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A generalized inf–sup stable variational formulation for the wave equation

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Abstract

In this paper, we consider a variational formulation for the Dirichlet problem of the wave equation with zero boundary and initial conditions, where we use integration by parts in space and time. To prove unique solvability in a subspace of $H^1(Q)$ with Q being the space–time domain, the classical assumption is to consider the right–hand side f in $L^2(Q)$. Here, we analyze a generalized setting of this variational formulation, which allows us to prove unique solvability also for f being in the dual space of the test space, i.e., the solution operator is an isomorphism between the ansatz space and the dual of the test space. This new approach is based on a suitable extension of the ansatz space to include the information of the differential operator of the wave equation at the initial time $t = 0$. These results are of utmost importance for the formulation and numerical analysis of unconditionally stable space–time finite element methods, and for the numerical analysis of boundary element methods to overcome the well–known norm gap in the analysis of boundary integral operators.

1 Introduction

For the analysis of hyperbolic partial differential equations, a variety of approaches such as Fourier methods, semigroups, or Galerkin methods are available, see, e.g., [18, 19, 20, 27, 30]. The theoretical results on existence and uniqueness of solutions also form the basis for the numerical analysis of related discretization schemes such as finite element methods, e.g., [5, 6, 7, 8, 10, 11, 14, 16, 21, 25, 26, 29, 31] or boundary element methods, e.g., [1, 4, 12, 13, 22].

As for elliptic second–order partial differential equations, we consider the weak solution of the inhomogeneous wave equation in the energy space $H^1(Q)$ with respect to the space–time domain $Q := \Omega \times (0,T)$. However, to ensure existence and uniqueness of a weak solution, we need to assume $f \in L^2(Q)$, see, e.g., [26, Theorem 5.1]. While this is a standard assumption to conclude sufficient regularity for the solution u, and therefore, to obtain linear convergence for piecewise linear finite element approximations u_h, stability of common finite element discretizations require in most cases some CFL condition, which relates the spatial and temporal mesh sizes to each other, see, e.g., [25, 26, 29, 31].

Although the variational formulation to find u in a suitable subspace of $H^1(Q)$ is well–defined for f being in the dual of the test space, this is not sufficient to establish unique solvability. This is mainly due to the missing information about the differential operator of the wave equation at $t = 0$ in the standard ansatz space. Hence, we are not able to define an isomorphism between the ansatz space and the dual of the test space. But such an isomorphism is an important ingredient in the analysis of equivalent boundary integral formulations for boundary value problems for the wave equation, and the numerical analysis of related boundary and finite element methods.

In this paper, we are interested in inf–sup stable variational formulations for the Dirichlet boundary value problem for the wave equation,

$$\left.\begin{array}{rcll} \Box u(x,t) & = & f(x,t) & \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) & = & 0 & \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) & = & 0 & \text{for } x \in \Omega, \\ \partial_t u(x,t)_{|t=0} & = & 0 & \text{for } x \in \Omega, \end{array}\right\} \tag{1.1}$$

where $\Box u := \partial_{tt} u - \Delta_x u$ is the wave operator, $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded domain with, for $d = 2,3$, Lipschitz boundary $\Gamma = \partial\Omega$, $T > 0$ is a finite time horizon, and f is some given source. For simplicity, we assume homogeneous Dirichlet boundary conditions as well as homogeneous initial conditions, see, e.g., [18, 29, 31] for the treatment of inhomogeneous initial conditions. A possible variational formulation of (1.1) is to find $u \in H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega))$ such that

$$a(u,v) = \int_0^T \int_\Omega f(x,t)\, v(x,t)\, \mathrm{d}x\, \mathrm{d}t \tag{1.2}$$

is satisfied for all $v \in H^{1,1}_{0;,0}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega))$ with the bilinear form $a(\cdot,\cdot)\colon\; H^{1,1}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$,

$$a(u,v) := -\int_0^T \int_\Omega \partial_t u(x,t)\, \partial_t v(x,t)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega \nabla_x u(x,t) \cdot \nabla_x v(x,t)\, \mathrm{d}x\, \mathrm{d}t \tag{1.3}$$

for $u \in H^{1,1}_{0;0,}(Q)$, $v \in H^{1,1}_{0;,0}(Q)$. In addition to the standard Bochner space $L^2(0,T;H^1_0(\Omega))$, we use the space $H^1_{0,}(0,T;L^2(\Omega))$ of all $v \in L^2(Q)$ with $\partial_t v \in L^2(Q)$, and $v(x,0) = 0$ for

$x \in \Omega$. Moreover, $H^1_{,0}(0,T;L^2(\Omega))$ is defined analogously with $v(x,T) = 0$ for $x \in \Omega$. The spaces $H^{1,1}_{0;0,}(Q)$, $H^{1,1}_{0;,0}(Q)$ are Hilbert spaces with the inner products

$$\langle w, z\rangle_{H^{1,1}_{0;0,}(Q)} := \langle w, z\rangle_{H^{1,1}_{0;,0}(Q)} := \langle \partial_t w, \partial_t z\rangle_{L^2(Q)} + \langle \nabla_x w, \nabla_x z\rangle_{L^2(Q)},$$

and the corresponding induced norms $\|\cdot\|_{H^{1,1}_{0;0,}(Q)}$, $\|\cdot\|_{H^{1,1}_{0;,0}(Q)}$. The bilinear form $a(\cdot,\cdot)$ in (1.3) is continuous, i.e., for all $u \in H^{1,1}_{0;0,}(Q)$ and $v \in H^{1,1}_{0;,0}(Q)$ we have

$$\begin{aligned} |a(u,v)| &\leq \|\partial_t u\|_{L^2(Q)}\|\partial_t v\|_{L^2(Q)} + \|\nabla_x u\|_{L^2(Q)}\|\nabla_x v\|_{L^2(Q)} \\ &\leq \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}}\sqrt{\|\partial_t v\|^2_{L^2(Q)} + \|\nabla_x v\|^2_{L^2(Q)}} \\ &= \|u\|_{H^{1,1}_{0;0,}(Q)}\|v\|_{H^{1,1}_{0;,0}(Q)}. \end{aligned}$$

Note that the first initial condition $u(\cdot,0) = 0$ in Ω is incorporated in the ansatz space $H^{1,1}_{0;0,}(Q)$, whereas the second initial condition $\partial_t u(\cdot,t)_{|t=0} = 0$ in Ω is considered as a natural condition in the variational formulation. Thus, an inhomogeneous condition $\partial_t u(\cdot,t)_{|t=0} = v_0$ in Ω with given v_0 could be realized with the right–hand side $f_{v_0} \in [H^{1,1}_{0;,0}(Q)]'$,

$$\langle f_{v_0}, v\rangle_Q = \langle v_0, v(\cdot,0)\rangle_\Omega, \quad v \in H^{1,1}_{0;,0}(Q). \tag{1.4}$$

However, known existence results for the variational formulation (1.2) do not allow right–hand sides in $[H^{1,1}_{0;,0}(Q)]'$, which is the aim of the new approach as given in Section 3. So, when assuming $f \in L^2(Q)$, we are able to construct a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation (1.2), satisfying the stability estimate [26, Theorem 5.1], see also [18, 25, 28],

$$\|u\|_{H^{1,1}_{0;0,}(Q)} = \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}} \leq \frac{1}{\sqrt{2}} T\, \|f\|_{L^2(Q)}\,. \tag{1.5}$$

The stability estimate (1.5) does not fit to the situation of the Banach–Nečas–Babuška theorem as stated, e.g., in [9, Theorem 2.6], see also [2, 3, 17]. The next theorem states that it is not possible to prove the corresponding inf–sup condition, i.e., (1.6), for the bilinear form (1.3).

Theorem 1.1 [28, Theorem 4.2.24] *There does not exist a constant $c > 0$ such that each right–hand side $f \in L^2(Q)$ and the corresponding solution $u \in H^{1,1}_{0;0,}(Q)$ of (1.2) satisfy*

$$\|u\|_{H^{1,1}_{0;0,}(Q)} \leq c\, \|f\|_{[H^{1,1}_{0;,0}(Q)]'}.$$

In particular, the inf–sup condition

$$c_S\, \|u\|_{H^{1,1}_{0;0,}(Q)} \leq \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|a(u,v)|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \quad \textit{for all } u \in H^{1,1}_{0;0,}(Q) \tag{1.6}$$

with a constant $c_S > 0$ does not hold true.

The proofs of the stability estimate (1.5) and Theorem 1.1 are based on an appropriate Fourier analysis, using the eigenfunctions of the spatial differential operator $-\Delta_x$, and the analysis of the related ordinary differential equation (1.7), which allows us also to present the essential ingredients for the new approach. So, for $\mu > 0$, we consider the scalar ordinary differential equation

$$\Box_\mu u(t) := \partial_{tt} u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(t)_{|t=0} = 0. \tag{1.7}$$

The related variational formulation is to find $u \in H^1_{0,}(0,T)$ for a given right–hand side $f \in [H^1_{,0}(0,T)]'$ such that

$$a_\mu(u,v) = \langle f, v\rangle_{(0,T)} \tag{1.8}$$

is satisfied for all $v \in H^1_{,0}(0,T)$. The bilinear form $a_\mu(\cdot,\cdot)\colon H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$ is defined by

$$a_\mu(u,v) := -\int_0^T \partial_t u(t)\, \partial_t v(t)\, \mathrm{d}t + \mu \int_0^T u(t)\, v(t)\, \mathrm{d}t \tag{1.9}$$

for $u \in H^1_{0,}(0,T)$, $v \in H^1_{,0}(0,T)$. As before, $u \in H^1_{0,}(0,T)$ covers the initial condition $u(0) = 0$, while $v \in H^1_{,0}(0,T)$ satisfies the terminal condition $v(T) = 0$, and where the inner product $\langle \partial_t(\cdot), \partial_t(\cdot)\rangle_{L^2(0,T)}$ makes both to Hilbert spaces. Note that the second initial condition $\partial_t u(t)_{|t=0} = 0$ enters the variational formulation (1.8) as natural condition. The dual space $[H^1_{,0}(0,T)]'$ is characterized as completion of $L^2(0,T)$ with respect to the Hilbertian norm

$$\|f\|_{[H^1_{,0}(0,T)]'} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle f, v\rangle_{(0,T)}|}{\|\partial_t v\|_{L^2(0,T)}},$$

where $\langle \cdot,\cdot\rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0,T)$, see, e.g., [27, Satz 17.3]. The continuity of the bilinear form (1.9) follows from

$$|a_\mu(u,v)| \leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) \|\partial_t u\|_{L^2(0,T)} \|\partial_t v\|_{L^2(0,T)} \tag{1.10}$$

for all $u \in H^1_{0,}(0,T)$ and $v \in H^1_{,0}(0,T)$, where the Cauchy–Schwarz and the Poincaré inequalities [28, Lemma 3.4.5] are used. Furthermore, the bilinear form (1.9) satisfies the inf–sup condition [26, Lemma 4.2]

$$\frac{2}{2 + \sqrt{\mu} T} \|\partial_t u\|_{L^2(0,T)} \leq \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|a_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} \quad \text{for all } u \in H^1_{0,}(0,T). \tag{1.11}$$

