{M}(\omega\sqrt{\varepsilon_\ell\mu_\ell}) & \mu_\ell\,\mathsf{S}(\omega\sqrt{\varepsilon_\ell\mu_\ell}) \\ \omega^2\varepsilon_\ell\,\mathsf{S}(\omega\sqrt{\varepsilon_\ell\mu_\ell}) & \mathsf{M}(\omega\sqrt{\varepsilon_\ell\mu_\ell}) \end{pmatrix}.$$

A function $\mathbf{u}_2 \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega^{\mathrm{e}})$ is an outgoing solution of Maxwell's equations in Ω^{e} with wavenumber $\omega\sqrt{\varepsilon_2\mu_2}$ if and only if it satisfies the exterior Calderón identity [9, Thm. 8]

$$\left(\frac{1}{2}I - A_2(\omega)\right) \begin{pmatrix} \gamma_\tau^{\mathrm{e}}\,\mathbf{u}_2 \\ \mu_2^{-1}\gamma_N^{\mathrm{e}}\mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \gamma_\tau^{\mathrm{e}}\,\mathbf{u}_2 \\ \mu_2^{-1}\gamma_N^{\mathrm{e}}\mathbf{u}_2 \end{pmatrix}.$$

We obtain the following boundary integral formulation of the transmission eigenvalue problem (35) by setting $(\boldsymbol{\varphi}, \boldsymbol{\psi}) := (\gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1) = (\gamma_\tau^{\mathrm{e}}\,\mathbf{E}_2, \mu_2^{-1}\gamma_N^{\mathrm{e}}\mathbf{E}_2)$ and by using the interior and exterior Calderón identity:

$$(A_1(\omega) + A_2(\omega)) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{37}$$

Note that the eigenvalue problems (35) and (37) are not equivalent. If in the eigenvalue problem (35) ε_1 is interchanged with ε_2 and μ_1 with μ_2, then one also obtains (37) as corresponding boundary integral formulation. However, the equivalence of (35) and (37) is guaranteed if for (37) a constraint is imposed, as shown next.

Theorem 4.1. *The following assertions hold true:*

(i) Suppose that $(\omega, \mathbf{E}_1, \mathbf{E}_2)$ *is a solution of the transmission eigenvalue problem* (35)*. Then* $(\omega, \gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1$ *is a solution of the eigenvalue problem* (37) *and satisfies* $(\frac{1}{2}I - A_1(\omega))(\gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1)^\top = 0$*.*

(ii) Conversely, suppose that $(\omega, \boldsymbol{\varphi}, \boldsymbol{\psi})$ *is a solution of the eigenvalue problem* (37) *satisfying* $(\frac{1}{2}I - A_1(\omega))(\boldsymbol{\varphi}, \boldsymbol{\psi})^\top = 0$*. Define*

$$\mathbf{E}_1 := \boldsymbol{\Psi}_{\mathrm{SL}}(\kappa_1)\mu_1\boldsymbol{\psi} + \boldsymbol{\Psi}_{\mathrm{DL}}(\kappa_1)\boldsymbol{\varphi} \quad \textit{and} \quad \mathbf{E}_2 := -\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa_2)\mu_2\boldsymbol{\psi} - \boldsymbol{\Psi}_{\mathrm{DL}}(\kappa_2)\boldsymbol{\varphi},$$

where $\kappa_\ell := \omega\sqrt{\varepsilon_\ell\mu_\ell}$*,* $\ell = 1, 2$*. Then* $(\omega, \mathbf{E}_1, \mathbf{E}_2)$ *is a solution of the eigenvalue problem* (35) *and* $(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1)$*.*

Proof. (i) We have already shown that if $(\omega, \mathbf{E}_1, \mathbf{E}_2)$ is a solution of the eigenvalue problem (35) that then $(\omega, \gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1$ is a solution of (37). The interior Calderón identity (36) implies that $\big(\frac{1}{2}I - A_1(\omega)\big)\,(\gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1, \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1)^\top = 0$ holds.

