Technische Universität Graz

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Stability of the Laplace single layer boundary integral operator in Sobolev spaces

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1 Introduction

As a model problem we consider the Dirichlet boundary value problem for the Laplace equation,

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz polyhedron. Using an indirect approach, the solution of (1.1) can be described as single layer potential

$$u(x) = (\widetilde{V}w)(x) := \int_\Gamma U^*(x,y)w(y)\,ds_y \quad \text{for } x \in \Omega, \tag{1.2}$$

where

$$U^*(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}$$

is the fundamental solution of the Laplacian. It is well known, see, e.g. [2], that $\widetilde{V} : H^{-1/2}(\Gamma) \to H^1(\Omega)$. The unknown density $w \in H^{-1/2}(\Gamma)$ is then found by applying the interior Dirichlet trace operator $\gamma_0^{\text{int}} : H^1(\Omega) \to H^{1/2}(\Gamma)$ to (1.2) which results in the boundary integral equation

$$(Vw)(x) := \int_\Gamma U^*(x,y)w(y)\,ds_y = g(x) \quad \text{for } x \in \Gamma, \tag{1.3}$$

and which is equivalent to a Galerkin–Bubnov formulation: Find $w \in H^{-1/2}(\Gamma)$ such that

$$\langle Vw, v\rangle_\Gamma = \langle g, v\rangle_\Gamma \quad \text{for all } v \in H^{-1/2}(\Gamma). \tag{1.4}$$

Since the single layer boundary integral operator $V = \gamma_0^{\text{int}}\widetilde{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is elliptic, [6],

$$\langle Vw, w\rangle_\Gamma \geq c_1^V \, \|w\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } w \in H^{-1/2}(\Gamma), \tag{1.5}$$

unique solvability of the variational formulation (1.4) follows. Moreover we can deduce a stability and error analysis of related boundary element discretization schemes, see, e.g., [7]. Error estimates then rely on the regularity of $w = V^{-1}g$, i.e. on the regularity of the given Dirichlet datum g, and on the mapping properties of the single layer boundary integral operator V. In the case of a Lipschitz domain Ω we have that $V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is bijective for all $s \in [-\frac{1}{2}, \frac{1}{2}]$, see [2, 8], while in the case of a polyhedral bounded domain this remains true for $|s| < s_0$ where $s_0 > \frac{1}{2}$ is determined by the related interior and exterior angles in corners and at edges, see, e.g., [5], and [4] for the two–dimensional case.

The error estimate for the Galerkin solution of the Galerkin–Bubnov variational formulation (1.4) is given, due to Cea's lemma, in the energy norm in $H^{-1/2}(\Gamma)$. Hence, to derive error estimates in stronger norms, e.g. in $L^2(\Gamma)$, we have to use an inverse inequality for the used boundary element space and where we have to assume a globally quasi–uniform boundary element mesh, see, e.g., [7] for a more detailed discussion. In fact, this excludes non–uniform and adaptive meshes as often used in practice.

Instead of the Galerkin–Bubnov variational formulation (1.4) we will consider a Galerkin–Petrov variational formulation which allows the use of different trial and test spaces, both in the continuous and discrete setting. In this case, the ellipticity estimate (1.5) has to be replaced by an appropriate stability condition, also known as inf sup condition. While the analysis of the Galerkin–Bubnov formulation (1.4) relies on a related domain variational formulation in $H^1(\Omega)$, our analysis is based on using a Galerkin–Petrov domain variational formulation for which we have to introduce suitable Sobolev spaces. With this we can not only conclude known mapping properties of the single layer boundary integral operator, but we can establish a new stability condition which ensures unique solvability of the Galerkin–Petrov variational formulation.

In this note we will not consider a stability and error analysis of related Galerkin–Petrov boundary element methods which will be a topic of further research. In fact, such an approach can also be used for Galerkin–Petrov variational formulations in weaker Sobolev spaces, e.g., when the given Dirichlet data have reduced regularity, for example if we have $g \in L^2(\Gamma)$ only, see, e.g., [1]. However, the main focus of future work will be on the extension of this concept to the mathematical and numerical analysis of boundary integral equation and boundary element methods for time–dependent problems such as the heat equation, see, e.g., [3] for related Galerkin–Bubnov formulations.