Together with the positivity condition [28, Lemma 4.2.4]

$$a_\mu(u_v, v) = \langle v, v\rangle_{L^2(0,T)} = \|v\|^2_{L^2(0,T)} > 0$$

for

$$u_v(t) = \frac{1}{\sqrt{\mu}} \int_0^t v(s) \sin(\sqrt{\mu}(t-s))\, \mathrm{d}s, \quad 0 \neq v \in L^2(0,T),$$

we conclude unique solvability of (1.8) as well as the stability estimate

$$\|\partial_t u\|_{L^2(0,T)} \leq \left(1 + \frac{1}{2}\sqrt{\mu}T\right) \|f\|_{[H^1_{,0}(0,T)]'} \tag{1.12}$$

with the help of the Banach–Nečas–Babuška theorem [9, Theorem 2.6]. As discussed in [26, Remark 4.4], the stability estimate (1.12) is sharp in $\sqrt{\mu}$ and T, respectively. It turns out, however, that the estimate (1.12) is not sufficient to prove a related stability estimate for the solution of the wave equation (1.1), see Theorem 1.1. This is mainly due to the appearance of $\sqrt{\mu}$ in the stability constant, i.e., (1.12) is not uniform in μ. Instead, when assuming $f \in L^2(0,T)$, we can prove the stability estimate [26, Lemma 4.5]

$$\|\partial_t u\|^2_{L^2(0,T)} + \mu \|u\|^2_{L^2(0,T)} \leq \frac{1}{2}T^2 \|f\|^2_{L^2(0,T)}, \tag{1.13}$$

which allows to prove the stability estimate (1.5) for the solution of the wave equation (1.1), see [26, Theorem 5.1] and [18, 28].

Since the variational formulation (1.8) is well–defined also for $f \in [H^1_{,0}(0,T)]'$, we are interested to establish, instead of (1.11), an inf–sup stability condition with a constant, which is independent of μ, and which later on can be generalized to the analysis of the variational problem (1.2).

The remainder of this paper is structured as follows: In Section 2, we present the main ideas in order to solve the ordinary differential equation (1.7). For this purpose, we introduce a suitable function space and prove a related inf–sup stability condition. Then, in Section 3, these results are generalized to analyze a variational formulation for the wave equation (1.1). The main result of this paper is given in Theorem 3.9, where we state bijectivity of the solution operator for the Dirichlet problem of the wave equation with zero boundary and initial conditions. Finally, in Section 4, we give some conclusions and comment on ongoing work.

2 Variational formulation for the ODE

In this section, we derive a different setting of a variational formulation for (1.7) in order to establish an inf–sup condition with a constant independent of μ. For this purpose, we first follow the approach as for the heat equation, see, e.g., [26]. So, for given $u \in H^1_{0,}(0,T)$, we define

$$\langle \partial_{tt} u + \mu u, v\rangle_{(0,T)} := a_\mu(u,v) \quad \text{for all } v \in H^1_{,0}(0,T).$$

Since the bilinear form $a_\mu(\cdot,\cdot)$ is continuous, see (1.10), $\partial_{tt}u + \mu u$ is a continuous functional in $[H^1_{,0}(0,T)]'$. Therefore, by the Riesz representation theorem there exists a unique element $w_u \in H^1_{,0}(0,T)$, satisfying

$$\langle \partial_{tt} u + \mu u, v\rangle_{(0,T)} = \langle \partial_t w_u, \partial_t v\rangle_{L^2(0,T)} \quad \text{for all } v \in H^1_{,0}(0,T),$$

and

$$\|\partial_{tt}u + \mu u\|^2_{[H^1_{,0}(0,T)]'} = a_\mu(u, w_u) = \|\partial_t w_u\|^2_{L^2(0,T)}. \tag{2.1}$$

At first glance, (2.1) implies the inf–sup condition

$$\|\partial_{tt}u + \mu u\|_{[H^1_{,0}(0,T)]'} = \sup_{0\neq v\in H^1_{,0}(0,T)} \frac{|a_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} \quad \text{for } u \in H^1_{0,}(0,T),$$

but $u \mapsto \|\partial_{tt}u + \mu u\|_{[H^1_{,0}(0,T)]'}$ only defines a semi–norm in $H^1_{0,}(0,T)$ since the differential operator $\partial_{tt}+\mu$ is treated only in $(0,T)$, i.e., its behavior in $t = 0$ is not covered in (2.1). As example, consider, e.g., $u \in H^1_{0,}(0,T)$ with $u(t) = \sin(\sqrt{\mu}t)$ and $\|\partial_{tt}u + \mu u\|_{[H^1_{,0}(0,T)]'} = 0$. Hence, we need to modify the ansatz space to determine u in a suitable way. For this purpose, we first introduce notations for additional function spaces and operators.

In this work, $C_0^\infty(A)$ is the set of infinitely differentiable real–valued functions with compact support in any domain $A \subset \mathbb{R}^d$, $d = 1, 2, 3, 4$. The set $C_0^\infty(A)$ is endowed with the, usual for distributions, locally convex topology and is called the space of test functions on A. The set of (Schwartz) distributions $[C_0^\infty(A)]'$ is given by all linear and sequentially continuous functionals on $C_0^\infty(A)$.

For given $u \in L^2(0,T)$, we define the extension $\widetilde{u} \in L^2(-T,T)$ by

$$\widetilde{u}(t) := \begin{cases} u(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T, 0]. \end{cases} \tag{2.2}$$

The application of the differential operator $\Box_\mu$ to $\widetilde{u}$ is defined as distribution on $(-T,T)$, i.e., for all test functions $\varphi \in C_0^\infty(-T,T)$, we define

$$\langle \Box_\mu \widetilde{u}, \varphi\rangle_{(-T,T)} := \int_{-T}^T \widetilde{u}(t)\,\Box_\mu\varphi(t)\,\mathrm{d}t = \int_0^T u(t)\,\Box_\mu\varphi(t)\,\mathrm{d}t. \tag{2.3}$$

This motivates to consider the dual space $[H^1_0(-T,T)]'$ of $H^1_0(-T,T)$, which is characterized as completion of $L^2(-T,T)$ with respect to the Hilbertian norm

$$\|g\|_{[H^1_0(-T,T)]'} := \sup_{0\neq z\in H^1_0(-T,T)} \frac{|\langle g, z\rangle_{(-T,T)}|}{\|\partial_t z\|_{L^2(-T,T)}},$$

where $\langle\cdot,\cdot\rangle_{(-T,T)}$ denotes the duality pairing as extension of the inner product in $L^2(-T,T)$, see, e.g., [27, Satz 17.3]. In other words, for $[H^1_0(-T,T)]'$, there exists an inner product $\langle\cdot,\cdot\rangle_{[H^1_0(-T,T)]'}$, inducing the norm $\|\cdot\|_{[H^1_0(-T,T)]'} = \sqrt{\langle\cdot,\cdot\rangle_{[H^1_0(-T,T)]'}}$, i.e., with this abstract inner product, $[H^1_0(-T,T)]'$ is a Hilbert space. Additionally, we define the subspace

$$H^{-1}_{[0,T]}(-T,T) := \Big\{ g \in [H^1_0(-T,T)]' \colon \\ \forall z \in H^1_0(-T,T) \text{ with } \operatorname{supp} z \subset (-T,0)\colon\ \langle g, z\rangle_{(-T,T)} = 0 \Big\} \subset [H^1_0(-T,T)]',$$

endowed with the Hilbertian norm $\|\cdot\|_{[H^1_0(-T,T)]'}$. To characterize the subspace $H^{-1}_{[0,T]}(-T,T)$, we introduce the following notations. Let $\mathcal{R} : H^1_0(-T,T) \to H^1_{,0}(0,T)$ be the continuous and surjective restriction operator, defined by $\mathcal{R}z = z_{|(0,T)}$ for $z \in H^1_0(-T,T)$, with its adjoint operator $\mathcal{R}' : [H^1_{,0}(0,T)]' \to [H^1_0(-T,T)]'$. Furthermore, let $\mathcal{E} : H^1_{,0}(0,T) \to H^1_0(-T,T)$ be any continuous and injective extension operator with its adjoint operator $\mathcal{E}' : [H^1_0(-T,T)]' \to [H^1_{,0}(0,T)]'$, satisfying

$$\|\mathcal{E}v\|_{H^1_0(-T,T)} \leq c_{\mathcal{E}} \|v\|_{H^1_{,0}(0,T)}$$

with a constant $c_{\mathcal{E}} > 0$ and $\mathcal{R}\mathcal{E}v = v$ for all $v \in H^1_{,0}(0,T)$. An example for such an extension operator is given by reflection in $t = 0$, i.e., consider the function $\overline{v}$, defined for $v \in H^1_{,0}(0,T)$ by

$$\overline{v}(t) = \begin{cases} v(t) & \text{for } t \in [0,T), \\ v(-t) & \text{for } t \in (-T,0), \end{cases}$$

which leads to the constant $c_{\mathcal{E}} = 2$ in this particular case. With this, we prove the following lemma.