(ii) Assume now that $(\omega, \boldsymbol{\varphi}, \boldsymbol{\psi})$, $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{V} \setminus \{0\}$, satisfies the eigenvalue problem (37) and that the equation $\big(\frac{1}{2}I - A_1(\omega)\big)\big(\boldsymbol{\varphi}, \boldsymbol{\psi}\big)^\top = 0$ holds. We first show that

$$\mathbf{E}_1 := \boldsymbol{\Psi}_{\mathrm{SL}}(\kappa_1)\mu_1\boldsymbol{\psi} + \boldsymbol{\Psi}_{\mathrm{DL}}(\kappa_1)\boldsymbol{\varphi} \quad \text{and} \quad \mathbf{E}_2 := -\boldsymbol{\Psi}_{\mathrm{SL}}(\kappa_2)\mu_2\boldsymbol{\psi} - \boldsymbol{\Psi}_{\mathrm{DL}}(\kappa_2)\boldsymbol{\varphi},$$

satisfy the eigenvalue problem (35), where $\kappa_\ell := \omega_\ell\sqrt{\varepsilon_\ell\mu_\ell}$, $\ell = 1, 2$. By construction $\mathbf{E}_1$ is a solution of the Maxwell's equations in Ω^{i} and $\mathbf{E}_2$ is an outgoing solution of the Maxwell's equations in Ω^{e}. Applying the trace operators to $\mathbf{E}_1$ and $\mathbf{E}_2$ yields

$$\begin{pmatrix} \gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1 \\ \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1 \end{pmatrix} = \left(\frac{1}{2}I + A_1(\omega)\right)\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \text{ and } \begin{pmatrix} \gamma_\tau^{\mathrm{e}}\,\mathbf{E}_1 \\ \mu_2^{-1}\gamma_N^{\mathrm{e}}\mathbf{E}_1 \end{pmatrix} = \left(\frac{1}{2}I - A_2(\omega)\right)\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix}. \tag{38}$$

Subtracting the second equation from the first equation in (38) and using that $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ satisfies the eigenvalue problem (37) gives

$$\begin{pmatrix} \gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1 - \gamma_\tau^{\mathrm{e}}\,\mathbf{E}_1 \\ \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1 - \mu_2^{-1}\gamma_N^{\mathrm{e}}\mathbf{E}_1 \end{pmatrix} = (A_1(\omega) + A_2(\omega))\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $\mathbf{E}_1$ and $\mathbf{E}_2$ satisfy the transmission conditions of the eigenvalue problem (37). From the assumption $\big(\frac{1}{2}I - A_1(\omega)\big)\big(\boldsymbol{\varphi}, \boldsymbol{\psi}\big)^\top = 0$ and the first equation in (38), we finally get

$$\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} = \left(\frac{1}{2}I + A_1(\omega)\right)\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \gamma_\tau^{\mathrm{i}}\,\mathbf{E}_1 \\ \mu_1^{-1}\gamma_N^{\mathrm{i}}\mathbf{E}_1 \end{pmatrix}.$$

□

For the analysis of the eigenvalue problem (37) we consider the antisymmetric pairing $B : \mathbf{V}^2 \times \mathbf{V}^2 \to \mathbb{C}$ defined by

$$B\left(\begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix}, \begin{pmatrix} \widetilde{\boldsymbol{\varphi}} \\ \widetilde{\boldsymbol{\psi}} \end{pmatrix}\right) = \langle \boldsymbol{\varphi}, \widetilde{\boldsymbol{\psi}} \rangle_{\tau,\Gamma} - \langle \widetilde{\boldsymbol{\varphi}}, \boldsymbol{\psi} \rangle_{\tau,\Gamma}.$$

Theorem 4.2. *Let $\omega \in \mathbb{C} \setminus \{0\}$. Then there exists a compact operator $C(\omega) : \mathbf{V}^2 \to \mathbf{V}^2$ and a constant $c(\omega) > 0$ such that for all $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{V}$ it holds*

$$\left| B\left((A_1(\omega) + A_2(\omega) + C(\omega)) \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix}, T(\omega) \begin{pmatrix} \overline{\boldsymbol{\varphi}} \\ \overline{\boldsymbol{\psi}} \end{pmatrix} \right) \right| \geq c(\omega) \left\| \begin{pmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\psi} \end{pmatrix} \right\|^2_{\mathbf{V}^2},$$

where $T(\omega) = \begin{pmatrix} \mathsf{T} & 0 \\ 0 & \mathsf{T} \end{pmatrix}$ *if* $\mathrm{Re}(\omega) \neq 0$ *and otherwise* $T(\omega) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$.