2 Strong domain variational formulation

For the Dirichlet boundary value problem (1.1) we consider, instead of a standard domain variational formulation in $H^1(\Omega)$ which is based on Green's first formula, a Galerkin–Petrov variational formulation. For this we introduce

$$H_\Delta(\Omega) := \Big\{ v \in H^1(\Omega) : \Delta v \in L^2(\Omega) \Big\} \subset H^1(\Omega),$$

with the norm

$$\|v\|^2_{H_\Delta(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \,.$$

Then we have to find $u \in H_\Delta(\Omega)$ satisfying $u(x) = g(x)$ for $x \in \Gamma$ such that

$$\int_\Omega [-\Delta u(x)] v(x)\, dx = 0 \quad \text{for all } v \in L^2(\Omega), \tag{2.1}$$

where we have to assume that the given Dirichlet datum g is in the Dirichlet trace space $\gamma_0^{\text{int}} H_\Delta(\Omega) \subset H^{1/2}(\Gamma)$. In particular, let $u_g \in H_\Delta(\Omega)$ be a bounded and norm preserving extension of $g \in \gamma_0^{\text{int}} H_\Delta(\Omega)$ with

$$\|g\|_{\gamma_0^{\text{int}} H_\Delta(\Omega)} = \min_{v \in H_\Delta(\Omega): v_{|\Gamma} = g} \|v\|_{H_\Delta(\Omega)} = \|u_g\|_{H_\Delta(\Omega)} \,. \tag{2.2}$$

It remains to find $u_0 \in X_S := H_\Delta(\Omega) \cap H_0^1(\Omega)$ such that

$$a_S(u_0, v) := \int_\Omega [-\Delta u_0(x)] v(x)\, dx = \int_\Omega [\Delta u_g(x)] v(x)\, dx \quad \text{for all } v \in Y_S := L^2(\Omega). \tag{2.3}$$

Related to the trial and test spaces we introduce the associated norms

$$\|u\|_{X_S} := \left[\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right]^{1/2}, \quad \|v\|_{Y_S} := \|v\|_{L^2(\Omega)} \,.$$

Lemma 2.1 *The bilinear form of the variational problem* (2.3), *is bounded, i.e.*

$$|a_S(u, v)| \le \|u\|_{X_S} \|v\|_{Y_S} \quad \text{for all } u \in X_S,\ v \in Y_S,$$

and satisfies the stability condition

$$c_S \, \|u\|_{X_S} \le \sup_{0 \ne v \in Y_S} \frac{a_S(u, v)}{\|v\|_{Y_S}} \quad \text{for all } u \in X_S, \quad c_S = \sqrt{\frac{\lambda_{\min}(\Omega)}{1 + \lambda_{\min}(\Omega)}}$$

where $\lambda_{\min}(\Omega)$ *is the minimal eigenvalue of the Dirichlet eigenvalue problem*

$$-\Delta u(x) = \lambda u(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma.$$

Proof. The boundedness of the bilinear form $a_S(\cdot, \cdot)$ is a direct consequence of the Cauchy–Schwarz inequality,

$$|a_S(u, v)| = \left| \int_\Omega [-\Delta u(x)] v(x)\, dx \right| \le \|\Delta u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|u\|_{X_S} \|v\|_{Y_S}.$$

To prove the stability condition we consider $u \in X_S$ and choose $v = u - \Delta u \in Y_S$. By using the minimal Dirichlet eigenvalue for the Laplacian in Ω,

$$\lambda_{\min}(\Omega) = \min_{0 \ne v \in H_0^1(\Omega)} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \,,$$

and Hölders inequality we have

$$\begin{aligned}
\|v\|_{Y_S} &= \|u-\Delta u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \\
&\leq \frac{1}{\sqrt{\lambda_{\min}(\Omega)}}\|\nabla u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \\
&\leq \Big(\frac{1}{\lambda_{\min}(\Omega)}+1\Big)^{1/2}\Big(\|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}\Big)^{1/2} = \sqrt{\frac{1+\lambda_{\min}(\Omega)}{\lambda_{\min}(\Omega)}}\,\|u\|_{X_S}\,.
\end{aligned}$$