Lemma 2.1 *The spaces* $(H^{-1}_{[0,T]}(-T,T), \|\cdot\|_{[H^1_0(-T,T)]'})$ *and* $([H^1_{,0}(0,T)]', \|\cdot\|_{[H^1_{,0}(0,T)]'})$ *are isometric, i.e., the mapping*

$$\mathcal{E}'_{|H^{-1}_{[0,T]}(-T,T)} \colon\ H^{-1}_{[0,T]}(-T,T) \to [H^1_{,0}(0,T)]'$$

is bijective with

$$\|g\|_{[H^1_0(-T,T)]'} = \|\mathcal{E}'g\|_{[H^1_{,0}(0,T)]'} \quad \textit{for all } g \in H^{-1}_{[0,T]}(-T,T).$$

In addition, for $g \in H^{-1}_{[0,T]}(-T,T)$*, the relation*

$$\langle g, z\rangle_{(-T,T)} = \langle \mathcal{E}'g, \mathcal{R}z\rangle_{(0,T)} \quad \textit{for all } z \in H^1_0(-T,T) \tag{2.4}$$

holds true, i.e., $\mathcal{R}'\mathcal{E}'g = g$*. In particular, the subspace* $H^{-1}_{[0,T]}(-T,T) \subset [H^1_0(-T,T)]'$ *is closed, i.e., complete.*

Proof. First, we prove that $\|g\|_{[H^1_0(-T,T)]'} = \|\mathcal{E}'g\|_{[H^1_{,0}(0,T)]'}$ for all $g \in H^{-1}_{[0,T]}(-T,T)$. For this purpose, let $g \in H^{-1}_{[0,T]}(-T,T)$ be arbitrary but fixed. The Riesz representation theorem gives the unique element $z_g \in H^1_0(-T,T)$ with

$$\langle g, z\rangle_{(-T,T)} = \langle \partial_t z_g, \partial_t z\rangle_{L^2(-T,T)} \quad \text{for all } z \in H^1_0(-T,T),$$

and $\|g\|_{[H^1_0(-T,T)]'} = \|\partial_t z_g\|_{L^2(-T,T)}$. It holds true that $z_{g|(-T,0)} = 0$, since we have

$$0 = \langle g, z\rangle_{(-T,T)} = \langle \partial_t z_g, \partial_t z\rangle_{L^2(-T,T)} = \langle \partial_t z_{g|(-T,0)}, \partial_t z_{|(-T,0)}\rangle_{L^2(-T,0)}$$

for all $z \in H^1_0(-T,T)$ with $\operatorname{supp} z \subset (-T,0)$. Hence, we have

$$\langle g, z\rangle_{(-T,T)} = \langle \partial_t z_g, \partial_t z\rangle_{L^2(-T,T)} = \langle \partial_t \mathcal{R} z_g, \partial_t \mathcal{R} z\rangle_{L^2(0,T)} \tag{2.5}$$

for all $z \in H^1_0(-T,T)$. So, using (2.5) with $z = \mathcal{E}v$ for $v \in H^1_{,0}(0,T)$ this gives

$$\langle \mathcal{E}' g, v\rangle_{(0,T)} = \langle g, \mathcal{E}v\rangle_{(-T,T)} = \langle \partial_t \mathcal{R} z_g, \partial_t \mathcal{R}\mathcal{E} v\rangle_{L^2(0,T)} = \langle \partial_t \mathcal{R} z_g, \partial_t v\rangle_{L^2(0,T)}, \tag{2.6}$$

i.e.,

$$\|\mathcal{E}' g\|_{[H^1_{,0}(0,T)]'} = \|\partial_t \mathcal{R} z_g\|_{L^2(0,T)} = \|\partial_t z_g\|_{L^2(-T,T)} = \|g\|_{[H^1_0(-T,T)]'}.$$

Second, we prove that $\mathcal{E}'_{|H^{-1}_{[0,T]}(-T,T)}$ is surjective. For this purpose, let $f \in [H^1_{,0}(0,T)]'$ be given. Set $g_f = \mathcal{R}' f \in [H^1_0(-T,T)]'$, i.e.,

$$\langle g_f, z\rangle_{(-T,T)} = \langle \mathcal{R}' f, z\rangle_{(-T,T)} = \langle f, \mathcal{R} z\rangle_{(0,T)}$$

for all $z \in H^1_0(-T,T)$. With this it follows immediately that $g_f \in H^{-1}_{[0,T]}(-T,T)$. Moreover, we have

$$\langle \mathcal{E}' g_f, v\rangle_{(0,T)} = \langle g_f, \mathcal{E} v\rangle_{(-T,T)} = \langle f, \mathcal{R}\mathcal{E} v\rangle_{(0,T)} = \langle f, v\rangle_{(0,T)}$$

for all $v \in H^1_{,0}(0,T)$, i.e., $\mathcal{E}' g_f = f$ in $[H^1_{,0}(0,T)]'$. In other words, $\mathcal{E}'_{|H^{-1}_{[0,T]}(-T,T)}$ is surjective. Third, the equality (2.4) follows from (2.5) and (2.6) for $v = \mathcal{R} z$ for any $z \in H^1_0(-T,T)$. The last assertion of the lemma is straightforward. ∎

The last lemma gives immediately the following corollary.

Corollary 2.2 *For all $g \in H^{-1}_{[0,T]}(-T,T)$, the norm representation*

$$\|g\|_{[H^1_0(-T,T)]'} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle g, \mathcal{E} v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}}$$

holds true.

Proof. Let $g \in H^{-1}_{[0,T]}(-T,T)$ be arbitrary but fixed. With Lemma 2.1, we have

$$\|g\|_{[H^1_0(-T,T)]'} = \|\mathcal{E}' g\|_{[H^1_{,0}(0,T)]'} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle \mathcal{E}' g, v\rangle_{(0,T)}|}{\|\partial_t v\|_{L^2(0,T)}} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle g, \mathcal{E} v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}},$$

i.e., the assertion is proven. ∎

Next, we introduce

$$\mathcal{H}(0,T) := \Big\{ u = \widetilde{u}_{|(0,T)} : \widetilde{u} \in L^2(-T,T),\ \widetilde{u}_{|(-T,0)} = 0,\ \Box_\mu \widetilde{u} \in [H^1_0(-T,T)]' \Big\}$$

with the norm

$$\|u\|_{\mathcal{H}(0,T)} := \sqrt{\|u\|^2_{L^2(0,T)} + \|\Box_\mu \widetilde{u}\|^2_{[H^1_0(-T,T)]'}}\,.$$

For a function $u \in \mathcal{H}(0,T)$, the condition $\Box_\mu \widetilde{u} \in [H_0^1(-T,T)]'$ involves that there exists an element $f_u \in [H_0^1(-T,T)]'$ with

$$\langle \Box_\mu \widetilde{u}, \varphi \rangle_{(-T,T)} = \langle f_u, \varphi \rangle_{(-T,T)} \quad \text{for all } \varphi \in C_0^\infty(-T,T).$$

Note that $\varphi \in H_0^1(-T,T)$ for $\varphi \in C_0^\infty(-T,T)$, and that $C_0^\infty(-T,T)$ is dense in $H_0^1(-T,T)$. Hence, the element $f_u \in [H_0^1(-T,T)]'$ is unique and therefore, in the following, we identify the distribution $\Box_\mu \widetilde{u}\colon C_0^\infty(-T,T) \to \mathbb{R}$ with the functional $f_u\colon H_0^1(-T,T) \to \mathbb{R}$.

Next, we state properties of the space $\mathcal{H}(0,T)$. Clearly, $(\mathcal{H}(0,T), \|\cdot\|_{\mathcal{H}(0,T)})$ is a normed vector space, and it is even a Banach space.

Lemma 2.3 *The normed vector space $(\mathcal{H}(0,T), \|\cdot\|_{\mathcal{H}(0,T)})$ is a Banach space.*

Proof. Consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}(0,T)$. Hence, $(u_n)_{n\in\mathbb{N}} \subset L^2(0,T)$ is also a Cauchy sequence in $L^2(0,T)$, and $(\Box_\mu \widetilde{u}_n)_{n\in\mathbb{N}} \subset [H_0^1(-T,T)]'$ is also a Cauchy sequence in $[H_0^1(-T,T)]'$. So, there exist $u \in L^2(0,T)$ and $f \in [H_0^1(-T,T)]'$ with

$$\lim_{n\to\infty} \|u_n - u\|_{L^2(0,T)} = 0, \quad \lim_{n\to\infty} \|\Box_\mu \widetilde{u}_n - f\|_{[H_0^1(-T,T)]'} = 0.$$

For $\varphi \in C_0^\infty(-T,T)$, we have

$$\begin{aligned}\langle \Box_\mu \widetilde{u}, \varphi \rangle_{(-T,T)} &= \langle \widetilde{u}, \Box_\mu \varphi \rangle_{L^2(-T,T)} = \int_0^T u(t)\, \Box_\mu \varphi(t)\, \mathrm{d}t = \lim_{n\to\infty} \int_0^T u_n(t)\, \Box_\mu \varphi(t)\, \mathrm{d}t \\ &= \lim_{n\to\infty} \langle \widetilde{u}_n, \Box_\mu \varphi \rangle_{L^2(-T,T)} = \lim_{n\to\infty} \langle \Box_\mu \widetilde{u}_n, \varphi \rangle_{(-T,T)} = \langle f, \varphi \rangle_{(-T,T)},\end{aligned}$$

i.e., $\Box_\mu \widetilde{u} = f \in [H_0^1(-T,T)]'$. Hence, $u \in \mathcal{H}(0,T)$ follows. ∎

With the abstract inner product $\langle \cdot, \cdot \rangle_{[H_0^1(-T,T)]'}$ of $[H_0^1(-T,T)]'$, the inner product

$$\langle u, v \rangle_{\mathcal{H}(0,T)} := \langle u, v \rangle_{L^2(0,T)} + \langle \Box_\mu \widetilde{u}, \Box_\mu \widetilde{v} \rangle_{[H_0^1(-T,T)]'}, \quad u, v \in \mathcal{H}(0,T),$$

induces the norm $\|\cdot\|_{\mathcal{H}(0,T)}$. Hence, the space $(\mathcal{H}(0,T), \langle\cdot,\cdot\rangle_{\mathcal{H}(0,T)})$ is even a Hilbert space, but this abstract inner product is not used explicitly in the remainder of this work.

Lemma 2.4 *For all $u \in \mathcal{H}(0,T)$ there holds $\Box_\mu \widetilde{u} \in H^{-1}_{[0,T]}(-T,T)$ and*

$$\|\Box_\mu \widetilde{u}\|_{[H_0^1(-T,T)]'} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle \Box_\mu \widetilde{u}, \mathcal{E} v \rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}}. \tag{2.7}$$

Proof. First, we prove that $\Box_\mu \widetilde{u} \in H^{-1}_{[0,T]}(-T,T)$. For this purpose, let $u \in \mathcal{H}(0,T)$ and $z \in H_0^1(-T,T)$ with $\operatorname{supp} z \subset (-T,0)$ be arbitrary but fixed. Due to $z_{|(-T,0)} \in H_0^1(-T,0)$ there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(-T,0)$ with $\|\partial_t z_{|(-T,0)} - \partial_t \psi_n\|_{L^2(-T,0)} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, define

$$\varphi_n(t) = \begin{cases} \psi_n(t) & \text{for } t \in (-T,0), \\ 0 & \text{for } t \in [0,T), \end{cases}$$

i.e., $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(-T,T)$ satisfies

$$\|\partial_t z - \partial_t \varphi_n\|_{L^2(-T,T)} = \|\partial_t z_{|(-T,0)} - \partial_t \psi_n\|_{L^2(-T,0)} \to 0$$

as $n \to \infty$. So, it follows that

$$\langle \Box_\mu \widetilde{u}, z\rangle_{(-T,T)} = \lim_{n\to\infty} \langle \Box_\mu \widetilde{u}, \varphi_n\rangle_{(-T,T)} = \lim_{n\to\infty} \int_0^T u(t)\, \Box_\mu \varphi_n(t)\, \mathrm{d}t = 0,$$

and therefore, the assertion. The norm representation follows from $\Box_\mu \widetilde{u} \in H^{-1}_{[0,T]}(-T,T)$ and Corollary 2.2. ■

Lemma 2.5 *It holds true that $H^1_{0,}(0,T) \subset \mathcal{H}(0,T)$. Furthermore, each $u \in H^1_{0,}(0,T)$ with zero extension $\widetilde{u}$, as defined in (2.2), satisfies*

$$\|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} \le \left(1 + \frac{4}{\pi^2}\mu T^2\right) \|\partial_t u\|_{L^2(0,T)}, \tag{2.8}$$

and

$$\langle \Box_\mu \widetilde{u}, z\rangle_{(-T,T)} = a_\mu(u, \mathcal{R}z) = -\langle \partial_t u, \partial_t \mathcal{R}z\rangle_{L^2(0,T)} + \mu\, \langle u, \mathcal{R}z\rangle_{L^2(0,T)} \tag{2.9}$$

for all $z \in H^1_0(-T,T)$, where $a_\mu(\cdot,\cdot)$ is the bilinear form (1.9).