Proof. A proof for positive ω is given in [10, Thm. 6.3]. This proof can be extended to $\omega \in \mathbb{C} \setminus \{0\}$ by using the results of Lem. 2.2. □

Thm. 4.2 shows that the operator $A_1(\omega) + A_2(\Omega)$ satisfies a T-Gårding's inequality if $\mathrm{Re}(\omega) \neq 0$ and a standard Gårding's inequality if $\omega = i\alpha$, $\alpha \in \mathbb{R} \setminus \{0\}$. A regular approximation of the operator is provided by a Galerkin approximation with ansatz spaces $\{\mathbf{V}_h \times \mathbf{V}_h\}_{h \in \mathbb{H}}$ if $\{\mathbf{V}_h\}_{h \in \mathbb{H}}$ satisfies the assumption (A1) and (A2) of Sect. 3. This follows immediately from the abstract result in [15, Lem. 2.7] or it can be shown analogously as for the Galerkin approximation of the operator $\mathsf{S}(\omega)$. As a consequence, the abstract convergence theory [15, 16, 17] can be applied to the Galerkin approximation of the boundary integral formulation of the transmission eigenvalue problem (37) and the same kinds of results as in Thm. 3.3 can be concluded.

5 Numerical Examples

In this section we report on results from some numerical experiments for the approximation of the eigenvalues of the boundary integral formulations of the interior and exterior eigenvalue problem problem (1) and (2), and of the transmission eigenvalue problem (35). In all experiments Raviart–Thomas elements of lowest order $k = 0$ are used. For the computations of the boundary element matrices the open-source library BEM++ [28] is employed.

The Galerkin approximations of the eigenvalue problems for S and for $A_1 + A_2$ result in holomorphic matrix eigenvalue problems in $\mathbb{C} \setminus \{0\}$. The related underlying matrix-valued functions are denoted by S_h and $A_{1,h} + A_{2,h}$, respectively. For the numerical solution of the matrix eigenvalue problems we use the contour integral method as given in [3]. For other variants of the contour integral method we refer to [1, 34]. The contour integral method is a reliable method for the approximation of all eigenvalues which lie inside of a given contour in the complex plane, and for the approximation of the corresponding eigenvectors. The method is based on the contour integration of the resolvent and utilizes that the eigenvalues of eigenvalue problems for holomorphic matrix-valued functions are poles of the resolvent. By contour integration of the resolvent a reduction of the holomorphic eigenvalue problem to an equivalent linear eigenvalue problem is possible such that the eigenvalues of the linear eigenvalue problem coincide with the eigenvalue of the nonlinear eigenvalue problem inside the contour. For the practical application of this method an efficient approximation

h	$\lvert\kappa_1 - \overline{\kappa}_{1,h}\rvert$	eoc	$\lvert\kappa_2 - \overline{\kappa}_{2,h}\rvert$	eoc	$\lvert\kappa_3 - \overline{\kappa}_{3,h}\rvert$	eoc	$\lvert\kappa_4 - \overline{\kappa}_{4,h}\rvert$	eoc
0.707	1.114e-02	-	2.201e-02	-	3.648e-02	-	6.027e-02	-
0.354	1.029e-03	3.43	2.110e-03	3.38	2.960e-03	3.62	5.111e-03	3.56
0.177	1.086e-04	3.24	2.308e-04	3.19	3.089e-04	3.26	5.509e-04	3.33
0.088	1.221e-05	3.15	2.682e-05	3.11	3.534e-05	3.13	6.439e-05	3.10

Table 1: Unit cube: Approximation error and experimental order of convergence (eoc) of the four smallest interior eigenvalues inside the ellipse.

of the contour integral is necessary. This can be achieved for example by the composite trapezoidal rule [3].