Then,

$$\begin{aligned}
a_S(u,v) = a_S(u, u-\Delta u) &= \int_\Omega [-\Delta u(x)]\,[u(x)-\Delta u(x)]\,dx \\
&= \int_\Omega [-\Delta u(x)]u(x)\,dx + \int_\Omega [\Delta u(x)]^2\,dx \\
&= \int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega [\Delta u(x)]^2\,dx \\
&= \|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} = \|u\|^2_{X_S} \geq \sqrt{\frac{\lambda_{\min}(\Omega)}{1+\lambda_{\min}(\Omega)}}\,\|u\|_{X_S}\|v\|_{Y_S}
\end{aligned}$$

implies the stability condition as claimed. ■

As a consequence of Lemma 2.1 we conclude unique solvability of the variational problem (2.3) to obtain $u = u_0 + u_g \in H_\Delta(\Omega)$. In particular, when chosing in (2.1) $v = -\Delta u \in L^2(\Omega)$, this gives

$$\|\Delta u\|^2_{L^2(\Omega)} = \int_\Omega [-\Delta u(x)]^2\,dx \,=\, 0\,. \tag{2.4}$$

For the solution $u \in H_\Delta(\Omega) \subset H^1(\Omega)$ of the variational formulation (2.1) we note that the interior Neumann trace

$$\gamma_1^{\rm int} u(x) := \lim_{\Omega\ni\widetilde{x}\to x\in\Gamma} n_x \cdot \nabla_{\widetilde{x}} u(\widetilde{x}) = \frac{\partial}{\partial n_x} u(x) \quad \text{for } x \in \Gamma \tag{2.5}$$

is well defined, at least we have $\gamma_1^{\rm int} u \in H^{-1/2}(\Gamma)$ due to duality arguments and the use of Green's first formula. To do a more detailed analysis, for $u \in H_\Delta(\Omega)$ we define $\psi = \Delta u \in L^2(\Omega)$ and we consider the Dirichlet boundary value problem

$$-\Delta\phi(x) = \psi(x) \quad \text{for } x \in \Omega, \quad \phi(x) = 0 \quad \text{for } x \in \Gamma.$$

In the case of a domain Ω with a sufficient smooth boundary Γ or in the case of a convex polyhedron we find $\phi \in H^2(\Omega)$, and therefore $H_\Delta(\Omega) = H^2(\Omega)$ follows. However, this is not true when the domain Ω is polyhedral bounded, but non–convex. In this case, $H_\Delta(\Omega)$ includes harmonic functions which are not in $H^2(\Omega)$ but in $H^s(\Omega)$, $s < s_i$, for some $s_i > \frac{3}{2}$, see, for example, [5, Corollary 2.6.7]. In any case, the Neumann trace operator

$\gamma_1^{\rm int} : H_\Delta(\Omega) \to \gamma_1^{\rm int} H_\Delta(\Omega)$ is well defined, implying the Neumann trace space $\gamma_1^{\rm int} H_\Delta(\Omega)$, and satisfying

$$\|\gamma_1^{\rm int} v\|_{\gamma_1^{\rm int} H_\Delta(\Gamma)} \leq c_N \, \|v\|_{H_\Delta(\Omega)} \quad \text{for all } v \in H_\Delta(\Omega). \tag{2.6}$$

Lemma 2.2 *Let $u \in H_\Delta(\Omega)$ be the unique solution of the variational formulation* (2.1)*. Then,*

$$c_i \, \|\gamma_1^{int} u\|^2_{\gamma_1^{int} H_\Delta(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)}. \tag{2.7}$$

Proof. Let $u = u_0 + u_g \in H_\Delta(\Omega)$ be the unique solution of (2.3). We then define

$$\widetilde{u}(x) = u(x) - \gamma, \quad \gamma = \frac{1}{|\Gamma|} \int_\Gamma g(x) \, ds_x, \quad \int_\Gamma \widetilde{u}(x) \, ds_x = 0 \, ,$$

and where $\widetilde{u}$ is the unique solution of the Dirichlet boundary value problem

$$-\Delta \widetilde{u}(x) = 0 \quad \text{for } x \in \Omega, \quad \widetilde{u}(x) = g(x) - \gamma \quad \text{for } x \in \Gamma.$$