Proof. First, we prove that $H^1_{0,}(0,T) \subset \mathcal{H}(0,T)$. For $u \in H^1_{0,}(0,T)$, we define the extension $\widetilde{u}$, see (2.2). By construction, we have $\widetilde{u} \in L^2(-T,T)$, and $\widetilde{u}_{|(-T,0)} = 0$. It remains to prove that $\Box_\mu \widetilde{u} \in [H^1_0(-T,T)]'$. For this purpose, define the functional $f_u \in [H^1_0(-T,T)]'$ by

$$\langle f_u, z\rangle_{(-T,T)} := a_\mu(u, \mathcal{R}z)$$

for all $z \in H^1_0(-T,T)$, where $a_\mu(\cdot,\cdot)$ is the bilinear form (1.9). The continuity of f_u follows from

$$\begin{aligned} |\langle f_u, z\rangle_{(-T,T)}| = |a_\mu(u, \mathcal{R}z)| &\le \left(1 + \frac{4}{\pi^2}\mu T^2\right) \|\partial_t u\|_{L^2(0,T)} \|\partial_t \mathcal{R}z\|_{L^2(0,T)} \\ &\le \left(1 + \frac{4}{\pi^2}\mu T^2\right) \|\partial_t u\|_{L^2(0,T)} \|\partial_t z\|_{L^2(-T,T)} \end{aligned}$$

for all $z \in H^1_0(-T,T)$, where the estimate (1.10) is used. Using the definition (2.3), and integration by parts, this gives

$$\begin{aligned} \langle \Box_\mu \widetilde{u}, \varphi\rangle_{(-T,T)} &= \int_0^T u(t)\, \Box_\mu \varphi(t)\, \mathrm{d}t = -\langle \partial_t u, \partial_t \mathcal{R}\varphi\rangle_{L^2(0,T)} + \mu \langle u, \mathcal{R}\varphi\rangle_{L^2(0,T)} \\ &= \langle f_u, \varphi\rangle_{(-T,T)} \end{aligned}$$

for all $\varphi \in C_0^\infty(-T,T)$, i.e., $\Box_\mu \widetilde{u} = f_u \in [H_0^1(-T,T)]'$. The equality (2.9) follows from the density of $C_0^\infty(-T,T)$ in $H_0^1(-T,T)$. The estimate (2.8) is proven by

$$\begin{aligned} \|\Box_\mu \widetilde{u}\|_{[H_0^1(-T,T)]'} &= \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle \Box_\mu \widetilde{u}, \mathcal{E}v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle f_u, \mathcal{E}v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} \\ &= \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|a_\mu(u, \mathcal{R}\mathcal{E}v)|}{\|\partial_t v\|_{L^2(0,T)}} \leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) \|\partial_t u\|_{L^2(0,T)}, \end{aligned}$$

when using the norm representation (2.7), the equality (2.9), and the bound (1.10). ∎

Next, by completion, we define the Hilbert space

$$\mathcal{H}_{0,}(0,T) := \overline{H^1_{0,}(0,T)}^{\|\cdot\|_{\mathcal{H}(0,T)}} \subset \mathcal{H}(0,T),$$

endowed with the Hilbertian norm $\|\cdot\|_{\mathcal{H}(0,T)}$, i.e.,

$$\mathcal{H}_{0,}(0,T) = \left\{ v \in \mathcal{H}(0,T) \colon \exists (v_n)_{n\in\mathbb{N}} \subset H^1_{0,}(0,T) \text{ with } \lim_{n\to\infty} \|v_n - v\|_{\mathcal{H}(0,T)} = 0 \right\}.$$

Lemma 2.6 *For $u \in \mathcal{H}_{0,}(0,T)$ there holds*

$$\|\Box_\mu \widetilde{u}\|_{[H_0^1(-T,T)]'} \geq \frac{\sqrt{2}}{T} \|u\|_{L^2(0,T)}.$$

Proof. For $0 \neq u \in \mathcal{H}_{0,}(0,T)$, there exists a non–trivial sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_{0,}(0,T)$, $u_n \not\equiv 0$, with

$$\lim_{n\to\infty} \|u - u_n\|_{\mathcal{H}(0,T)} = 0.$$

For each $u_n \in H^1_{0,}(0,T)$, we define $w_n \in H^1_{,0}(0,T)$ as unique solution of the backward problem

$$\partial_{tt} w_n(t) + \mu w_n(t) = u_n(t) \quad \text{for } t \in (0,T), \quad w_n(T) = \partial_t w_n(t)_{|t=T} = 0, \tag{2.10}$$

i.e., $w_n \in H^1_{,0}(0,T)$ solves the variational problem

$$a_\mu(v, w_n) = \langle u_n, v\rangle_{L^2(0,T)} \quad \text{for all } v \in H^1_{0,}(0,T)$$

with the bilinear form (1.9). In particular for $v = u_n$, this gives

$$a_\mu(u_n, w_n) = \|u_n\|^2_{L^2(0,T)}.$$

Analogously to the estimate (1.13) for the solution of (1.7), we conclude

$$\|\partial_t w_n\|^2_{L^2(0,T)} + \mu \|w_n\|^2_{L^2(0,T)} \leq \frac{1}{2} T^2 \|u_n\|^2_{L^2(0,T)}$$

for the solution w_n of (2.10), i.e.,

$$\|\partial_t w_n\|_{L^2(0,T)} \le \frac{1}{\sqrt{2}} T \, \|u_n\|_{L^2(0,T)}.$$

For the zero extension $\widetilde{u}_n \in L^2(-T,T)$ of $u_n \in H^1_{0,}(0,T)$, we obtain, when using the norm representation (2.7), and (2.9), that

$$\begin{aligned} \|\Box_\mu \widetilde{u}_n\|_{[H^1_0(-T,T)]'} &= \sup_{0\neq v \in H^1_{,0}(0,T)} \frac{|\langle \Box_\mu \widetilde{u}_n, \mathcal{E}v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} \ge \frac{|\langle \Box_\mu \widetilde{u}_n, \mathcal{E}w_n\rangle_{(-T,T)}|}{\|\partial_t w_n\|_{L^2(0,T)}} \\ &= \frac{|a_\mu(u_n, w_n)|}{\|\partial_t w_n\|_{L^2(0,T)}} = \frac{\|u_n\|^2_{L^2(0,T)}}{\|\partial_t w_n\|_{L^2(0,T)}} \ge \frac{\sqrt{2}}{T} \|u_n\|_{L^2(0,T)}, \end{aligned}$$

and the assertion follows by completion for $n \to \infty$. ■

Corollary 2.7 *The inner product space* $\left(\mathcal{H}_{0,}(0,T), \langle \Box_\mu \widetilde{(\cdot)}, \Box_\mu \widetilde{(\cdot)}\rangle_{[H^1_0(-T,T)]'}\right)$ *is complete, i.e., a Hilbert space.*

Proof. The assertion follows immediately from Lemma 2.6. ■

In the following, $\mathcal{H}_{0,}(0,T)$ is endowed with the Hilbertian norm $\|\Box_\mu \widetilde{(\cdot)}\|_{[H^1_0(-T,T)]'}$. With this new Hilbert space, the bilinear form

$$\widetilde{a}_\mu(\cdot,\cdot)\colon \mathcal{H}_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}, \quad \widetilde{a}_\mu(u,v) := \langle \Box_\mu \widetilde{u}, \mathcal{E}v\rangle_{(-T,T)},$$

is continuous, i.e.,

$$|\widetilde{a}_\mu(u,v)| = |\langle \Box_\mu \widetilde{u}, \mathcal{E}v\rangle_{(-T,T)}| \le \|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} \|\partial_t v\|_{L^2(0,T)} \tag{2.11}$$

for all $u \in \mathcal{H}_{0,}(0,T)$ and $v \in H^1_{,0}(0,T)$, and fulfills the inf–sup condition

$$\|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} = \sup_{0\neq v \in H^1_{,0}(0,T)} \frac{|\langle \Box_\mu \widetilde{u}, \mathcal{E}v\rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} = \sup_{0\neq v \in H^1_{,0}(0,T)} \frac{|\widetilde{a}_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} \tag{2.12}$$

for all $u \in \mathcal{H}_{0,}(0,T)$, where the norm representation (2.7) is used. In addition, Lemma 2.5 yields the representation

$$\widetilde{a}_\mu(u,v) = a_\mu(u,v) \tag{2.13}$$

for all $u \in H^1_{0,}(0,T) \subset \mathcal{H}_{0,}(0,T)$, $v \in H^1_{,0}(0,T)$, which is used in the following lemma.

Lemma 2.8 *For all* $0 \neq v \in H^1_{,0}(0,T)$, *there exists a function* $u_v \in \mathcal{H}_{0,}(0,T)$ *such that*

$$\widetilde{a}_\mu(u_v, v) > 0\,.$$

Proof. For $0 \neq v \in H^1_{,0}(0,T)$, there exists the unique solution $u_v \in H^1_{0,}(0,T)$ satisfying

$$a_\mu(u_v, w) = \langle v, w\rangle_{L^2(0,T)} \quad \text{for all } w \in H^1_{,0}(0,T).$$

Using the representation (2.13), this gives

$$\widetilde{a}_\mu(u_v, w) = \langle v, w\rangle_{L^2(0,T)} \quad \text{for all } w \in H^1_{,0}(0,T),$$

and in particular for $w = v$, we obtain

$$\widetilde{a}_\mu(u_v, v) = \|v\|^2_{L^2(0,T)} > 0,$$

i.e., the assertion. ∎

Next, we state the new variational setting for the scalar ordinary differential equation (1.7). For given $f \in [H^1_{,0}(0,T)]'$, we consider the variational formulation to find $u \in \mathcal{H}_{0,}(0,T)$ such that

$$\widetilde{a}_\mu(u, v) = \langle f, v\rangle_{(0,T)} \quad \text{for all } v \in H^1_{,0}(0,T), \tag{2.14}$$

i.e., the operator equation

$$\mathcal{E}'\Box_\mu \widetilde{u} = f \quad \text{ in } [H^1_{,0}(0,T)]'.$$

With the properties of the bilinear form $\widetilde{a}_\mu(\cdot,\cdot)$, the unique solvability of the variational formulation (2.14), i.e., the main theorem of this section, is proven.