For the analysis of the contour integral method it is usually assumed that the underlying matrix-valued function of the eigenvalue problem is holomorphic inside the contour. However, the contour integral method is also suitable for eigenvalue problems where the underlying function has isolated singularities in the case of that the resolvent has a holomorphic continuation in the neighborhood of the singularities. The reason for that is that the contour integral method operates on the resolvent of the eigenvalue problem and approximates the poles of the resolvent. Our numerical experiments indicates that there is a holomorphic continuation of $(S_h(\kappa))^{-1}$ as well as of $(A_{1,h}(\omega) + A_{2,h}(\omega))^{-1}$ in the neighborhood of 0, see Fig. 1–2 and Fig. 3–4, respectively. This property is for practical application of importance where often the eigenvalues of smallest modulus are of interest.

In the following numerical examples the eigenvalues inside of a contour are numbered in ascending order with respect to their real part.

5.1 Interior and exterior eigenvalue problem

In the first numerical examples we consider the Galerkin approximation (20) of the boundary integral formulation (10) of the interior and exterior eigenvalue problem (1) and (2). The unit cube and the unit ball are chosen for Ω^{i}. In all examples in this subsection we take $\varepsilon = \mu = 1.0$.

5.1.1 Interior eigenvalue problem for the unit cube

For the unit cube only the interior eigenvalues are known analytically. Therefore we restrict ourselves to the approximation of the eigenvalues of the interior problem. The interior eigenvalues have the form $\kappa = \pi k$, where $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$ with $k_1, k_2, k_3 \in \mathbb{N}_0$ and $k_1k_2 + k_2k_3 + k_3k_1 > 0$ [5, Sect. 6]. For the contour integral method we choose as contour the ellipse $\varphi(t) = c + a\cos(t) + ib\sin(t)$, $t \in [0, 2\pi]$, with $c = 5.0$, $a = 5.3$ and $b = 0.5$. Our numerical experiments indicate that there are only interior eigenvalues inside this ellipse. In Tab. 1 the errors of the approximations of the four smallest eigenvalues are given for different mesh sizes. The experimental orders of convergence suggest a cubic asymptotic convergence order, which confirm the theoretical convergence order (26), and (34) with

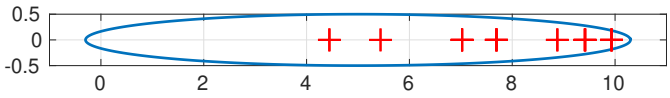

Figure 1: Unit cube: Computed interior eigenvalues by the contour integral method for $h = 0.177$.

h	$\lvert\kappa_1^{\mathrm{e}} - \overline{\kappa}_{1,h}^{\mathrm{e}}\rvert$	eoc	$\lvert\kappa_2^{\mathrm{e}} - \overline{\kappa}_{2,h}^{\mathrm{e}}\rvert$	eoc	$\lvert\kappa_1^{\mathrm{i}} - \overline{\kappa}_{1,h}^{\mathrm{i}}\rvert$	eoc	$\lvert\kappa_2^{\mathrm{i}} - \overline{\kappa}_{2,h}^{\mathrm{i}}\rvert$	eoc
0.707	4.293e-02	-	8.767e-02	-	1.202e-01	-	1.584e-01	-
0.354	6.773e-03	2.66	1.337e-02	2.71	1.878e-02	2.68	2.598e-02	2.61
0.177	1.535e-03	2.14	3.000e-03	2.16	4.230e-03	2.15	5.918e-03	2.13
0.088	3.552e-04	2.11	6.913e-04	2.12	9.765e-04	2.12	1.372e-03	2.11

Table 2: Unit ball: Approximation error and experimental order of convergence (eoc) of the two smallest interior and exterior eigenvalues in magnitude inside the ellipse.

$s = 1$. Further, we observe that all exact eigenvalues are approximated with the right multiplicities and that no spurious eigenvalues occur. The experiments also confirm the mentioned conjecture that the resolvent $\mathsf{S}_h(\kappa)^{-1}$ can be holomorphically extended to $\kappa = 0$. A plot of the computed eigenvalues for the mesh with mesh size $h = 0.177$ is given in Fig. 1.