Obviously, $\widetilde{u} \in H_\Delta(\Omega)$, and (2.6) together with (2.4) then implies

$$\begin{aligned} \frac{1}{c_N} \, \|\gamma_1^{\rm int} \widetilde{u}\|^2_{\gamma_1^{\rm int} H_\Delta(\Omega)} \leq \|\widetilde{u}\|^2_{H_\Delta(\Omega)} &= \|\widetilde{u}\|^2_{L^2(\Omega)} + \|\nabla \widetilde{u}\|^2_{L^2(\Omega)} + \|\Delta \widetilde{u}\|^2_{L^2(\Omega)} \\ &= \|\widetilde{u}\|^2_{L^2(\Omega)} + \|\nabla \widetilde{u}\|^2_{L^2(\Omega)} \; = \; \|\widetilde{u}\|^2_{H^1(\Omega)} \, . \end{aligned}$$

Since an equivalent norm in $H^1(\Omega)$ is given by

$$\|v\|^2_{H^1_\Gamma(\Omega)} := \left[\int_\Gamma v(x) \, ds_x \right]^2 + \|\nabla v\|^2_{L^2(\Omega)},$$

we immediately conclude

$$\frac{1}{c_N} \, \|\gamma_1^{\rm int} \widetilde{u}\|^2_{\gamma_1 H_\Delta(\Omega)} \leq c \, \|\widetilde{u}\|^2_{H^1_\Gamma(\Omega)} = c \, \|\nabla \widetilde{u}\|^2_{L^2(\Omega)} \, .$$

Inserting $\widetilde{u} = u - \gamma$ concludes the proof. ■

In addition to the interior Dirichlet boundary value problem (1.1) we also consider the exterior Dirichlet problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega^c := \mathbb{R}^n \backslash \overline{\Omega}, \quad u(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.8}$$

where in addition we have to impose a suitable radiation condition,

$$u(x) = \mathcal{O}(1/|x|) \quad \text{as } |x| \to \infty. \tag{2.9}$$

When introducing the bounded domain $\Omega_r := B_r \backslash \overline{\Omega}$ with $B_r := \{x \in \mathbb{R}^3 : |x| < r\}$, and chosing $r > 0$ such that $\Omega \subset B_r$, we can proceed as in the case of the interior Dirichlet boundary value problem (1.1), when considering the limit $r \to \infty$ and the radiation condition (2.9). As in (2.7) we may define the exterior Neumann trace of the solution u satisfying

$$c_e \, \|\gamma_1^{\rm ext} u\|^2_{\gamma_1^{\rm ext} H_\Delta(\Omega^c)} \leq \|\nabla u\|^2_{L^2(\Omega^c)}. \tag{2.10}$$

Note that $H_\Delta(\Omega^c) \subseteq H^s(\Omega^c)$, $s < s_e$, for some $s_e > \frac{3}{2}$ which may differ from s_i.

3 Ultra week domain variational formulation

To derive mapping properties of the single layer boundary integral operator V we may also consider the ultra week domain variational formulation, see, e.g., [1]. Multiplying the partial differential equation in (1.1) with a test function $v \in H_\Delta(\Omega) \cap H_0^1(\Omega)$, integrating over Ω, and applying integration by parts twice, this gives

$$\begin{aligned} 0 = \int_\Omega [-\Delta u(x)] v(x)\, dx &= \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx \\ &= \int_\Omega u(x)[-\Delta v(x)]\, dx + \int_\Gamma u(x) \frac{\partial}{\partial n_x} v(x)\, ds_x\,. \end{aligned}$$