Theorem 2.9 *For each given $f \in [H^1_{,0}(0,T)]'$, there exists a unique solution $u \in \mathcal{H}_{0,}(0,T)$ of the variational formulation* (2.14). *Furthermore,*

$$\mathcal{L}_\mu \colon\ [H^1_{,0}(0,T)]' \to \mathcal{H}_{0,}(0,T), \qquad \mathcal{L}_\mu f := u,$$

is an isomorphism satisfying

$$\|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} = \|\Box_\mu \widetilde{\mathcal{L}_\mu f}\|_{[H^1_0(-T,T)]'} = \|f\|_{[H^1_{,0}(0,T)]'}.$$

Proof. With the help of the Banach–Nečas–Babuška theorem [9, Theorem 2.6], the results in (2.11), (2.12) and Lemma 2.8 yield the existence and uniqueness of a solution $u \in \mathcal{H}_{0,}(0,T)$. In addition, with the variational formulation (2.14), the equalities

$$\|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} = \sup_{0\neq v\in H^1_{,0}(0,T)} \frac{|\widetilde{a}_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} = \sup_{0\neq v\in H^1_{,0}(0,T)} \frac{|\langle f, v\rangle_{(0,T)}|}{\|\partial_t v\|_{L^2(0,T)}} = \|f\|_{[H^1_{,0}(0,T)]'}$$

hold true, and therefore, the assertion. ∎

While the unique solution u of the variational formulation (2.14) is considered in $\mathcal{H}_{0,}(0,T)$, it turns out that indeed $u \in H^1_{0,}(0,T)$. In fact, the following lemma clarifies the relation between $\mathcal{H}_{0,}(0,T)$ and $H^1_{0,}(0,T)$.

Lemma 2.10 *There holds $\mathcal{H}_{0,}(0,T) = H^1_{0,}(0,T)$ with the norm equivalence inequalities*

$$\left(1+\frac{4}{\pi^2}\mu T^2\right)^{-1} \|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} \;\leq\; \|\partial_t u\|_{L^2(0,T)} \;\leq\; \left(1+\frac{1}{2}\sqrt{\mu}T\right) \|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'}$$

for all $u \in H^1_{0,}(0,T)$.

Proof. We first prove that $\mathcal{H}_{0,}(0,T) = H^1_{0,}(0,T)$. As $\mathcal{H}_{0,}(0,T) \supset H^1_{0,}(0,T)$, see Lemma 2.5, it remains to prove that $\mathcal{H}_{0,}(0,T) \subset H^1_{0,}(0,T)$. For this purpose, let $u \in \mathcal{H}_{0,}(0,T)$ be fixed. Consider the unique solution $\hat{u} \in H^1_{0,}(0,T)$ of the variational formulation (1.8) for the right–hand side $f = \mathcal{L}_\mu^{-1} u \in [H^1_{,0}(0,T)]'$, where $\mathcal{L}_\mu$ is the solution operator of Theorem 2.9. So, using Lemma 2.5 and the variational formulations (1.8), (2.14) this yields

$$\widetilde{a}_\mu(\hat{u}, v) = a_\mu(\hat{u}, v) = \langle f, v\rangle_{(0,T)} = \widetilde{a}_\mu(u, v)$$

for all $v \in H^1_{,0}(0,T)$, i.e., $u = \hat{u} \in H^1_{0,}(0,T)$. Thus, we have $\mathcal{H}_{0,}(0,T) \subset H^1_{0,}(0,T)$. The upper norm equivalence inequality is proven by

$$\begin{aligned} \|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} &= \sup_{0\neq v\in H^1_{,0}(0,T)} \frac{|\widetilde{a}_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} \\ &= \sup_{0\neq v\in H^1_{,0}(0,T)} \frac{|a_\mu(u,v)|}{\|\partial_t v\|_{L^2(0,T)}} \geq \frac{2}{2+\sqrt{\mu}T} \|\partial_t u\|_{L^2(0,T)} \end{aligned}$$

for all $u \in \mathcal{H}_{0,}(0,T) = H^1_{0,}(0,T)$, where the inf–sup conditions (2.12), (1.11) are used. The lower inequality follows from (2.8). ■

Corollary 2.11 *For all $u \in \mathcal{H}_{0,}(0,T)$ and all $v \in H^1_{,0}(0,T)$, the equality*

$$\widetilde{a}_\mu(u, v) = a_\mu(u, v)$$

is valid, i.e., the variational formulations (1.8) *and* (2.14) *are equivalent.*

Proof. The assertion follows immediately from Lemma 2.10 and (2.13). ■

Remark 2.12 *Functions $u \in C^2([0,T])$ with $u(0) = 0$ are contained in $\mathcal{H}_{0,}(0,T)$, since such functions are in $H^1_{0,}(0,T)$. Note that the second initial condition $\partial_t u(t)_{|t=0} = 0$ is not incorporated in the ansatz space $\mathcal{H}_{0,}(0,T)$.*

Remark 2.13 *The function u, defined by $u(t) = \sin(\sqrt{\mu}t)$ for $t \in (0,T)$, is obviously in $H^1_{0,}(0,T)$ and so, in $\mathcal{H}_{0,}(0,T)$. For this function, we have*

$$\|\partial_{tt} u + \mu u\|_{[H^1_{,0}(0,T)]'} = 0.$$

For

$$\widetilde{u}(t) = \begin{cases} 0 & \text{for } t \in (-T, 0], \\ \sin(\sqrt{\mu} t) & \text{for } t \in (0, T), \end{cases}$$

the first–order distributional derivative is identified with the function

$$\partial_t \widetilde{u}(t) = \begin{cases} 0 & \text{for } t \in (-T, 0], \\ \sqrt{\mu}\cos(\sqrt{\mu} t) & \text{for } t \in (0, T), \end{cases}$$

i.e., it is a regular distribution. To compute the second–order distributional derivative of $\widetilde{u}$*, we consider*

$$\begin{aligned} \langle \partial_{tt} \widetilde{u}, \varphi \rangle_{(-T,T)} &= -\int_{-T}^{T} \partial_t \widetilde{u}(t)\, \partial_t \varphi(t)\, \mathrm{d}t \\ &= -\int_{0}^{T} \sqrt{\mu}\cos(\sqrt{\mu} t)\, \partial_t \varphi(t)\, \mathrm{d}t \\ &= -\sqrt{\mu}\cos(\sqrt{\mu} t)\, \varphi(t)\Big|_0^T + \int_0^T \Big(-\mu \sin(\sqrt{\mu} t)\Big)\, \varphi(t)\, \mathrm{d}t \\ &= \sqrt{\mu}\, \varphi(0) - \mu \langle \widetilde{u}, \varphi \rangle_{(-T,T)} \end{aligned}$$

for all $\varphi \in C_0^\infty(-T,T)$*. Hence,*

$$\Box_\mu \widetilde{u} = \partial_{tt} \widetilde{u} + \mu \widetilde{u} = \sqrt{\mu}\, \delta_0$$

is a singular distribution with the Dirac distribution $\delta_0 \in H^{-1}_{[0,T]}(-T,T) \subset [H^1_0(-T,T)]'$*. Furthermore, it follows that*

$$\begin{aligned} \|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} &= \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{|\langle \Box_\mu \widetilde{u}, \mathcal{E} v \rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} = \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{\sqrt{\mu}\, |\langle \delta_0, \mathcal{E} v \rangle_{(-T,T)}|}{\|\partial_t v\|_{L^2(0,T)}} \\ &= \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{\sqrt{\mu}\, |v(0)|}{\|\partial_t v\|_{L^2(0,T)}} > 0 \end{aligned}$$

for, e.g., $v(t) = \frac{1}{T}(T-t)$*, where the norm representation* (2.7) *is used. To summarize, the function* $u(t) = \sin(\sqrt{\mu} t)$ *for* $t \in (0,T)$ *with*

$$\|\Box_\mu \widetilde{u}\|_{[H^1_0(-T,T)]'} > 0 \quad \text{and} \quad \|\partial_{tt} u + \mu u\|_{[H^1_{,0}(0,T)]'} = 0$$

solves the variational formulations (1.8) *and* (2.14) *for the right–hand side* $f_{v_0} \in [H^1_{,0}(0,T)]'$*,*

$$\langle f_{v_0}, v \rangle_{(0,T)} = \sqrt{\mu} v(0), \quad v \in H^1_{,0}(0,T),$$

which realizes the initial condition $\partial_t u(t)_{|t=0} = v_0 := \sqrt{\mu}$*.*

Remark 2.14 *The variational formulation* (2.14) *is the weak formulation of the differential equation*

$$\partial_{tt}\widetilde{u}(t) + \mu\widetilde{u}(t) = \widetilde{f}(t) = \begin{cases} 0 & \text{for } t \in (-T,0), \\ f(t) & \text{for } t \in [0,T), \end{cases}$$

which can be written as coupled system, using $u(t) = \widetilde{u}(t)$ *for* $t \in (0,T)$*, and* $u_-(t) = \widetilde{u}(t)$ *for* $t \in (-T,0)$*,*

$$\begin{aligned} \partial_{tt}u(t) + \mu u(t) &= f(t) && \text{for } t \in (0,T), \\ \partial_{tt}u_-(t) + \mu u_-(t) &= 0 && \text{for } t \in (-T,0), \quad u_-(-T) = \partial_t u_-(t)_{|t=-T} = 0 \end{aligned}$$

together with the transmission interface conditions

$$u(0) = u_-(0), \quad \partial_t u(t)_{|t=0} - \partial_t u_-(t)_{|t=0} = v_0$$

with given $v_0 \in \mathbb{R}$*, satisfying* $\langle \widetilde{f}, z\rangle_{(-T,T)} = v_0 z(0)$ *for all* $z \in H^1_0(-T,T)$*. The conditions* $u_-(-T) = \partial_t u_-(t)_{|t=-T} = 0$ *lead to* $u_-(t) = \widetilde{u}(t) = 0$ *for* $t \in (-T,0)$*, which finally implies the initial conditions*

$$u(0) = 0, \quad \partial_t u(t)_{|t=0} = v_0.$$

3 A generalized variational formulation for the wave equation

In this section, we generalize the approach, as introduced for the solution of the ordinary differential equation (1.7), to end up with a generalized inf–sup stable variational formulation for the Dirichlet boundary value problem for the wave equation (1.1). For this purpose, we first introduce notations analogously to them of Section 2.