5.1.2 Interior and exterior eigenvalue problem for the unit ball

The interior and exterior eigenvalues of the unit ball can be represented as zeros using the spherical Bessel and Hankel functions [26]. We denote by j_n the spherical Bessel functions of the first kind and as before by $h_n^{(1)}$ the spherical Hankel functions of the first kind. The set of the interior eigenvalues is given by

$$\{\kappa \in \mathbb{R} : j_n(\kappa) = 0 \text{ or } j_n(\kappa) + \kappa j_n'(\kappa) = 0, n \in \mathbb{N}\},$$

and that of the exterior eigenvalues by

$$\{\kappa \in \mathbb{C} : h_n^{(1)}(\kappa) = 0 \text{ or } h_n^{(1)}(\kappa) + \kappa h_n^{(1)\prime}(\kappa) = 0, n \in \mathbb{N}\}.$$

As contour for the contour integral method the ellipse $\varphi(t) = c + a\cos(t) + ib\sin(t)$, $t \in [0, 2\pi]$, is chosen with $c = 5.0$, $a = 5.3$ and $b = 1.0$. In Tab. 2 the errors of the two smallest eigenvalues in modulus of the interior and exterior eigenvalue problem are given. The experimental convergence order is in contrast to the cube of one order reduced since the sphere is approximated by flat triangles. Again, all exact eigenvalues are approximated with the right multiplicities and no spurious eigenvalues occur. In Fig. 2 the computed eigenvalues by the contour integral method for the mesh with mesh size $h = 0.177$ are plotted.

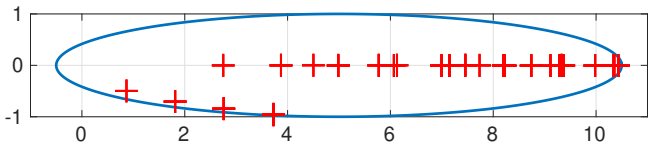

Figure 2: Unit ball: Computed interior and exterior eigenvalues by the contour integral method for $h = 0.177$.

h	$\lvert\omega_1 - \overline{\omega}_{1,h}\rvert$	eoc	$\lvert\omega_2 - \overline{\omega}_{2,h}\rvert$	eoc	$\lvert\omega_3 - \overline{\omega}_{3,h}\rvert$	eoc	$\lvert\omega_4 - \overline{\omega}_{4,h}\rvert$	eoc
0.707	7.94e-2	-	1.24e-1	-	2.05e-1	-	1.58e-1	-
0.354	6.14e-3	3.69	1.21e-2	3.37	2.30e-2	3.16	3.94e-2	2.00
0.177	6.05e-4	3.34	1.16e-3	3.41	2.21e-3	3.46	3.25e-3	3.60
0.088	6.21e-5	3.28	1.17e-4	3.42	2.18e-4	3.25	3.15e-4	3.36

Table 3: Transmission eigenvalue problem for the unit cube: Error and experimental order of convergence (eoc) of the four eigenvalues with smallest real part inside the ellipse.

5.2 Transmission eigenvalue problem

In this subsection we consider the Galerkin approximation of the boundary integral formulation (37) of the transmission eigenvalue problem (35). The domains Ω^{i} for the numerical examples are again the unit cube and the unit ball. As approximation space we choose $\mathbf{V}_h \times \mathbf{V}_h = \mathrm{RT}_0(\mathcal{T}_h) \times \mathrm{RT}_0(\mathcal{T}_h)$. The material parameters are set to $\varepsilon_1 = 4.0$ and $\varepsilon_2 = \mu_1 = \mu_2 = 1.0$.

5.2.1 Transmission eigenvalue problem for the unit cube

For the cube the exact eigenvalues of the transmission eigenvalue problem are not known. As reference eigenvalues the computed eigenvalues of a very fine mesh with mesh size $h = 0.03125$ are taken. For the experiments we chose the ellipse $\varphi(t) = c + a\cos(t) + ib\sin(t)$, $t \in [0, 2\pi]$, with $c = 2.5$, $a = 2.7$ and $b = 0.5$. The numerical experiments suggest a cubic asymptotic convergence order, see Tab. 3, which is in accordance with the theoretical results. The computed approximations of the eigenvalues by the contour integral method inside the ellipse for $h = 0.177$ are plotted in Fig. 3. The numerical experiments indicate that the resolvent $(A_{1,h}(\omega) + A_{2,h}(\omega))^{-1}$ can be holomorphically extended to $\omega = 0$.