When inserting the Dirichlet boundary condition, this results in the Galerkin–Petrov variational formulation to find $u \in X_U = L^2(\Omega)$ such that

$$a_U(u,v) := \int_\Omega u(x)[\Delta v(x)]\, dx = \int_\Gamma g(x) \frac{\partial}{\partial n_x} v(x)\, ds_x \tag{3.1}$$

is satisfied for all $v \in Y_U = H_\Delta(\Omega) \cap H_0^1(\Omega)$. In this case we have to assume that the given Dirichlet datum g is in the dual of the interior Neumann trace space, i.e. $g \in [\gamma_1^{\rm int} H_\Delta(\Omega)]'$. We obviously have $X_U = Y_S$ and $Y_U = X_S$, respectively, with the associated norms

$$\|u\|_{X_U} = \|u\|_{L^2(\Omega)}, \quad \|v\|_{Y_U} = \Big[\|\nabla v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \Big]^{1/2}.$$

Similar as in Lemma 2.1 we can prove boundedness,

$$|a_U(u,v)| \leq \|u\|_{X_U} \|v\|_{Y_U} \quad \text{for all } u \in X_U,\ v \in Y_U,$$

and the stability condition

$$c_S\, \|u\|_{X_U} \leq \sup_{0 \neq v \in Y_U} \frac{a_U(u,v)}{\|v\|_{Y_U}} \quad \text{for all } u \in X_U.$$

As a consequence, we conclude unique solvability of the variational problem (3.1) to obtain $u \in X_U = L^2(\Omega)$.

4 Single layer potential

We now consider the single layer potential (1.2), $u(x) = (\widetilde{V} w)(x)$, $x \in \mathbb{R}^3 \backslash \Gamma$. When defining $g = \gamma_0^{\rm int} \widetilde{V} w$, we observe that u is a solution of the Dirichlet boundary value problem (1.1) being also the unique solution of the strong Galerkin–Petrov formulation (2.1). To ensure $u \in H_\Delta(\Omega)$, we chose $\psi \in [H_\Delta(\Omega)]'$ and we consider

$$\begin{aligned} \langle \widetilde{V} w, \psi \rangle_{H_\Delta(\Omega) \times [H_\Delta(\Omega)]'} &= \int_\Omega \psi(x) \int_\Gamma U^*(x,y) w(y)\, ds_y\, dx \\ &= \int_\Gamma w(y) \int_\Omega U^*(x,y) \psi(x)\, dx\, ds_y = \langle \varphi_{|\Gamma}, w \rangle_\Gamma \end{aligned}$$

where the duality pairing has to be specified. Using the Newton potential

$$\varphi(y) = (N_0\psi)(y) = \int_\Omega U^*(x,y)\psi(x)dx \quad \text{for } y \in \Omega$$

and the Dirichlet datum $\phi(y) = \varphi(y) = (N_0\psi)(y)$ for $y \in \Gamma$, we note that $\varphi \in X_U$ is the solution of the Dirichlet boundary value problem

$$-\Delta_y \varphi(y) = \psi(y) \quad \text{for } y \in \Omega, \quad \varphi(y) = \phi(y) \quad \text{for } y \in \Gamma.$$

In fact, $\varphi \in X_U$ solves, due to $\psi \in [H_\Delta(\Omega)]' \subset Y_U' = [H_\Delta(\Omega) \cap H_0^1(\Omega)]'$, the ultra–week variational formulation

$$\int_\Omega \varphi(x)\Delta v(x)\,dx = \int_\Gamma \phi(x)\frac{\partial}{\partial n_x} v(x)\,ds_x - \int_\Omega \psi(x)v(x)dx \quad \text{for all } v \in Y_U.$$

This variational formulation implies

$$\phi = \varphi_{|\Gamma} \in \left[\gamma_1^{\text{int}} Y_U\right]' = \left[\gamma_1^{\text{int}}[H_0^1(\Omega) \cap H_\Delta(\Omega)]\right]'$$

from which we further conclude

$$w \in \gamma_1^{\text{int}}[H_0^1(\Omega) \cap H_\Delta(\Omega)] = \gamma_1^{\text{int}} H_\Delta(\Omega)$$

as well as

$$\widetilde{V} : \gamma_1^{\text{int}} H_\Delta(\Omega) \to H_\Delta(\Omega)\,,$$

and finally,

$$V : \gamma_1^{\text{int}} H_\Delta(\Omega) \to \gamma_0^{\text{int}} H_\Delta(\Omega) \tag{4.1}$$

follows. In particular, when $H_\Delta(\Omega) \subseteq H^s(\Omega)$is satisfied for $\frac{3}{2} < s < s_i$, we have

$$V : H^{s-\frac{3}{2}}(\Gamma) \to H^{s-\frac{1}{2}}(\Gamma)\,.$$

Since we can do the same considerations subject to the exterior problem, we finally conclude

$$V : H^{s-1}(\Gamma) \to H^s(\Gamma) \quad \text{for all } s \in (1, \min\{s_i - 1/2, s_e - 1/2\})\,. \tag{4.2}$$

Note that in the case of a polygonal bounded domain $\Omega \subset \mathbb{R}^2$ this result was already given in [4].