In addition to the space–time domain $Q := \Omega \times (0,T)$ we consider the extended domain $Q_- := \Omega \times (-T,T)$. The dual space $[H^{1,1}_{0;,0}(Q)]'$ is characterized as completion of $L^2(Q)$ with respect to the Hilbertian norm

$$\|f\|_{[H^{1,1}_{0;,0}(Q)]'} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, v\rangle_Q|}{\|v\|_{H^{1,1}_{0;,0}(Q)}},$$

where $\langle \cdot,\cdot\rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$. Note that $[H^{1,1}_{0;,0}(Q)]'$ is a Hilbert space, see Section 2. For given $u \in L^2(Q)$, we define the extension $\widetilde{u} \in L^2(Q_-)$ by

$$\widetilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in Q, \\ 0 & \text{for } (x,t) \in Q_- \setminus Q. \end{cases} \tag{3.1}$$

The application of the wave operator $\Box := \partial_{tt} - \Delta_x$ to $\widetilde{u}$ is defined as a distribution on Q_-, i.e., for all test functions $\varphi \in C_0^\infty(Q_-)$, we define

$$\langle \Box \widetilde{u}, \varphi \rangle_{Q_-} := \int_{-T}^{T} \int_\Omega \widetilde{u}(x,t)\, \Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega u(x,t)\,\Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2}$$

This motivates to consider the dual space $[H_0^1(Q_-)]'$ of $H_0^1(Q_-)$, which is characterized as completion of $L^2(Q_-)$ with respect to the Hilbertian norm

$$\|g\|_{[H_0^1(Q_-)]'} := \sup_{0 \neq z \in H_0^1(Q_-)} \frac{|\langle g, z\rangle_{Q_-}|}{\|z\|_{H_0^1(Q_-)}},$$

where the inner product

$$\langle z_1, z_2\rangle_{H_0^1(Q_-)} = \langle \partial_t z_1, \partial_t z_2\rangle_{L^2(Q_-)} + \langle \nabla_x z_1, \nabla_x z_2\rangle_{L^2(Q_-)}, \quad z_1, z_2 \in H_0^1(Q_-),$$

induces the norm $\|\cdot\|_{H_0^1(Q_-)}$, and $\langle \cdot, \cdot\rangle_{Q_-}$ denotes the duality pairing as extension of the inner product in $L^2(Q_-)$, see [27, Satz 17.3]. Note that $[H_0^1(Q_-)]'$ is a Hilbert space, see Section 2. In addition we define the subspace

$$H^{-1}_{|\overline{Q}}(Q_-) := \left\{ g \in [H_0^1(Q_-)]' \colon \forall z \in H_0^1(Q_-) \text{ with } \operatorname{supp} z \subset \Omega \times (-T, 0)\colon\ \langle g, z\rangle_{Q_-} = 0 \right\}$$

of $[H_0^1(Q_-)]'$, endowed with the Hilbertian norm $\|\cdot\|_{[H_0^1(Q_-)]'}$. To characterize the subspace $H^{-1}_{|\overline{Q}}(Q_-)$, we introduce the following notations. Let $\mathcal{R} \colon H_0^1(Q_-) \to H^{1,1}_{0;,0}(Q)$ be the continuous and surjective restriction operator, defined by $\mathcal{R}z = z_{|Q}$ for $z \in H_0^1(Q_-)$, with its adjoint operator $\mathcal{R}' \colon [H^{1,1}_{0;,0}(Q)]' \to [H_0^1(Q_-)]'$. Furthermore, let $\mathcal{E} \colon H^{1,1}_{0;,0}(Q) \to H_0^1(Q_-)$ be any continuous and injective extension operator with its adjoint operator $\mathcal{E}' \colon [H_0^1(Q_-)]' \to [H^{1,1}_{0;,0}(Q)]'$, satisfying

$$\|\mathcal{E}v\|_{H_0^1(Q_-)} \leq c_{\mathcal{E}}\, \|v\|_{H^{1,1}_{0;,0}(Q)}$$

with a constant $c_{\mathcal{E}} > 0$, and $\mathcal{R}\mathcal{E}v = v$ for all $v \in H^{1,1}_{0;,0}(Q)$. An example for such an extension operator is given by reflection in $\Omega \times \{0\}$, i.e., consider the function $\overline{v}$, defined by

$$\overline{v}(x,t) = \begin{cases} v(x,t) & \text{for } (x,t) \in \Omega \times [0,T), \\ v(x,-t) & \text{for } (x,t) \in \Omega \times (-T,0) \end{cases}$$

for $(x,t) \in Q_-$, and a given function $v \in H^{1,1}_{0;,0}(Q)$, which leads to a constant $c_{\mathcal{E}} = 2$ in this particular case. With this, we prove the following lemma as the counter part of Lemma 2.1.

Lemma 3.1 *The spaces $(H^{-1}_{|\overline{Q}}(Q_-), \|\cdot\|_{[H_0^1(Q_-)]'})$ and $([H^{1,1}_{0;,0}(Q)]', \|\cdot\|_{[H^{1,1}_{0;,0}(Q)]'})$ are isometric, i.e., the mapping*

$$\mathcal{E}'_{|H^{-1}_{|\overline{Q}}(Q_-)} \colon\ H^{-1}_{|\overline{Q}}(Q_-) \to [H^{1,1}_{0;,0}(Q)]'$$

is bijective with

$$\|g\|_{[H^1_0(Q_-)]'} = \|\mathcal{E}'g\|_{[H^{1,1}_{0;,0}(Q)]'} \quad \textit{for all } g \in H^{-1}_{|\overline{Q}}(Q_-).$$

In addition, for $g \in H^{-1}_{|\overline{Q}}(Q_-)$, *the relation*

$$\langle g, z\rangle_{Q_-} = \langle \mathcal{E}'g, \mathcal{R}z\rangle_Q \quad \textit{for all } z \in H^1_0(Q_-) \tag{3.3}$$

i.e., $\mathcal{R}'\mathcal{E}'g = g$, *holds true. In particular, the subspace* $H^{-1}_{|\overline{Q}}(Q_-) \subset [H^1_0(Q_-)]'$ *is closed, i.e., complete.*

Proof. First, we prove that $\|g\|_{[H^1_0(Q_-)]'} = \|\mathcal{E}'g\|_{[H^{1,1}_{0;,0}(Q)]'}$ for all functionals $g \in H^{-1}_{|\overline{Q}}(Q_-)$. For this purpose, let $g \in H^{-1}_{|\overline{Q}}(Q_-)$ be arbitrary but fixed. The Riesz representation theorem gives the unique element $z_g \in H^1_0(Q_-)$ with

$$\langle g, z\rangle_{Q_-} = \langle z_g, z\rangle_{H^1_0(Q_-)} \quad \text{for all } z \in H^1_0(Q_-),$$

and $\|g\|_{[H^1_0(Q_-)]'} = \|z_g\|_{H^1_0(Q_-)}$. It holds true that $z_{g|\Omega\times(-T,0)} = 0$, since we have

$$0 = \langle g, z\rangle_{Q_-} = \langle z_g, z\rangle_{H^1_0(Q_-)} = \int_{-T}^0 \int_\Omega \Big[\partial_t z_g(x,t)\,\partial_t z(x,t) + \nabla_x z_g(x,t)\cdot\nabla_x z(x,t)\Big]\,\mathrm{d}x\,\mathrm{d}t$$

for all $z \in H^1_0(Q_-)$ with $\operatorname{supp} z \subset \Omega \times (-T, 0)$. Hence, we have

$$\langle g, z\rangle_{Q_-} = \langle z_g, z\rangle_{H^1_0(Q_-)} = \langle \mathcal{R}z_g, \mathcal{R}z\rangle_{H^{1,1}_{0;,0}(Q)} \tag{3.4}$$

for all $z \in H^1_0(Q_-)$. So, using (3.4) with $z = \mathcal{E}v$ for $v \in H^{1,1}_{0;,0}(Q)$ this gives

$$\langle \mathcal{E}'g, v\rangle_Q = \langle g, \mathcal{E}v\rangle_{Q_-} = \langle \mathcal{R}z_g, \mathcal{R}\mathcal{E}v\rangle_{H^{1,1}_{0;,0}(Q)} = \langle \mathcal{R}z_g, v\rangle_{H^{1,1}_{0;,0}(Q)}, \tag{3.5}$$

i.e.,

$$\|\mathcal{E}'g\|_{[H^{1,1}_{0;,0}(Q)]'} = \|\mathcal{R}z_g\|_{H^{1,1}_{0;,0}(Q)} = \|z_g\|_{H^1_0(Q_-)} = \|g\|_{[H^1_0(Q_-)]'}.$$

Next, we prove that $\mathcal{E}'_{|H^{-1}_{|\overline{Q}}(Q_-)}$ is surjective. For this purpose, let $f \in [H^{1,1}_{0;,0}(Q)]'$ be given. Set $g_f = \mathcal{R}'f$, i.e.,

$$\langle g_f, z\rangle_{Q_-} = \langle \mathcal{R}'f, z\rangle_{Q_-} = \langle f, \mathcal{R}z\rangle_Q$$

for all $z \in H^1_0(Q_-)$. With this it follows immediately that $g_f \in H^{-1}_{|\overline{Q}}(Q_-)$. Moreover, we have

$$\langle \mathcal{E}'g_f, v\rangle_Q = \langle g_f, \mathcal{E}v\rangle_{Q_-} = \langle f, \mathcal{R}\mathcal{E}v\rangle_Q = \langle f, v\rangle_Q$$

for all $v \in H^{1,1}_{0;,0}(Q)$, i.e., $\mathcal{E}'g_f = f$ in $[H^{1,1}_{0;,0}(Q)]'$. In other words, $\mathcal{E}'_{|H^{-1}_{|\overline{Q}}(Q_-)}$ is surjective. Finally, (3.3) follows from (3.4) and (3.5) for $v = \mathcal{R}z$ for any $z \in H^1_0(Q_-)$. The last assertion of the lemma is straightforward. ∎

The last lemma gives immediately the following corollary.

Corollary 3.2 *For all $g \in H^{-1}_{|\overline{Q}}(Q_-)$, the norm representation*

$$\|g\|_{[H^1_0(Q_-)]'} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle g, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}}$$

holds true.