5.2.2 Transmission eigenvalue problem for the unit ball

The exact eigenvalues of the transmission eigenvalue problem for the unit ball can be determined by the Mie series method [4, Chapt. 4]. A number $\omega \in \mathbb{C}$ is an eigenvalue if it satisfies either

$$\frac{h_n^{(1)}(\delta_2\omega) + \delta_2\omega h_n^{(1)\prime}(\delta_2\omega)}{h_n^{(1)}(\delta_2\omega)} = \frac{\varepsilon_2}{\varepsilon_1}\frac{j_n(\delta_1\omega) + \delta_1\omega j_n'(\delta_1\omega)}{j_n(\delta_1\omega)}$$

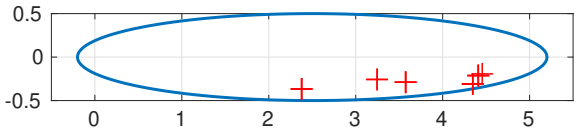

Figure 3: Transmission eigenvalue problem for the unit cube: Computed eigenvalues of the transmission eigenvalue problem by the contour integral method for $h = 0.177$.

h	$\lvert\omega_1 - \overline{\omega}_{1,h}\rvert$	eoc	$\lvert\omega_2 - \overline{\omega}_{2,h}\rvert$	eoc	$\lvert\omega_3 - \overline{\omega}_{3,h}\rvert$	eoc	$\lvert\omega_4 - \overline{\omega}_{4,h}\rvert$	eoc
0.707	6.38e-2	-	8.67e-2	-	9.92e-2	-	8.56e-2	-
0.354	9.96e-3	2.68	1.40e-2	2.63	1.78e-2	2.50	2.10e-2	2.03
0.177	2.24e-3	2.15	3.18e-3	2.14	4.09e-3	2.12	4.95e-3	2.08
0.088	5.15e-4	2.11	7.37e-4	2.11	9.53e-4	2.10	1.16e-3	2.09

Table 4: Transmission eigenvalue problem for the unit ball: Error and experimental order of convergence (eoc) of the four eigenvalues with smallest real part inside the ellipse.

or

$$\frac{h_n^{(1)}(\delta_2\omega) + \delta_2\omega h_n^{(1)\prime}(\delta_2\omega)}{h_n^{(1)}(\delta_2\omega)} = \frac{\mu_2}{\mu_1}\frac{j_n(\delta_1\omega) + \delta_1\omega j_n'(\delta_1\omega)}{j_n(\delta_1\omega)} \tag{39}$$

for some $n \in \mathbb{N}$, where $\delta_i = \sqrt{\varepsilon_i\mu_i}$, $i = 1, 2$. In Tab. 4 the error and the experimental convergence order for the approximation of the four eigenvalues $\omega_1, \ldots, \omega_4$ are given which have the smallest real part inside the ellipse $\varphi(t) = c + a\cos(t) + ib\sin(t)$, $t \in [0, 2\pi]$, with $c = 2.5$, $a = 2.7$ and $b = 0.5$. The eigenvalues ω_i, $i = 1, \ldots, 4$, are solutions of the equation (39) for $n = i$. The convergence order is compared to the cube of one order reduced as expected since the sphere is approximated by flat triangles. Again, all eigenvalues inside the ellipse are approximated with the right multiplicity and no spurious eigenvalues occur.

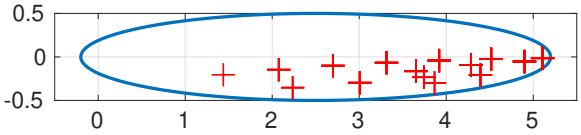

Figure 4: Transmission eigenvalue problem for the unit ball: Computed eigenvalues by the contour integral method for $h = 0.177$.

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