Due to the mapping properties (4.2) and when assuming $g \in H^s(\Gamma)$ for some $s \in (1, \min\{s_i, s_e\})$ we may consider the Galerkin–Petrov variational formulation to find $w \in H^{s-1}(\Gamma)$ such that

$$\langle Vw, v\rangle_\Gamma = \langle g, v\rangle_\Gamma \quad \text{for all } v \in H^{-s}(\Gamma). \tag{4.3}$$

To prove unique solvability of the Galerkin–Petrov formulation (4.3) we need to establish an appropriate stability condition.

Theorem 4.1 *Let $w \in H^{s-1}(\Gamma)$ be given for some $s \in (1, \min\{s_i, s_e\})$. Then there holds the stability condition*

$$c_V \, \|w\|_{H^{s-1}(\Gamma)} \leq \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{\langle Vw, v\rangle_\Gamma}{\|v\|_{H^{-s}(\Gamma)}} \tag{4.4}$$

with a positive constant $c_V > 0$ independent of w.

Proof. In fact, we follow the standard approach to prove the ellipticity estimate (1.5), see, e.g., [6, 7]. Since $u = \widetilde{V}w$ is harmonic in Ω, Green's first formula implies

$$\int_\Gamma \frac{\partial}{\partial n_x} u(x) u(x) \, ds_x = \int_\Omega |\nabla u(x)|^2 \, dx \,. \tag{4.5}$$

With the jump relations for the interior Dirichlet and Neumann trace operators for the single layer potential we have

$$\gamma_0^{\rm int} u(x) = (Vw)(x), \quad \gamma_1^{\rm int} u(x) = \frac{1}{2} w(x) + (K'w)(x) \quad \text{for } x \in \Gamma,$$

where in addition to the single layer boundary integral operator V we used the adjoint double layer boundary integral operator,

$$(K'w)(x) = \int_\Gamma \frac{\partial}{\partial n_x} U^*(x,y) w(y) ds_y, \quad x \in \Gamma.$$

Hence, (4.5) gives

$$\langle (\frac{1}{2} I + K')w, Vw\rangle_\Gamma = \|\nabla u\|^2_{L^2(\Omega)} \,.$$

When doing the same considerations subject to the exterior Dirichlet boundary value problem, this gives

$$\langle (\frac{1}{2} I - K')w, Vw\rangle_\Gamma = \|\nabla u\|^2_{L^2(\Omega^c)}.$$

Note that in $\mathbb{R}^3$ the single layer potential $u = \widetilde{V}w$ satisfies the radiation condition (2.9). Now we conclude, by using (2.7) and (2.10),

$$\langle w, Vw\rangle_\Gamma = \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega^c)} \geq c_i \|\gamma_1^{\rm int} u\|^2_{H^{s-1}(\Gamma)} + c_e \|\gamma_1^{\rm ext} u\|^2_{H^{s-1}(\Gamma)}.$$

On the other hand we have

$$\|w\|^2_{H^{s-1}(\Gamma)} = \|\gamma_1^{\rm int} u - \gamma_1^{\rm ext} u\|^2_{H^{s-1}(\Gamma)} \leq 2 \left(\|\gamma_1^{\rm int} u\|^2_{H^{s-1}(\Gamma)} + \|\gamma_1^{\rm ext} u\|^2_{H^{s-1}(\Gamma)} \right)$$

and therefore we obtain

$$\langle w, Vw\rangle_\Gamma \geq c_S \, \|w\|^2_{H^{s-1}(\Gamma)} \geq c_V \, \|w\|_{H^{s-1}(\Gamma)} \|w\|_{H^{-s}(\Gamma)}$$

due to $H^{s-1}(\Gamma) \subset H^{-s}(\Gamma)$ for $s > 1$, which finally gives (4.4). ■

From the stability condition (4.4) we can conclude unique solvability of the Galerkin–Petrov formulation (4.3).