Proof. Let $g \in H^{-1}_{|\overline{Q}}(Q_-)$ be arbitrary but fixed. With Lemma 3.1, we have

$$\|g\|_{[H^1_0(Q_-)]'} = \|\mathcal{E}'g\|_{[H^{1,1}_{0;,0}(Q)]'} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle \mathcal{E}'g, v\rangle_Q|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle g, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}},$$

i.e., the assertion is proven. ■

Next, we introduce

$$\mathcal{H}(Q) := \Big\{ u = \widetilde{u}_{|Q} : \widetilde{u} \in L^2(Q_-),\ \widetilde{u}_{|\Omega \times (-T,0)} = 0,\ \Box\widetilde{u} \in [H^1_0(Q_-)]' \Big\},$$

with the norm

$$\|u\|_{\mathcal{H}(Q)} := \sqrt{\|u\|^2_{L^2(Q)} + \|\Box\widetilde{u}\|^2_{[H^1_0(Q_-)]'}}\,.$$

For a function $u \in \mathcal{H}(Q)$, the condition $\Box\widetilde{u} \in [H^1_0(Q_-)]'$ involves that there exists an element $f_u \in [H^1_0(Q_-)]'$ with

$$\langle \Box\widetilde{u}, \varphi\rangle_{Q_-} = \langle f_u, \varphi\rangle_{Q_-} \quad \text{for all } \varphi \in C^\infty_0(Q_-).$$

Note that $\varphi \in H^1_0(Q_-)$ for $\varphi \in C^\infty_0(Q_-)$, and that $C^\infty_0(Q_-)$ is dense in $H^1_0(Q_-)$. Hence, the element $f_u \in [H^1_0(Q_-)]'$ is unique and therefore, in the following, we identify the distribution $\Box\widetilde{u}\colon C^\infty_0(Q_-) \to \mathbb{R}$ with the functional $f_u\colon H^1_0(Q_-) \to \mathbb{R}$.

Next, we state properties of the space $\mathcal{H}(Q)$. Clearly, $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a normed vector space and it is even a Banach space.

Lemma 3.3 *The normed vector space $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a Banach space.*

Proof. Consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}(Q)$. Hence, $(u_n)_{n\in\mathbb{N}} \subset L^2(Q)$ is also a Cauchy sequence in $L^2(Q)$, and $(\Box\widetilde{u}_n)_{n\in\mathbb{N}} \subset [H^1_0(Q_-)]'$ is also a Cauchy sequence in $[H^1_0(Q_-)]'$. So, there exist $u \in L^2(Q)$ and $f \in [H^1_0(Q_-)]'$ with

$$\lim_{n\to\infty} \|u_n - u\|_{L^2(Q)} = 0, \quad \lim_{n\to\infty} \|\Box\widetilde{u}_n - f\|_{[H^1_0(Q_-)]'} = 0.$$

For $\varphi \in C^\infty_0(Q_-)$, we have

$$\begin{aligned}
\langle \Box\widetilde{u}, \varphi\rangle_{Q_-} = \langle \widetilde{u}, \Box\varphi\rangle_{L^2(Q_-)} &= \int_0^T \int_\Omega u(x,t)\Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega u_n(x,t)\,\Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{n\to\infty} \langle \widetilde{u}_n, \Box\varphi\rangle_{L^2(Q_-)} = \lim_{n\to\infty} \langle \Box\widetilde{u}_n, \varphi\rangle_{Q_-} = \langle f, \varphi\rangle_{Q_-},
\end{aligned}$$

i.e., $\Box\widetilde{u} = f \in [H_0^1(Q_-)]'$. Hence, $u \in \mathcal{H}(Q)$ follows. ∎

With the abstract inner product $\langle\cdot,\cdot\rangle_{[H_0^1(Q_-)]'}$ of $[H_0^1(Q_-)]'$, the inner product

$$\langle\cdot,\cdot\rangle_{\mathcal{H}(Q)} := \langle\cdot,\cdot\rangle_{L^2(Q)} + \langle\Box\widetilde{(\cdot)}, \Box\widetilde{(\cdot)}\rangle_{[H_0^1(Q_-)]'}$$

induces the norm $\|\cdot\|_{\mathcal{H}(Q)}$. Hence, the space $(\mathcal{H}(Q), \langle\cdot,\cdot\rangle_{\mathcal{H}(Q)})$ is even a Hilbert space, but this abstract inner product is not used explicitly in the remainder of this work.

Lemma 3.4 *For all $u \in \mathcal{H}(Q)$ there holds $\Box\widetilde{u} \in H^{-1}_{|\overline{Q}}(Q_-)$ with*

$$\|\Box\widetilde{u}\|_{[H_0^1(Q_-)]'} = \sup_{0\neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle\Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}}. \tag{3.6}$$

Proof. First, we prove that $\Box\widetilde{u} \in H^{-1}_{|\overline{Q}}(Q_-)$. For this purpose, let $u \in \mathcal{H}(Q)$ and $z \in H_0^1(Q_-)$ with $\operatorname{supp} z \subset \Omega \times (-T, 0)$ be arbitrary but fixed. Due to $z_{|\Omega\times(-T,0)} \in H_0^1(\Omega \times (-T, 0))$ there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega \times (-T, 0))$ with $\|z_{|\Omega\times(-T,0)} - \psi_n\|_{H_0^1(\Omega\times(-T,0))} \to 0$ as $n \to \infty$, where

$$\|w\|_{H_0^1(\Omega\times(-T,0))} = \left(\int_{-T}^0\int_\Omega \left[|\partial_t w(x,t)|^2 + |\nabla_x w(x,t)|^2\right] \mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

for $w \in H_0^1(\Omega \times (-T, 0))$. For $n \in \mathbb{N}$, define

$$\varphi_n(x,t) = \begin{cases} \psi_n(x,t) & \text{for } (x,t) \in \Omega \times (-T,0), \\ 0 & \text{for } (x,t) \in \Omega \times [0,T), \end{cases}$$

i.e., $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(Q_-)$ satisfies

$$\|z - \varphi_n\|_{H_0^1(Q_-)} = \|z_{|\Omega\times(-T,0)} - \psi_n\|_{H_0^1(\Omega\times(-T,0))} \to 0$$

as $n \to \infty$. So, it follows that

$$\langle\Box\widetilde{u}, z\rangle_{Q_-} = \lim_{n\to\infty}\langle\Box\widetilde{u}, \varphi_n\rangle_{Q_-} = \lim_{n\to\infty}\int_0^T\int_\Omega u(x,t)\,\Box\varphi_n(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and therefore, the assertion. The norm representation follows from $\Box\widetilde{u} \in H^{-1}_{|\overline{Q}}(Q_-)$ and Corollary 3.2. ∎

Lemma 3.5 *It holds true that $H^{1,1}_{0;0,}(Q) \subset \mathcal{H}(Q)$. Furthermore, each $u \in H^{1,1}_{0;0,}(Q)$ with zero extension $\widetilde{u}$, as defined in (3.1), satisfies*

$$\|\Box\widetilde{u}\|_{[H_0^1(Q_-)]'} \le \|u\|_{H^{1,1}_{0;0,}(Q)}, \tag{3.7}$$

and

$$\langle\Box\widetilde{u}, z\rangle_{Q_-} = a(u, \mathcal{R}z) = -\langle\partial_t u, \partial_t\mathcal{R}z\rangle_{L^2(Q)} + \langle\nabla_x u, \nabla_x\mathcal{R}z\rangle_{L^2(Q)} \tag{3.8}$$

for all $z \in H_0^1(Q_-)$, where $a(\cdot,\cdot)$ is the bilinear form (1.3).

Proof. First, we prove that $H^{1,1}_{0;0,}(Q) \subset \mathcal{H}(Q)$. For $u \in H^{1,1}_{0;0,}(Q)$, we define the extension $\widetilde{u}$, see (3.1). By construction, we have $\widetilde{u} \in L^2(Q_-)$, and $\widetilde{u}_{|\Omega\times(-T,0)} = 0$. It remains to prove that $\Box\widetilde{u} \in [H^1_0(Q_-)]'$. For this purpose, define the functional $f_u \in [H^1_0(Q_-)]'$ by

$$\langle f_u, z\rangle_{Q_-} = a(u, \mathcal{R}z)$$

for all $z \in H^1_0(Q_-)$, where $a(\cdot,\cdot)$ is the bilinear form (1.3). The continuity of f_u follows from

$$|\langle f_u, z\rangle_{Q_-}| = |a(u, \mathcal{R}z)| \leq \|u\|_{H^{1,1}_{0;0,}(Q)} \|\mathcal{R}z\|_{H^{1,1}_{0;,0}(Q)} \leq \|u\|_{H^{1,1}_{0;0,}(Q)} \|z\|_{H^1_0(Q_-)}$$

for all $z \in H^1_0(Q_-)$, where the estimate (1.4) is used. Using the definition (3.2) and integration by parts, this gives

$$\begin{aligned}\langle \Box\widetilde{u}, \varphi\rangle_{Q_-} &= \int_0^T \int_\Omega u(x,t)\, \Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\langle \partial_t u, \partial_t \mathcal{R}\varphi\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{R}\varphi\rangle_{L^2(Q)} = \langle f_u, \varphi\rangle_{Q_-}\end{aligned}$$

for all $\varphi \in C^\infty_0(Q_-)$, i.e., $\Box\widetilde{u} = f_u \in [H^1_0(Q_-)]'$. The equality (3.8) follows from the density of $C^\infty_0(Q_-)$ in $H^1_0(Q_-)$. The estimate (3.7) is proven by

$$\begin{aligned}\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} &= \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box\widetilde{u}, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle f_u, \mathcal{E}v\rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \\ &= \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|a(u, \mathcal{R}\mathcal{E}v)|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \leq \|u\|_{H^{1,1}_{0;0,}(Q)}\end{aligned}$$

when using the norm representation (3.6), the equality (3.8), and (1.4). ■

Next, by completion, we define the Hilbert space

$$\mathcal{H}_{0,}(Q) := \overline{H^{1,1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q),$$

endowed with the Hilbertian norm $\|\cdot\|_{\mathcal{H}(Q)}$, i.e.,

$$\mathcal{H}_{0,}(Q) = \Big\{ v \in \mathcal{H}(Q) \colon\ \exists (v_n)_{n\in\mathbb{N}} \subset H^{1,1}_{0;0,}(Q) \text{ with } \|v_n - v\|_{\mathcal{H}(Q)} \to 0 \Big\}.$$

Lemma 3.6 *For $u \in \mathcal{H}_{0,}(Q)$ there holds*

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} \geq \frac{\sqrt{2}}{T}\, \|u\|_{L^2(Q)}.$$