By using

$$\langle Vw, v\rangle_\Gamma = \langle w, Vv\rangle_\Gamma \quad \text{for } w \in H^{s-1}(\Gamma), v \in H^{-s}(\Gamma)$$

we can define the single layer boundary integral operator $V : H^{-s}(\Gamma) \to H^{1-s}(\Gamma)$, which satisfies the following stability condition.

Lemma 4.2 *Let $v \in H^{-s}$ be given for some $s \in (1, \min\{s_i, s_e\})$. Then there holds the stability condition*

$$c_V \|v\|_{H^{-s}(\Gamma)} \le \sup_{0 \neq w \in H^{s-1}(\Gamma)} \frac{\langle Vv, w\rangle_\Gamma}{\|w\|_{H^{s-1}(\Gamma)}} \tag{4.6}$$

with the positive constant $c_V > 0$ as used in (4.4).

Proof. For $g \in H^s(\Gamma)$ we find, by solving (4.3), $w \in H^{s-1}(\Gamma)$, and the stability condition (4.4) gives

$$c_V \|w\|_{H^{s-1}(\Gamma)} \le \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{\langle Vw, v\rangle_\Gamma}{\|v\|_{H^{-s}(\Gamma)}} = \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{\langle g, v\rangle_\Gamma}{\|v\|_{H^{-s}(\Gamma)}} \le \|g\|_{H^s(\Gamma)} .$$

Using duality we then conclude the stability estimate

$$\|v\|_{H^{-s}(\Gamma)} = \sup_{0 \neq g \in H^s(\Gamma)} \frac{\langle v, g\rangle_\Gamma}{\|g\|_{H^s(\Gamma)}} \le \frac{1}{c_V} \sup_{0 \neq w \in H^{s-1}(\Gamma)} \frac{\langle v, Vw\rangle_\Gamma}{\|w\|_{H^{s-1}(\Gamma)}}$$

as claimed. ■

Using the indirect single layer potential $u(x) = (\widetilde{V}v)(x)$, $x \in \Omega$ we can describe the solution of the Dirichlet boundary value problem (1.1) with a given Dirichlet datum $g \in H^{1-s}(\Gamma)$, i.e. $v \in H^{-s}(\Gamma)$ is the unique solution of the Galerkin–Petrov formulation

$$\langle Vv, w\rangle_\Gamma = \langle g, w\rangle_\Gamma \quad \text{for all } w \in H^{s-1}(\Gamma). \tag{4.7}$$

Due to $s > 1$ it is possible to consider $g \in L^2(\Gamma) \subset H^{1-s}(\Gamma)$ within the variational formulation (4.7) which can be seen as the boundary integral equation counter part of the ultra–week finite element formulation [1].

Remark 4.1 *In the two–dimensional case $\Omega \subset \mathbb{R}^2$ we need to have $w \in H^{s-1}(\Gamma)$ with the constraint $\langle w, 1\rangle_\Gamma = 0$ to satisfy the radiation condition* (2.9) *for the single layer potential $u(x) = (\widetilde{V}w)(x)$ as $|x| \to \infty$. To ensure solvability of the Galerkin–Petrov formulation* (4.3)*, g has to satisfy the solvability condition $\langle g, w_{eq}\rangle_\Gamma = 0$ with the natural density $w_{eq} = V^{-1}1$. So solvability of the Dirichlet boundary value problem (1.1) for general g can always be guaranteed when considering an appropriate additive splitting of $g(x) = \gamma_g + \widetilde{g}(x)$, $\gamma_g = \frac{\langle g, w_{eq}\rangle_\Gamma}{\langle 1, w_{eq}\rangle_\Gamma}$, where we have to assume diam$\Omega < 1$ to ensure $\langle 1, w_{eq}\rangle_\Gamma > 0$. All other results then remain true when considering appropriate factor spaces.*

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