Proof. For $0 \neq u \in \mathcal{H}_{0,}(Q)$, there exists a non–trivial sequence $(u_n)_{n\in\mathbb{N}} \subset H^{1,1}_{0;0,}(Q)$, $u_n \not\equiv 0$, with

$$\lim_{n\to\infty} \|u - u_n\|_{\mathcal{H}(Q)} = 0.$$

For each $u_n \in H^{1,1}_{0;0,}(Q)$, we define $w_n \in H^{1,1}_{0;,0}(Q)$ as unique solution of the variational formulation

$$a(v, w_n) = \langle u_n, v \rangle_{L^2(Q)} \quad \text{for all } v \in H^{1,1}_{0;0,}(Q)$$

with the bilinear form (1.3). In particular for $v = u_n$, this gives

$$a(u_n, w_n) = \|u_n\|^2_{L^2(Q)}.$$

Analogously to the estimate (1.5) for the solution of (1.1), we conclude

$$\|w_n\|_{H^{1,1}_{0;,0}(Q)} \leq \frac{1}{\sqrt{2}} T \, \|u_n\|_{L^2(Q)} \, .$$

For the zero extension $\widetilde{u}_n \in L^2(Q_-)$ of $u_n \in H^{1,1}_{0;0,}(Q)$, we obtain, when using the norm representation (3.6) and (3.8), that

$$\begin{aligned}\|\Box \widetilde{u}_n\|_{[H^1_0(Q_-)]'} &= \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box \widetilde{u}_n, \mathcal{E} v \rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \geq \frac{|\langle \Box \widetilde{u}_n, \mathcal{E} w_n \rangle_{Q_-}|}{\|w_n\|_{H^{1,1}_{0;,0}(Q)}} \\ &= \frac{|a(u_n, w_n)|}{\|w_n\|_{H^{1,1}_{0;,0}(Q)}} = \frac{\|u_n\|^2_{L^2(Q)}}{\|w_n\|_{H^{1,1}_{0;,0}(Q)}} \geq \frac{\sqrt{2}}{T} \|u_n\|_{L^2(Q)},\end{aligned}$$

and the assertion follows by completion for $n \to \infty$. ■

Corollary 3.7 *The inner product space* $\left(\mathcal{H}_{0,}(Q), \langle \Box \widetilde{(\cdot)}, \Box \widetilde{(\cdot)} \rangle_{[H^1_0(Q_-)]'}\right)$ *is complete, i.e., a Hilbert space.*

Proof. The assertion follows immediately from Lemma 3.6. ■

In the following, $\mathcal{H}_{0,}(Q)$ is endowed with the Hilbertian norm $\|\Box \widetilde{(\cdot)}\|_{[\overline{H}^1_0(Q_-)]'}$. With this new Hilbert space, the bilinear form

$$\widetilde{a}(\cdot, \cdot) \colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}, \quad \widetilde{a}(u, v) := \langle \Box \widetilde{u}, \mathcal{E} v \rangle_{Q_-},$$

is continuous, i.e.,

$$|\widetilde{a}(u, v)| \;=\; |\langle \Box \widetilde{u}, \mathcal{E} v \rangle_{Q_-}| \;\leq\; \|\Box \widetilde{u}\|_{[H^1_0(Q_-)]'} \|v\|_{H^{1,1}_{0;,0}(Q)} \tag{3.9}$$

for all $u \in \mathcal{H}_{0,}(Q)$ and $v \in H^{1,1}_{0;,0}(Q)$, and fulfills the inf–sup condition

$$\|\Box \widetilde{u}\|_{[H^1_0(Q_-)]'} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box \widetilde{u}, \mathcal{E} v \rangle_{Q_-}|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0 \neq v \in H^{1,1}_{0;,0}(Q)} \frac{|\widetilde{a}(u, v)|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} \tag{3.10}$$

for all $u \in \mathcal{H}_{0,}(Q)$, where the norm representation (3.6) is used. In addition, Lemma 3.5 yields the representation

$$\widetilde{a}(u, v) = a(u, v) \tag{3.11}$$

for all $u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$, $v \in H^{1,1}_{0;,0}(Q)$, which is used in the following lemma.

Lemma 3.8 *For all* $0 \neq v \in H^{1,1}_{0;,0}(Q)$*, there exists a function* $u_v \in \mathcal{H}_{0,}(Q)$ *such that*

$$\widetilde{a}(u_v, v) > 0\,.$$

Proof. For $0 \neq v \in H^{1,1}_{0;,0}(Q)$, there exists a unique solution $u_v \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$, satisfying

$$a(u_v, w) = \langle v, w\rangle_{L^2(Q)} \quad \text{for all } w \in H^{1,1}_{0;,0}(Q).$$

Using the representation (3.11), this gives

$$\widetilde{a}(u_v, w) = \langle v, w\rangle_{L^2(Q)} \quad \text{for all } w \in H^{1,1}_{0;,0}(Q),$$

and in particular for $w = v$, we obtain

$$\widetilde{a}(u_v, v) = \|v\|^2_{L^2(Q)} > 0,$$

i.e., the assertion. ■

Next, we state the new variational setting for the wave equation (1.1). For given $f \in [H^{1,1}_{0;,0}(Q)]'$, we consider the variational formulation to find $u \in \mathcal{H}_{0,}(Q)$ such that

$$\widetilde{a}(u, v) = \langle f, v\rangle_Q \quad \text{for all } v \in H^{1,1}_{0;,0}(Q), \tag{3.12}$$

i.e., the operator equation

$$\mathcal{E}'\Box\widetilde{u} = f \quad \text{in } [H^{1,1}_{0;,0}(Q)]'.$$

With the properties of the bilinear form $\widetilde{a}(\cdot,\cdot)$, the unique solvability of the variational formulation (3.12), i.e., the main theorem of this paper, is proven.

Theorem 3.9 *For each given* $f \in [H^{1,1}_{0;,0}(Q)]'$*, there exists a unique solution* $u \in \mathcal{H}_{0,}(Q)$ *of the variational formulation* (3.12)*. Furthermore,*

$$\mathcal{L}\colon\ [H^{1,1}_{0;,0}(Q)]' \to \mathcal{H}_{0,}(Q), \qquad \mathcal{L}f = u,$$

is an isomorphism satisfying

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} = \|\Box\widetilde{\mathcal{L}f}\|_{[H^1_0(Q_-)]'} = \|f\|_{[H^{1,1}_{0;,0}(Q)]'}.$$

Proof. With the help of the Banach–Nečas–Babuška theorem [9, Theorem 2.6], the results in (3.9), (3.10) and Lemma 3.8 yield the existence and uniqueness of the solution $u \in \mathcal{H}_{0,}(Q)$. In addition, with the variational formulation (3.12), the equalities

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]'} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\widetilde{a}(u,v)|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0\neq v\in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, v\rangle_Q|}{\|v\|_{H^{1,1}_{0;,0}(Q)}} = \|f\|_{[H^{1,1}_{0;,0}(Q)]'}.$$

hold true and, therefore, the assertion. ■

While for the ordinary differential equation (1.7), this new approach leads to the same variational setting as already considered in [26, Section 4] and [25], see Lemma 2.10 and Corollary 2.11, the situation is different for the wave equation (1.1). In greater detail, for the variational formulation (3.12), the Banach–Nečas–Babuška theorem [9, Theorem 2.6] is applicable, whereas the variational formulation (1.2) does not fit in this framework, see Theorem 1.1. Additionally, Lemma 3.5 and (3.11) show that the new variational formulation (3.12) is a generalization of the variational formulation (1.2). Next, the following functions are given to get a first impression of the solution space $\mathcal{H}_{0,}(Q)$.

Remark 3.10 *For $u \in C^2(\overline{Q})$ with $u_{|\Omega\times\{0\}} = u_{|\Sigma} = 0$ there holds $u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$. Note that the second initial condition*

$$\partial_t u(\cdot,t)_{|t=0} = 0 \quad \textit{in } \Omega$$

is not incorporated in the ansatz space $\mathcal{H}_{0,}(Q)$, see (1.4).

Remark 3.11 *Consider the smooth function*

$$u(x,t) = \sin(\pi x)\sin(\pi t) \qquad \textit{for } (x,t) \in (0,1)\times(0,1) = Q,$$

satisfying $u_{|\Omega\times\{0\}} = u_{|\Sigma} = 0$ and $\Box u = 0$ in Q. But there is $\Box\widetilde{u} \neq 0$ with

$$\widetilde{u}(x,t) = \begin{cases} u(x,t) & \textit{for } (x,t) \in Q, \\ 0 & \textit{for } (x,t) \in Q_- \setminus Q, \end{cases}$$

since the distributional derivative fulfills

$$\langle \Box\widetilde{u}, \varphi\rangle_{Q_-} = \int_0^T \int_\Omega u(x,t)\,\Box\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \pi \int_\Omega \sin(\pi x)\,\varphi(x,0)\,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(Q_-)$. Thus, the function $u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$ solves the variational formulation (3.12) *with the right–hand side $f_{v_0} \in [H^{1,1}_{0;,0}(Q)]'$,*

$$\langle f_{v_0}, v\rangle_Q = \pi \int_\Omega \sin(\pi x)\,v(x,0)\,\mathrm{d}x, \quad v \in H^{1,1}_{0;,0}(Q),$$

i.e., the function u satisfies the inhomogeneous initial condition

$$\partial_t u(x,t)_{|t=0} = v_0(x) := \pi\sin(\pi x), \quad x \in \Omega,$$

see (1.4).

4 Conclusions and outlook

In this paper, we presented a new approach to set up a bijection for the solution of the wave equation, when the right–hand side is considered in the dual space of the test space of the variational formulation. For this, we had to enlarge the ansatz space to prove a related inf–sup stability condition. Based on these results, we aim to derive a space–time finite element method for the numerical solution of the wave equation, and of related problems, which is unconditionally stable, and which also allows for an adaptive resolution of the solution simultaneously in space and time, and for an efficient solution, which is also parallel in time. First numerical results are very promising, see [15], and the related numerical analysis is ongoing work, and will be published elsewhere.

The presented results on the existence and uniqueness of solutions for the wave equation, in particular the bijectivity results for the solution operator in related function spaces, are of utmost importance for the analysis of related boundary integral equations for the approximate solution of the wave equation by boundary element methods. Using the appropriate Dirichlet and Neumann trace operators, we are able to analyze the mapping properties of related boundary integral operators [23], i.e., boundedness and coercivity, to close the existing gap in using different norms, see, e.g., [22]. Note that this norm gap also results in error estimates, which are not optimal, see also [24] for first numerical results.

We end this paper with an outlook for possible extensions of the approach in Section 3. Since the constructions of the spaces $\mathcal{H}(Q)$, $\mathcal{H}_{0,}(Q)$ and the proofs in this section mainly rely on the treatment of the second–order temporal differential operator $\partial_{tt} + \mu$ with a parameter μ, a generalization of the results of this section to differential operators $\partial_{tt} + \mathcal{A}_x$, acting on vector fields or scalar fields is possible, where the second–order spatial differential operator $\mathcal{A}_x$ has to fulfill certain properties, e.g., boundedness and ellipticity. A more detailed discussion is left for future work.

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